

## Scalar variance decay in chaotic advection and Batchelor-regime turbulence

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The decay of the variance of a diffusive scalar in chaotic advection flow (or equivalently Batchelor-regime turbulence) is analyzed using a model in which the advection is represented by an inhomogeneous baker's map on the unit square. The variance decays exponentially at large times, with a rate that has a finite limit as the diffusivity  $\kappa$  tends to zero and is determined by the action of the inhomogeneous map on the gravest Fourier modes in the scalar field. The decay rate predicted by recent theoretical work that follows scalar evolution in linear flow and then averages over all stretching histories is shown to be incorrect. The exponentially decaying scalar field is shown to have a spatial power spectrum of the form  $P(k) \sim k^{-\sigma}$  at wave numbers small enough for diffusion to be neglected, with  $\sigma < 1$ .

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The mixing of a passive scalar advected by an incompressible time-dependent flow that is smoothly varying in space and subject to weak diffusion remains a problem of fundamental theoretical interest, as well as being of practical importance in environmental and industrial flows. Such flows are sometimes described, when the time dependence is deterministic, as “chaotic advection” flows or, when the time dependence is random, as the “Batchelor regime” of turbulence. Scalar evolution in turbulent flows has recently been the focus of much research [1], and the Batchelor regime represents an important limiting case.

One measure of mixing is the rate of decay of the scalar variance. Numerical simulations of globally chaotic flows (i.e., flows without transport barriers) have shown that the variance decays exponentially at large times [2–4]. There is some evidence that the decay rate is asymptotically independent of diffusivity for small diffusivity [2,4], though it has been argued recently in [5] that this is not always the case. Pierrehumbert [2] has suggested that the scalar has an equilibrium spatial structure during the period of exponential decay associated with a “strange eigenmode” of the advection-diffusion equation and some evidence for persistent spatial structure in decaying scalar in a chaotic advection flow has been presented in [6]. This paper discusses the mechanism that determines the asymptotic decay rate of the scalar variance.

Antonsen *et al.* [4], working in the context of “chaotic advection,” and Son [7] and Balkovsky and Fouxon [8], working in the context of “Batchelor turbulence,” have argued that the rate of exponential decay may be predicted from the statistics of the stretching history experienced by different fluid parcels. These statistics are conventionally expressed by the probability density function  $P(h, t)$  for the finite-time Liapunov exponent distribution. The finite-time Liapunov exponent  $h$  at time  $t$  is simply the maximal average rate of stretching experienced following a given fluid particle up to that time.

The theoretical work mentioned above is based on the assumption of scale separation between the flow (which varies on the large scale) and the scalar field (which varies on a smaller scale). This allows the scalar within each fluid element to evolve, independently of all other elements, as if the flow were a linear function of space. In this description the scalar variance decays within each fluid element at a rate depending on the time history of the local flow. Taking the ensemble average over all such histories gives the decay rate of the total scalar variance.

In this paper we argue that the above description is inadequate for quantitative prediction of exponential decay rates. We describe a different mechanism that leads to exponential decay of the variance of a diffusive scalar in a chaotic advection flow. This new mechanism involves the gravest spatial Fourier modes in the system and cannot be captured by any “local” theory that follows the evolution of small scale structures within each fluid element. We shall therefore refer to the new mechanism as a “global” mechanism to distinguish from the “local” mechanism for decay envisaged in the papers mentioned above. We here illustrate the “global” mechanism in a model where the advecting flow is represented by an area-preserving baker's map.

The inhomogeneous baker's map has been described in [9,10]. The unit square is divided by a cut parallel to the  $x$  axis into two rectangles of area  $\alpha$  and  $\beta$ , with  $\alpha + \beta = 1$  and, without loss of generality,  $\alpha \leq \beta$ . The two rectangles are stretched in the  $y$  direction by, respectively, factors  $\alpha^{-1}$  and  $\beta^{-1}$  and then reassembled into a unit square.

The baker's map can be taken to represent the effect of a two-dimensional flow on a scalar field in the unit square, in the sense that it can be approximated arbitrarily closely by the effect of a continuous two-dimensional flow. For our purposes it is sufficient to consider the effect of the map on a scalar field that varies only in the contracting direction of the map, i.e., in the  $x$  direction. This reduces the problem from two dimensions to one dimension.

Repeated application of the baker's map may be regarded as equivalent to the advective effect of a time-periodic flow that is present only for an instant during each period, but which leads to finite particle displacements during that instant. The effect of diffusion during the instant that the flow is present may be neglected, but the scalar may evolve diffusively during the remainder of each period.

