Thursday June 20

All presentations will take place in Huxley Building Room 140 (basement)

- 08.30 – 09.00: REGISTRATION (Huxley Building Reception, 180 Queen's Gate)
- 09.00 – 09.15: Darren Crowdy – Welcome and Opening Remarks
- 09.15 – 10.20: Xianfeng (David) Gu: "Computational methods in conformal geometry based on Hodge theory and Ricci flow"
- 10.20 – 10.50 : Tea/coffee break
- 10.50 – 11.55: Ronald Lok Ming Lui: "Surface registration using extremal Teichmüller maps"
- 11.55 – 14.00: Lunch break/Discussions [Room 747, 7th Floor, Huxley Building]
- 14.00 – 15.00: Kenneth Stephenson: "Curvature flow in conformal mapping"
- 15.00 – 16.00: Monica Hurdal: "Investigating disease in the human brain with conformal maps and conformal invariants"
- 16.00 – 16:30: Tea/coffee break
- 16:30 – 17.00: Brock Williams: "Applications of circle packing moduli"
- 17.00 – 17:25: Timothy McNicholl: "Computable and incomputable boundary extensions"
- 17.30 – 18.30: Welcome Reception [Room 747, 7th Floor, Huxley Building]

Organizers: Darren Crowdy, Thomas DeLillo, Darryl Holm

Workshop jointly sponsored by:

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LONDON MATHEMATICAL SOCIETY

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Friday June 21

All presentations will take place in Huxley Building Room 140 (basement)

- 09.15 – 09.45: Thomas DeLillo: "Fourier series methods for numerical conformal mapping of smooth domains"
- 09.45 – 10.15: Everett Kropf: "Efficient numerical calculation of unbounded Schwarz-Christoffel transformations"
- 10.20 – 10.50 : Tea/coffee break
- 10.50 – 11.55: Mohamed M. S. Nasser: "A fast numerical method for conformal mapping of multiply connected regions"
- 11.55 – 14.00: Lunch break/Discussions [Room 747, 7th Floor, Huxley Building]
- 14.00 – 15.00: Sergey Kushnarev: "Geodesic equation on the universal Teichmüller space, Teichons and imaging"
- 15.00 – 16.00: Dmitry Khavinson: "Two-dimensional shapes and lemniscates"
- 16.00 – 16.30: Tea/coffee break
- 16.30 – 16.55: Sergei Rogosin: "Hele-Shaw Flow with a small obstacle"
- 16.55 – 17.20: Alan Elcrat: "Hollow vortices behind a step"
- 17.20 – 17.45: Vladimir Mityushev: "Poincaré $\boldsymbol{\alpha}$–series for classical Schottky groups and its applications"

End of workshop

Computational methods in conformal geometry based on Hodge theory and Ricci flow

Xianfeng (David) Gu
Department of Computer Science, SUNY Stony Brook

Computational conformal geometry is an emerging interdisciplinary field, combing modern geometry with computer science. In mathematics, conformal geometry is the intersection of complex analysis, Riemann surface theory, differential geometry, algebraic topology and Partial differential equation; in computer science, conformal geometry plays fundamental roles in computer graphics, geometric modeling, computer vision, networking and medical imaging.

Conformal geometry studies the invariants under angle-preserving transformations. Fundamental problems in the field include computing conformal structures of metric surfaces, conformal invariants (conformal modules), conformal mappings between surfaces, uniformization Riemannian metrics, extremal quasi-conformal mappings and so on. The solutions to these problems will be introduced.

Computational strategies including surface harmonic mapping method, holomorphic differential method and discrete surface Ricci flow. Ricci flow is the process to deform the Riemannian metric proportional to the curvature, such that the curvature evolves according to a heat diffusion process and becomes constant eventually. Surface Ricci flow leads to the uniformization metric without curvature blowing up. The discrete surface Ricci flow theory and algorithm will be covered in details.

Computational Conformal geometry has been applied in many engineering fields, including global parameterization in computer graphics, deformable surface registration in computer vision, manifold splines in geometric modeling, efficient routing in networking, brain mapping and virtual colonoscopy in medical imaging. These applications will be briefly introduced.

