PHYSICS OF FLUIDS VOLUME 16, NUMBER 1 JANUARY 2004

A new Eulerian–Lagrangian length-scale in turbulent flows

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(Received 19 June 2003; accepted 8 October 2003; published online 5 December 2003)

We introduce a time-dependent Eulerian–Lagrangian length-scale and an inverse locality hypothesis which explain scalings of second order one-particle Lagrangian structure functions observed in kinematic simulations (KS) of homogeneous isotropic turbulence. Our KS results are consistent with the physical picture that particle trajectories are more/less autocorrelated if they are smoother/rougher as a result of encountering less/more straining stagnation points, thus leading to enhanced/reduced turbulent diffusion. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1630325]

Turbulent diffusion is a phenomenon of central importance in oceanic, atmospheric, engineering, and astrophysical flows. It is known, since the seminal work of Taylor,[1] that the turbulent diffusivity (of contaminants, pollutants, or other substances advected by the turbulent flow) is proportional to the time-integral of the Lagrangian velocity autocorrelation function $R_L(\tau)$ of the turbulence. The Lagrangian study of fluid element trajectories is therefore central to understanding and calculating turbulent diffusion.

A consequence of Taylor's (1921)[1] theory is that the turbulent diffusivity is larger/smaller depending on whether Lagrangian velocities are correlated over longer/shorter times along trajectories. Such long/short correlation times reflect large/small Lagrangian velocity variations, and Lagrangian velocity variations over a time interval τ are characterized by the second order Lagrangian structure function which is proportional to $1-R_L(\tau)$. In statistically stationary and isotropic turbulence it is sufficient to consider only one component of the Lagrangian velocity, say $v(t)$; its second order structure function is $\langle\delta v^2(\tau)\rangle\equiv\langle[v(t)-v(t+\tau)]^2\rangle$ corresponding to the Lagrangian velocity power spectrum $E_L(\omega)$ in the frequency (ω) domain. As noted by Inoue[2] (see also Tennekes and Lumley),[3] Kolmogorov similarity arguments imply that, in the inertial range of times and frequencies, $\langle\delta v^2(\tau)\rangle\sim\epsilon\tau$ and $E_L(\omega)\sim\epsilon\omega^{-2}$ where ϵ is the kinetic energy dissipation rate per unit mass. Support for these scalings has recently been obtained in highly turbulent oceanic environments by Lien *et al.*,[4] in the laboratory by Mordant *et al.*,[5] and in direct numerical simulations (DNS) by Yeung.[6] However, Kolmogorov similarity arguments provide very little understanding of the underlying processes responsible for these scaling laws. What properties of the spatiotemporal flow structure of the turbulence determine that $\langle\delta v^2(\tau)\rangle$ is proportional to τ, and what do these properties imply for turbulent diffusion?

In this Brief Communication we address this question by use of kinematic simulation (KS, see Nicolleau and Vassilicos,[7] Davila and Vassilicos,[8] and references therein). KS is a Lagrangian model of turbulent diffusion based on kinematically simulated turbulent velocity fields which are incompressible and consistent with up to second order Eulerian statistics of the turbulence such as energy spectra $E(k)$ in the wave-number (k) domain. There is no assumption of Markovianity or δ correlation in time made at any level. Instead, a parameter λ controls the degree of unsteadiness of the turbulence. It might be worth mentioning that, when the energy spectrum input has the Kolmogorov $k^{-5/3}$ form, KS is in good agreement with laboratory experiments for one-particle statistics,[9] two-particle statistics,[7] three-particle statistics,[10] and concentration variances[11] (particle is used here to mean fluid element). KS is also in good agreement with DNS for two-particle statistics when the KS energy spectrum has the same overall shape as that of the DNS.[12] In this Brief Communication, we modify the spatiotemporal flow structure of the KS turbulence by changing the input energy spectrum $E(k)$ and the input unsteadiness parameter λ and study how $\langle\delta v^2(\tau)\rangle$ changes as a result.