Let the scalar field be represented by  $\Theta(x)$ . Under the baker's map

$$\{\Theta(x):0\leq x<1\}\rightarrow\begin{cases}\Theta(\alpha^{-1}x) & :0\leq x<\alpha, \\ \Theta(\beta^{-1}(x-\alpha)) & : \alpha\leq x<1. \end{cases} \quad (1)$$

This map is applied to the scalar field at intervals of time  $T$ . Between applications of the map the scalar field evolves according to the one-dimensional diffusion equation

$$\Theta_t = \kappa \Theta_{xx}, \quad (2)$$

where  $\kappa$  is the diffusivity. Periodic boundary conditions are imposed at  $x=0$  and  $x=1$ . The evolution of the scalar concentration, as measured by the scalar variance  $\mathcal{E}(t)$ , for example, may be followed numerically by applying Eq. (1) on a grid, Fourier transforming to solve Eq. (2) and then applying the inverse transform to return to the grid. It is found, as in [2,4] that there is a first stage in which the decay of the variance is super-exponential and then a second stage in which the decay is exponential.

The behavior in the first stage may be captured by a model analogous to that in [4] which follows the evolution of the scalar field within each fluid element and then integrates over all possible stretching histories for fluid elements. According to that model, the variance  $\mathcal{E}$  of a scalar field that initially varies sinusoidally with wave number  $k_0$  is given by

$$\mathcal{E}(t) = \mathcal{E}(0) \int_0^\infty dh P(h,t) \int_0^\infty d\tau Q(\tau,t) \exp(-\kappa k_0^2 e^{2ht} \tau), \quad (3)$$

where  $\tau = e^{-2ht} \int_0^t \exp(2h(t')t')$  depends on the history of stretching and  $Q(\tau,t)$  is its distribution. At large times  $h$  and  $\tau$  may be regarded as independent [4]. The stretching properties of the inhomogeneous baker's map are analyzed in detail in [10] and the function  $P(h,t)$  is derived explicitly. Here it is sufficient to note that the minimum possible value of stretching rate  $h$  is  $-\log(\beta)/T$  and the minimum possible value of  $\tau$ , at large  $t$ , is  $T\alpha^2/(1-\alpha^2)$ . It follows from Eq. (3) that the variance  $\mathcal{E}(t)$  decays superexponentially for all time. In [4] an additional integration over orientation of wave number with respect to stretching direction allows exponential, rather than super-exponential decay in time. However that is not relevant in the one-dimensional problem considered here and a different explanation of the exponential decay is needed.

It is useful to write the scalar field as a Fourier series in  $x$ ,

$$\Theta(x,t) = \sum_{n=-\infty}^{\infty} \Theta_n(t) e^{2\pi i n x}, \quad (4)$$

with  $\Theta_n = \overline{\Theta_{-n}}$ , ( $\overline{\phantom{x}}$ ) denoting a complex conjugate. The scalar variance,  $\mathcal{E}(t)$ , may then be evaluated as  $\sum_{n=-\infty}^{\infty} |\Theta_n(t)|^2$ . If we write  $\Theta_n(1T) = \Theta_n^l$  then the action of a single application of the baker's map plus diffusion acting over a time  $T$  may be expressed in terms of a transfer matrix  $M$  acting on the Fourier coefficients as

$$\Theta_n^{l+1} = \sum_{m=-\infty}^{\infty} M_{nm} \Theta_m^l. \quad (5)$$

The components of the transfer matrix may be straightforwardly evaluated as

$$M_{nm} = \frac{\sin(n\pi\alpha)}{\pi} \frac{(\beta-\alpha)m e^{-4\pi^2\kappa T n^2} e^{-ni\pi\alpha}}{(m-\alpha n)(m-\beta n)}. \quad (6)$$

Two special cases are  $\alpha=0$ , when  $M_{nm} = e^{-4\pi^2\kappa T n^2} \delta_{nm}$ , and  $\alpha=\frac{1}{2}$ , when  $M_{nm}=0$  unless  $m=\frac{1}{2}n$ , in which case  $M_{nm} = e^{-4\pi^2\kappa T n^2}$ . For the case  $\alpha=0$ , i.e., no stretching, an initial condition of a single wave with only  $\Theta_{\pm 1}$  nonzero implies scalar variance decay according to  $\mathcal{E}(1T) = \mathcal{E}(0) \exp(-8\pi^2\kappa T)$ , i.e., exponential decay, but at a rate proportional to diffusivity. For the case  $\alpha=\frac{1}{2}$ , i.e., homogeneous stretching, the corresponding initial condition gives variance decay according to  $\mathcal{E}(1T) = \mathcal{E}(0) \exp(-32\pi^2\kappa T(4^l-1)/3)$ , i.e., super-exponential decay and consistent with Eq. (3).