This work is collaborated with Shing-Tung Yau, Feng Luo, Ronald Lok Lui and many other mathematicians, computer scientists and medical doctors.

Surface registration using extremal Teichmuller maps

Ronald Lok Ming Lui
Chinese University of Hong Kong (CUHK)

Registration, which aims to find an optimal 1-1 correspondence between shapes, is an important process in different research areas. Conformal mappings have been widely used to obtain a diffeomorphism between shapes that minimizes angular distortion. Conformal registrations are beneficial since it preserves the local geometry well. However, when constraints are enforced (e.g. landmarks), conformal mappings generally do not exist. This motivates us to look for a unique constrained quasiconformal registration, which minimizes the conformality distortion. Under suitable condition on the constraints, a unique diffeomorphism, called the extremal Teichmüller map between two surfaces can be obtained, which minimizes the maximal conformality distortion. In this talk, an efficient iterative algorithm, called the Quasi-conformal (QC) iterations, to compute the Teichmüller map will be presented. The basic idea is to represent the set of diffeomorphisms using Beltrami coefficients (BCs), and look for an optimal BC associated to the desired Teichmüller map. The associated diffeomorphism can be efficiently reconstructed from the optimal BC using the Linear Beltrami Solver(LBS). Using BCs to represent diffeomorphisms guarantees the diffeomorphic property of the registration. Using the proposed method, the Teichmüller map can be accurately and efficiently computed within 10 seconds. The obtained registration is guaranteed to be bijective. This proposed algorithm can also be practically applied to real applications. In the second part of my talk, I will present how extremal Teichmüller map can be used for brain landmark matching registration, constrained texture mapping and face recognition.

Curvature flow in conformal mapping

Kenneth Stephenson & Joan Lind,
University of Tennessee, Knoxville, USA

This talk will explore the notion of "curvature flow" in conformal mapping, first observed in the conformal flattening of polygonal Riemann surfaces via circle packing (see C. Collins, T. Driscoll, and K. S., *Computational Methods and Function Theory* 3 (2003), p 325–347.) This flow has a concrete meaning in the discrete setting, where it concerns the patterns of angle movement during circle packing computations. In preliminary joint work reported here, Joan Lind and the speaker study the classical interpretation. The definition is simple enough: curvature flow of a locally one-to-one mapping ϕ is the gradient flow of $\log|\phi'|$ on the domain of ϕ. But what is "flowing" and how can that flow be used in conformal mapping. The talk will illustrate both discrete and classical flows, concentrating in particular on classical Loewner slit mappings.

Investigating disease in the human brain with conformal maps and conformal invariants

Monica Hurdal
Florida State University, USA

Conformal maps offer many nice mathematical properties for applications, including angle preservation, uniqueness, and the ability to create conformal maps in different geometries. Such features are important for medical applications, and in particular, conformal maps are being used to create maps of the brain. In this presentation, I will discuss how we create quasi-conformal ”flat” maps of the brain from MRI brain scans using image segmentation, topology, and the computational method of circle packing. We are using these conformal brain maps to investigate the progression of brain diseases and illnesses, including Alzheimer’s, schizophrenia and depression. Conformal invariants, such as extremal lengths, and their role in studying the shape of regions of the human brain will also be discussed.

Applications of circle packing moduli

Brock Williams
Texas Tech University, USA

The conformal module of a ring or quadrilateral encodes basic information about its shape. Similarly, a module for triangulations of rings or quadrilaterals can be computed using circle packing. We will describe our progress in using circle packing moduli in applications including the quantification of cotton fiber maturity, the effect of geography on the spread of disease, and the calculation of nearness of persons in a social network.

Computable and incomputable boundary extensions

Timothy McNicholl
Iowa State University, USA

How does the computability of the boundary extension of a conformal map relate to the computability of the map itself and the boundary of its range? Using the tools of computability theory, we can draw some definite conclusions. For example, we show that there is a computable conformal map with an incomputable boundary extension. We then discuss what additional information is required to compute a boundary extension. In particular, we show that a computable surjection of the unit interval onto the boundary of the range does not provide sufficient information for computing a boundary extension.

