Fractal dimensions of lines in chaotic advection

J. C. H. Fung and J. C. Vassilicos*
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England

(Received 28 August 1990)

We release a line in a flow produced by a blinking vortex[1] which is known to advect fluid elements "chaotically" for a certain range of parameters. This is a similar problem to that of a line evolving in phase space through the action of an area-preserving map, addressed a decade ago by Berry *et al.*[2] These authors have classified the convolutions of such a line as being either "tendril shaped" or "whorl shaped." Recent numerical simulations of lines released in 2-D turbulence[3] have shown that they only develop "whorls," i.e., spiral structures. These spiral structures are produced by the eddying regions of the flow[4] and are responsible for the noninteger value of the "fractal" dimension $\mathscr{D}_K$ of the line, as measured by the box-counting algorithm. This "fractal" dimension is actually a Kolmogorov capacity. It has also been shown recently[3] that the Kolmogorov capacity is a measure of local self-similarity, whereas the Hausdorff dimension $\mathscr{D}_H$ is a measure of global self-similarity. Spirals are a good example of locally self-similar objects, for which $\mathscr{D}_K > 1$ but $\mathscr{D}_H = 1$. For conciseness we call a line H fractal when $\mathscr{D}_H > 1$, and K fractal when $\mathscr{D}_K > 1$. Most experimental and numerical evidence to date for "fractal" interfaces in turbulent flows is in fact evidence showing these interfaces to be K fractal. In fact, in the numerical simulations of Vassilicos,[3] lines have been found not to be H fractal. Whether a line in chaotic advection becomes K fractal or H fractal is not a trivial question. If one neglects the effect of the unsteadiness of the flow, and thinks in terms of a single vortex at a fixed point in space wrapping the line around it, then it is easy to show that the spiral thus created has a $\mathscr{D}_K > 1$ but $\mathscr{D}_H = 1$ (and, in particular, that $\mathscr{D}_K < 2$; the question of whether a line in chaotic advection is space filling is therefore not a trivial one either). But if one concentrates on the similarity between Aref's blinking vortex and a two-dimensional map, then one may be reminded of the Hénon attractor[5] which is known to have a transversal Cantor-like structure that is H fractal. In fact, pictures of the line in the blinking vortex flow show that line to have a comparable stretched and folded structure to that of the Hénon attractor. We measure $\mathscr{D}_H$ by measuring the length of the line with various resolutions and find that $\mathscr{D}_H$ grows with time above 1. By zooming into the pictures of our line we can see its self-similar structure, and are therefore inclined to conclude that lines in chaotic advection do become H fractal. We also measure $\mathscr{D}_K$ by the box-counting algorithm, and find that it also grows with time above 1, but is not equal to $\mathscr{D}_H$. It is a known mathematical fact that in general $\mathscr{D}_K \geqslant \mathscr{D}_H$, and our findings are consistent with this requirement. But we do not yet understand what this nonvanishing difference between $\mathscr{D}_K$ and $\mathscr{D}_H$ means for a line in chaotic advection. Furthermore, we find that both $\mathscr{D}_K$ and $\mathscr{D}_H$ increase as the switching of the vortex from one location to the other becomes faster. It is not clear whether these two fractal dimensions tend, asymptotically with time, toward a value strictly smaller than 2 or not. The interest of this work is to show how efficient unsteadiness (which is the central component of 2-D chaotic advection) can be for creating H fractal structures through a process of folding that it adds to stretching of the flow.[6] We compare with numerical simulations of 2-D turbulence[3] where the simulated, self-similar cascade of eddies fails to produce H fractal structures, and only produces K fractals.

[1] H. Aref, J. Fluid Mech. **143**, 1 (1984).
[2] M. V. Berry, N. C. Balazs, M. Tabor, and A. Voros, Ann. Phys. **122**, 26 (1979).
[3] J. C. Vassilicos, Ph. D. dissertation, University of Cambridge, 1990.
[4] A. A. Wray and J. C. Hunt, in *Topological Fluid Mechanics*, Proceedings of the IUTAM Symposium (Cambridge U. P., Cambridge, 1990).
[5] M. Hénon, Commun. Math. Phys. **50**, 69 (1976).
[6] J. M. Ottino, *The Kinematics of Mixing: Stretching, Chaos and Transport* (Cambridge U. P., Cambridge, 1989).

 0899-8213/91/051433-01$02.00