In our KS we use three-dimensional turbulent-like velocity fields of the form

$$\mathbf{u}=\sum_{n=1}^{N_k}\mathbf{A}_n\wedge\hat{\mathbf{k}}_n\cos(\mathbf{k}_n\cdot\mathbf{x}+\omega_n t)+\mathbf{B}_n\wedge\hat{\mathbf{k}}_n\sin(\mathbf{k}_n\cdot\mathbf{x}+\omega_n t), \quad (1)$$

where N_k (typically of order 100) is the number of modes, $\hat{\mathbf{k}}_n$ is a random unit vector ($\mathbf{k}_n=k_n\hat{\mathbf{k}}_n$), and the directions and orientations of $\mathbf{A}_n$ and $\mathbf{B}_n$ are chosen randomly under the constraint that they be normal to $\hat{\mathbf{k}}_n$ and uncorrelated with the directions and orientations of all other wave modes. Note that the velocity field (1) is incompressible by construction, and also statistically stationary, homogeneous, and isotropic as shown by Fung *et al.*[13] and Fung and Vassilicos.[14] The amplitudes A_n and B_n of the vectors $\mathbf{A}_n$ and $\mathbf{B}_n$ are determined by $A_n^2=B_n^2=2/3E(k_n)\Delta k_n$ where $E(k)$ is the energy spectrum prescribed to be of the form

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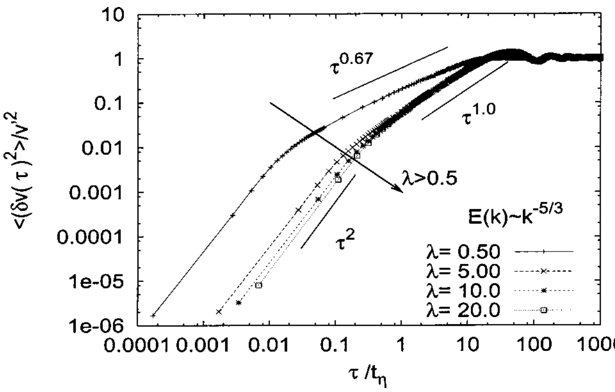

FIG. 1. The time evolution of the velocity increment $\langle \delta v^2(\tau)\rangle$ in a log-log plot.

$$E(k)=\frac{3v'^2(p-1)}{2(L/2\pi)^{p-1}}k^{-p} \tag{2}$$

in the range $2\pi/L=k_1\leq k\leq k_{N_k}=2\pi/\eta$ assumed large (i.e., $L/\eta\gg 1$), and $E(k)=0$ otherwise; v' is the rms of one velocity component of the KS turbulent-like flow; $\Delta k_n=(k_{n+1}-k_{n-1})/2$ for $2\leq n\leq N_k-1$, $\Delta k_1=(k_2-k_1)/2$, and $\Delta k_{N_k}=(k_{N_k}-k_{N_{k-1}})/2$. The distribution of wave numbers is geometric (see Ref. 11), specifically $k_n=k_1a^{n-1}$ with a constant a determined by $L/\eta=a^{N_k-1}$. The frequencies ω_n in (1) are proportional to the eddy-turnover frequency of mode n, and the dimensionless constant of proportionality is λ, i.e., $\omega_n=\lambda\sqrt{k_n^3E(k_n)}$. Particle trajectories are numerically integrated in (1) and the velocities of the particles are recorded along their flight. By averaging over 5×10^3 trajectories in 5×10^3 realizations of the velocity field we obtain $\langle \delta v^2(\tau)\rangle$ for different values of p and λ (see Fig. 1 which has been obtained for $L/\eta=1690$, but note that we corroborated these results for much higher values of L/η too). We try values of p larger than 1 to ensure that the total kinetic energy of the turbulence is finite in the large L/η limit, and values of p smaller than 2 for the power spectrum $k^2E(k)$ of the vorticity and strain rate fields to be an increasing function of k (see Ref. 8).