Geodesic equation on the Universal Teichmüller space, Teichons and Imaging

Sergey Kushnarev
Singapore University of Technology and Design (SUTD), Singapore

In this talk I will describe a way to parametrize a space of planar shapes by members of a coset space $PSL_2(R)\backslash\mathrm{Diff}(S^1)$. These functions are also known as welding maps in the Teichmüller theory. The geodesic equation on this space with the Weil-Petersson metric is the EPDiff(S^1), the Euler-Poincaré equation on the group of diffeomorphisms of the circle S^1. It admits a class of soliton-like solutions, Teichons. These solutions have momenta as a linear combination of delta functions. The resulting simpler geodesic equation is more tractable from the numerical point of view. Applications of image matching with Teichons on the database of hippocampi will be demonstrated.

Two-dimensional shapes and lemniscates

Dmitry Khavinson
University of South Florida, USA

The newly emerging field of vision and pattern recognition often focuses on the study of two dimensional “shapes”, i.e. simple, closed smooth curves. A common approach to describing shapes consists in defining a “natural” embedding of the space of curves into a metric space and studying the mathematical structure of the latter. Another idea that has been pioneered by Kirillov and developed recently among others by Mumford and Sharon consists of representing each shape by its “fingerprint”, a diffeomorphism of the unit circle. Kirillov’s theorem states that the correspondence between shapes and fingerprints is a bijection modulo conformal automorphisms of the disk. In this talk we shall discuss the joint work with P. Ebenfelt and Harold S. Shapiro outlining an alternative interpretation of the problem of shapes and Kirillov’s theorem based on finding a set of natural and simple fingerprints that is dense in the space of all diffeomorphisms of the unit circle. This approach is inspired by the celebrated theorem of Hilbert regarding approximation of smooth curves by lemniscates. We shall outline proofs of the main results and discuss some interesting function-theoretic ramifications and open-ended questions regarding possibilities of numerical applications of this idea.

A fast numerical method for conformal mapping of multiply connected regions

Mohamed M. S. Nasser
King Khalid University, Saudi Arabia

In this talk, a fast boundary integral equation method for approximating the conformal mapping from multiply connected regions of finite connectivity onto Koebe's thirty nine canonical slit regions as well as the canonical region obtained by removing rectilinear slits from a strip will be presented. The method requires $O((m+1)n\ln n)$ operations where $m+1$ is the connectivity of the region and n is the number of nodes in the discretization of each boundary component. The method is based on reformulating the conformal mapping problem as a Riemann-Hilbert problem which is solved by a uniquely solvable boundary integral equation with the generalized Neumann kernel. Discretizing the integral equation by the Nyström method yields a dense and non-symmetric linear system which is solved by an iterative method based on the restarted version of the generalized minimal residual (GMRES) method. Each iteration of the GMRES method requires a matrix-vector product which is computed using the Fast Multipole Method (FMM). Two numerical examples will be presented to illustrate that the presented method has the ability to handle regions with complex geometry and very high connectivity. A multiply connected region of connectivity 46 with piecewise smooth boundaries will be considered in the first example and a multiply connected circular region of connectivity 1096 will be considered in the second example.

A brief description of computing the conformal mapping of multiply connected regions as well as its inverse using a boundary integral equation with the adjoint generalized Neumann kernel will also be presented in this talk. The integral equation is derived by reformulating the conformal mapping problem as an adjoint Riemann-Hilbert problem.

Fourier series methods for numerical conformal mapping of smooth domains

Thomas DeLillo
Wichita State University, USA

We will give an overview of a class of numerical methods for approximating domains bounded by smooth curves. The methods use FFTs to compute Fourier and Laurent coefficients for the mapping functions and are extensions of Fornberg's original method for simply connected maps. Newton-like methods are developed to find the boundary correspondences and centers and radii of the circles. We show that the inner linear systems are discretizations of the identity plus a compact operator and can be solved efficiently with the conjugate gradient method. Maps from canonical slit domains are used to provide orthogonal grids. This is joint work with Everett Kropf.