For  $0 < \alpha < \frac{1}{2}$  the evolution must be calculated numerically. Within the Fourier representation this is done by truncating the Fourier series, so that  $-N \leq n \leq N$ . We denote the corresponding truncation of the transfer matrix by  $M^{(N)}$ . In the long-time limit the variance  $\mathcal{E}(1T)$  will, according to a calculation at finite truncation  $N$ , vary as  $|\mu|^{2l}$ , where  $\mu$  is the eigenvalue of the truncated transfer matrix  $M^{(N)}$  with largest modulus. Explicit numerical calculation shows that all eigenvalues have modulus less than 1, implying decay of the variance at large times. The symmetry properties of the matrix  $M$  ( $M_{nm} = \overline{M_{-n,-m}}$ ), imply that if  $\mu$  is an eigenvalue, with eigenvector  $\Theta_n = \chi_n$ , say, then  $\overline{\mu}$  is also an eigenvalue, with eigenvector  $\Theta_n = \overline{\chi_{-n}}$ . After a large number of iterations the Fourier coefficients of the scalar field will be dominated by these two eigenvectors, so that

$$\Theta_n^l \simeq \mu^l \chi_n + \overline{\mu}^l \overline{\chi_{-n}}. \quad (7)$$

From the form of the components of the transfer matrix given in Eq. (6) the largest eigenvalue is expected to be independent of the truncation  $N$  provided that  $\exp(-4\pi^2\kappa T N^2) \ll 1$ . This was verified to be the case and the corresponding decay rates checked against the explicit numerical solution of Eqs. (1) and (2). It was also verified that for each  $\alpha$  the decay factor  $|\mu|$  appeared to tend to a limit as  $\kappa \rightarrow 0$ . Indeed it was also shown that, for fixed  $\kappa$ ,  $|\mu|$  tended to a limit as the truncation  $N$  increased. These results are

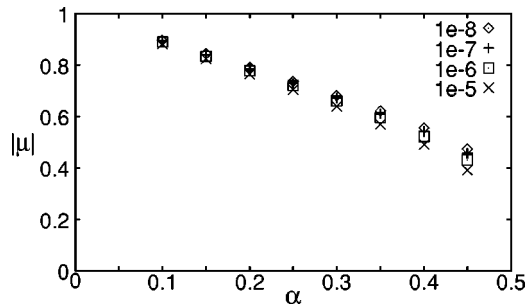


FIG. 1. Decay factor  $|\mu|$  as a function of  $\alpha$ , for diffusivity  $\kappa = 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}$  for truncation  $N=800$ .

summarized in Figs. 1 and 2, which show the variation of  $\mu$  with  $\alpha$ , in 1 for fixed truncation  $N=800$  and for different values of  $\kappa$  and in 2 for fixed  $\kappa=10^{-7}$  and varying  $N$ , from  $N=1$  to  $N=800$ . These results confirm that, whilst non-zero  $\kappa$  is essential for the decay of the scalar variance (if there were no diffusion then there would be no decay), the value of the decay rate (in the problem considered here) is essentially independent of  $\kappa$  in the small- $\kappa$  limit. The decay rate is largest when the baker's map is close to homogeneous ( $|\mu| \rightarrow 0$  as  $\alpha \rightarrow \frac{1}{2}$ ) and decreases as the map becomes more inhomogeneous ( $|\mu| \rightarrow 1$  as  $\alpha \rightarrow 0$ ).

The results shown in Fig. 2 show that the largest eigenvalue of the truncated transfer matrices  $M^{(N)}$  converges rapidly as  $N$  increases and is close to its limiting value even when  $N$  is quite small. Indeed, provided that  $\alpha$  is not too close to  $\frac{1}{2}$ , a useful estimate of the decay rate is obtained by truncating to the gravest Fourier mode ( $N=1$ ). This suggests that it is primarily the effect of the map on the gravest spatial Fourier modes in the system that determines the decay rate.

The transfer matrix shows that one application of the inhomogeneous baker map disperses scalar variance initially at wave number  $k$  to a range of wave numbers, but concentrated in peaks about the two wave numbers  $\alpha^{-1}k$  and  $\beta^{-1}k$ . Of particular importance when  $k$  is small is the finite width of the peaks, which implies that one iteration of the map leaves some of the scalar variance initially at wave number  $k$  remaining at that wave number (indeed some may move to wave numbers less than  $k$ ). This allows the possibility that the transfer matrix has a nonzero eigenvalue and therefore that there is exponential, rather than super-exponential, decay. (Note that when  $\alpha = \frac{1}{2}$  the position of the nonzero elements in the transfer matrix implies that the eigenvalues of

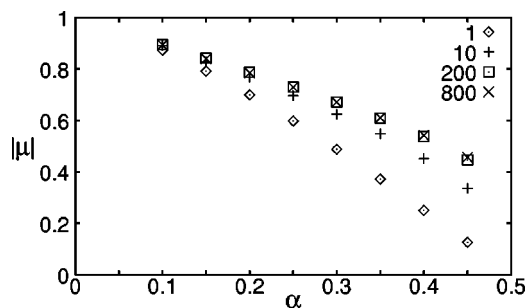


FIG. 2. Decay factor  $|\mu|$  as a function of  $\alpha$ , for truncation  $N = 1, 10, 200, 800$ , with diffusivity  $\kappa = 10^{-7}$ .

any finite truncation of the matrix must all be zero.) When  $k$  is large the dispersive effect of the width of the peaks is less important, since the width remains constant as  $k$  increases.

It may be shown that the transfer in wave number space is more dispersive when the baker's map is less homogeneous in physical space, i.e., when  $\alpha$  is smaller, i.e., the dispersion arises from large-scale spatial variations in the stretching effects of the inhomogeneous baker's map. (The same effect has been noted previously in the context of vortical flows [11].) It follows that the decay rate should then also be smaller, since the greater dispersion implies slower transfer of scalar variance out of the lowest spatial wave numbers in the system, and indeed this is the behavior observed.

While the decay rate appears to be governed by the action of the baker's map on the large scales, the exponentially decaying mode that emerges has a complex structure at small scales. A simple theory for the wave number spectrum of the decaying mode may be developed as follows.

After many iterations the scalar variance in each wave number  $n$  is, from Eq. (7),

$$\begin{aligned} |\Theta_n^l|^2 + |\Theta_{-n}^l|^2 &= 2|\Theta_n^l|^2 \\ &= 2|\mu|^{2l}(|\chi_n|^2 + |\chi_{-n}|^2) + 2(\bar{\mu}^{2l}\bar{\chi}_{-n}\bar{\chi}_n \\ &\quad + \mu^{2l}\chi_n\chi_{-n}). \end{aligned} \quad (8)$$

The ratio  $(\bar{\mu}^{2l}\bar{\chi}_{-n}\bar{\chi}_n + \mu^{2l}\chi_n\chi_{-n})/(2|\mu|^{2l}(|\chi_n|^2 + |\chi_{-n}|^2))$  will, as  $l$  increases, oscillate about an average value of 0 and it is useful to consider  $2|\mu|^{2l}(|\chi_n|^2 + |\chi_{-n}|^2)$  as the ‘‘average scalar variance’’ in each wave number  $n$ . We now regard wave number as continuously varying (and denoted by  $k$ ) and define by analogy the ‘‘average spectral density’’  $\bar{E}^l(k)$ . We shall assume that the important effect of the baker's map in determining the spectrum is to move scalar variance from wave number  $k$  to wave numbers  $\alpha^{-1}k$  and  $\beta^{-1}k$  in proportions  $\alpha$  and  $\beta$ , respectively. This suggests that  $\bar{E}^l(k)$  satisfies the recurrence relation

$$\bar{E}^{l+1}(k) = e^{-\kappa T k^2} \{ \alpha^2 \bar{E}^l(k\alpha) + \beta^2 \bar{E}^l(k\beta) \}. \quad (9)$$

The extra factors of  $\alpha$  and  $\beta$  take account of the fact that if the wave number is multiplied by a certain factor then the power spectral density is multiplied by the inverse so that the total variance is conserved. Furthermore in the exponentially decaying stage of evolution then  $\bar{E}^{l+1}(k) = |\mu|^{2l} \bar{E}^l(k)$ , hence

$$|\mu|^{2l} \bar{E}^l(k) = e^{-\kappa T k^2} \{ \alpha^2 \bar{E}^l(k\alpha) + \beta^2 \bar{E}^l(k\beta) \}. \quad (10)$$

This serves as an equation for  $\bar{E}^l(k)$ . If we consider wave numbers for which diffusion is unimportant, so that  $e^{-\kappa T k^2} \simeq 1$ , then Eq. (10) has the power law solution  $\bar{E}^l(k) \sim |\mu|^{2l} k^{-\sigma}$ , provided that the spectral slope  $\sigma$  satisfies the condition

$$|\mu|^2 = \alpha^{2-\sigma} + \beta^{2-\sigma}. \quad (11)$$

Note that  $\sigma$  is therefore determined by the decay factor  $|\mu|$ . In particular, if  $|\mu| < 1$  then  $\sigma < 1$ . For example, if  $|\mu|$