Efficient numerical calculation of unbounded Schwarz-Christoffel transformations

Everett Kropf
Harvard University, USA

We discuss recently developed numerics for the Schwarz-Christoffel transformation for unbounded, multiply connected domains. The original infinite product representation for the derivative of the mapping function is replaced by a finite product where the factors satisfy certain boundary conditions. These factors are approximated by truncated Laurent series where coefficients are given by a least squares method applied to the boundary conditions. This results in a much more efficient approach than the method based directly on the infinite product representation, making the accurate mapping of domains of higher connectivity feasible. This is joint work with Thomas DeLillo, Alan Elcrat, and John Pfaltzgraff.

Hele-Shaw flow with a small obstacle

Sergei V.Rogosin [joint report with G.Mishuris, Aberystwyth University, UK]
Belarusian State University, Belarus

The report is devoted to an asymptotic analysis of the Hele-Shaw moving boundary value problem with a small obstacle in the flow. Classical Hele-Shaw problem deals with the description of the free boundary encircled the domain occupied by incompressible fluid in so called Hele-Shaw cell (see, e.g. [1]), i.e. in a narrow space between two closely related plates. Different driving mechanism are supposed for the fluid flow, e.g. presence of a source/sink in the fluid domain. Application of asymptotic methods for approximation of Green's function goes back to the classical paper by J.Hadamard, where the method of regular perturbation is performed. Recently, V.Maz'ya and A.Movchan obtain a number of asymptotic formulas for Green's and Robin functions, related to different boundary value problems for a number of differential operators were obtained in the case of singular perturbation of domains (see [2], [3], [4] and references therein). Our asymptotic analysis of the Hele-Shaw model is based on the asymptotic formulas by V.Maz'ya and A.Movchan.

References: 1. B. Gustafsson, A. Vasil'ev, Conformal and Potential Analysis in Hele-Shaw cells, Birkhuser Verlag, Basel-Boston-Berlin, 2006. 2. V.Maz'ya, A.Movchan, Uniform asymptotics of Green's kernels for mixed and Neumann problems in domains with small holes and inclusions. Isakov, Victor (ed.), Sobolev spaces in mathematics. III: Applications in mathematical physics. New York, NY: Springer; Novosibirsk: Tamara Rozhkovskaya Publisher. International Mathematical Series, 10, 2009, 277–316. 3. V.Maz'ya, A.Movchan, Uniform asymptotics of Green's kernels in perforated domains and meso-scale approximation // Complex Var. Elliptic Equ., 57, No. 2-4, 137-154 (2012). 4. V.Maz'ya, A.Movchan, M.Nieves, Michael, Green's kernels and meso-scale approximations in perforated domains. Lecture Notes in Mathematics 2077. Berlin: Springer. xvii, 254 p. (2013).

Hollow vortices behind a step

Alan Elcrat
Wichita State University, USA

Closed and open hollow wakes are considered as analytic models for the 2D inviscid steady flow past a plate normal to the stream. It is shown that only open configurations exist which satisfy the Kutta condition. The main argument is based on considering the plate located on the edge of a step with varying height. It is shown that solutions for open wakes exist for backward, null and forward-facing steps, while closed wakes only exist for backward-facing steps. The occurrence of secondary separation has been modeled by adding a hollow region attached to the downstream corner. Peculiar accuracy issues of the problem are pointed out which may explain other contradictory results from the literature. It is shown how the Kirchhoff wake is a limiting solution for values of the governing parameters.

Poincaré $\boldsymbol{\alpha}$–series for classical Schottky groups and its applications

Vladimir Mityushev & Natalia Rylko
Pedagogical University of Cracow, Poland

The Poincaré $\boldsymbol{\alpha}$–series ($\boldsymbol{\alpha} \in \mathbb{R}^n$) for classical Schottky groups is introduced and used to solve Riemann–Hilbert problems for n–connected circular domains. The classical Poincaré θ_2–series can be obtained from the $\boldsymbol{\alpha}$–series by the substitution $\boldsymbol{\alpha} = \mathbf{0}$. The real Jacobi inversion problem and its generalisations are investigated via the Poincaré $\boldsymbol{\alpha}$–series. In particular, it is shown that the Riemann theta–function coincides with the Poincaré $\boldsymbol{\alpha}$–series. Relations to conformal mappings to slit domains and the Schottky–Klein prime function are established. A fast algorithm to compute Poincaré series for disks close to each other is outlined.