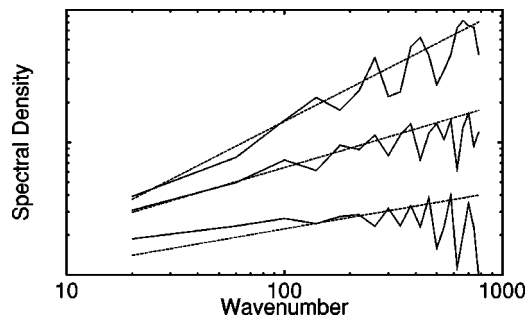


FIG. 3. Scaled average spectral density  $\bar{E}^l(k)$  (solid lines) and power law predicted by Eq. (11) (dashed lines) for (from top to bottom)  $\alpha=0.4$ , 0.3, and 0.2.

$=\sqrt{(\alpha^2+\beta^2)}$  then  $\sigma=0$ . In the limiting case of no decay  $|\mu|=1$  and  $\sigma=1$ . This happens to correspond to the case where the scalar variance is maintained constant, e.g., by a forcing at small wave numbers, and the predicted power-law spectrum is then just the Batchelor  $k^{-1}$  spectrum. The spectrum of the decaying mode being shallower than  $k^{-1}$  implies strongly singular behavior of the scalar field at small scales, but in practice the singularity is resolved by diffusion. Figure 3 shows the average spectral density for the slowest decaying mode for  $\alpha=0.2,0.3,0.4$  and also that predicted by Eq. (11), which is in good qualitative agreement.

We now return to regarding the baker's map as representing the effect of a two-dimensional flow on a two-dimensional field. The prediction of a "local" theory for the decay factor (based on the distribution function for the

stretching rates) is, following [10],  $1-2\alpha+2\alpha^2$ . For  $0<\alpha<0.42$  this is less than the "global" decay factor (for  $\kappa\rightarrow 0$ ) shown in Fig. 1, for a  $y$ -independent scalar field. But the slowest decaying structure in the two-dimensional problem must have decay factor greater than or equal to that shown in Fig. 1. Hence the "local" theory cannot be correct and it must be the "global" mechanism that controls the rate of decay. Support for this conclusion is provided by recent numerical simulations of two-dimensional flows [12], which show that the decay rate of the  $n$ th moment of the scalar is proportional to  $n$ . This is inconsistent with the predictions of "local" theories, but consistent with the scalar taking the form of a globally determined decaying eigenmode, as described here.

This paper illustrates, using the baker map as the simplest possible relevant model, how the rate of scalar decay is determined. The map acts on the large-scale contrast in the scalar field and, in each iteration, reduces the amplitude of this contrast by a fixed factor. Similarly in a two-dimensional flow, action of the flow over some suitable fixed time interval also reduces the amplitude of the scalar field by a fixed factor. A fuller study, analyzing this process in detail in two-dimensional flow, is in preparation and will be reported elsewhere.

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- [1] B.I. Shraiman and E.D. Siggia, *Nature (London)* **405**, 639 (2000).
- [2] R.T. Pierrehumbert, in *Chaos Applied to Fluid Mixing*, edited by H. Aref and M.S. El Naschie (Pergamon/Elsevier, New York, 1995), pp. 347–365.
- [3] V. Toussaint, Ph. Carrière, and F. Raynal, *Phys. Fluids* **7**, 2587 (1995).
- [4] T.M. Antonsen, Z. Fan, E. Ott, and E. Garcia-Lopez, *Phys. Fluids* **8**, 3094 (1996).
- [5] V. Toussaint, Ph. Carrière, J. Scott, and J.-N. Gence, *Phys. Fluids* **12**, 2834 (2000).
- [6] D. Rothstein, E. Henry, and J.P. Gollub, *Nature (London)* **401**, 770 (1999).
- [7] D.T. Son, *Phys. Rev. E* **59**, 3811 (1999).
- [8] E. Balkovsky and A. Fouxon, *Phys. Rev. E* **60**, 4164 (1999).
- [9] J.D. Farmer, E. Ott, and J.A. Yorke, *Physica D* **7**, 153 (1983).
- [10] E. Ott and T.M. Antonsen, *Phys. Rev. A* **39**, 3660 (1989).
- [11] G. Pedrizzetti and J.C. Vassilicos, *J. Fluid Mech.* **406**, 109 (2000).
- [12] R.T. Pierrehumbert, *Chaos* **10**, 61 (2000).