A first observation is that when $E(k)$ has the Kolmogorov similarity dependence on wave number (i.e., $p=5/3$), $\langle \delta v^2(\tau)\rangle$ exhibits the Kolmogorov similarity scaling $\langle \delta v^2(\tau)\rangle\sim\tau$ over an intermediate range of time scales only for large enough values of λ and not otherwise. This intermediate range of scales turns out to be $t_\eta\ll\tau\ll T_L$ where $t_\eta=2\pi/\omega_{N_k}$. For values of λ smaller than 1 and close to 0 as well as equal to 0, we find $\langle \delta v^2(\tau)\rangle\sim\tau^{2/3}$ over the intermediate range $\tau_\eta\ll\tau\ll T_L$ where $\tau_\eta\equiv 2\pi/\sqrt{(2\pi/\eta)^3E(2\pi/\eta)}$ and $T_L\equiv\int_0^\infty R_L(\tau)d\tau$ ($t_\eta=\tau_\eta/\lambda$). Fung *et al.*[13] have already observed this 2/3 scaling in frozen ($\lambda=0$) KS turbulence and added to (1) a sweeping of the small-scales (1) by larger scales which restored the Kolmogorov similarity scaling $\langle \delta v^2(\tau)\rangle\sim\tau$. In the absence of sweeping, large values of the unsteadiness parameter λ lead to the scaling that $\langle \delta v^2(\tau)\rangle$ should have in the presence of sweeping. In the context of KS, Kolmogorov similarity is only retrieved for large values of the dimensionless parameter λ.

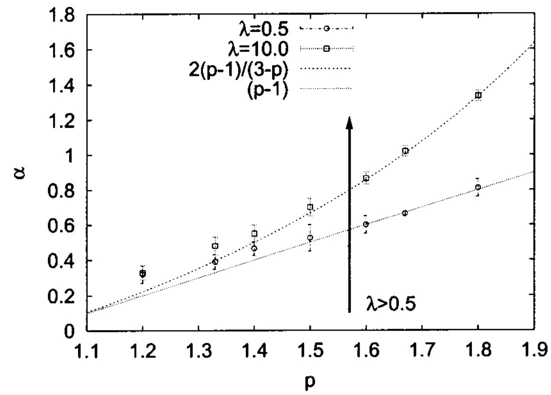

FIG. 2. Plot of the change in the power law of the Lagrangian velocity increment $[\langle \delta v(\tau)^2\rangle\sim\tau^\alpha$ as a function of $p]$ in KS for different λ's. The exponent α and its error bars are calculated from a least-squares fit over the range of scales where $\langle \delta v(\tau)^2\rangle\sim\tau^\alpha$ is observed. This power law is harder to detect accurately for values of p closer to 1 because a larger proportion of the energy is then distributed to higher wave numbers and both τ_η and t_η increase indefinitely as $p\to1$ thus requiring an ever larger ratio L/η and therefore value of T_L. The fact that more energy is distributed at the higher wave numbers when p decreases means that trajectories are more irregular, a fact reflected in the smaller values of α. With our extremely good time resolutions and ratios L/η up to 10^4 we still cannot accurately measure values of α below about 0.4.

Running simulations for a variety of values of p between 1 and 2 and a variety of values of λ we find

$$\langle \delta v^2(\tau)\rangle\sim\tau^\alpha \tag{3}$$

in the intermediate ranges mentioned above and $\alpha=\alpha(p,\lambda)$ (see Figs. 1 and 2). For values of λ smaller than 1 including $\lambda=0$,

$$\alpha=p-1. \tag{4}$$

This result is consistent with $\langle \delta v^2(\tau)\rangle\sim\langle \delta u^2(r)\rangle$ where $\langle \delta u^2(r)\rangle$ is the second order structure function of one component of the Eulerian velocity field and $r\sim v'\tau$. It is well known that $\langle \delta u^2(r)\rangle\sim r^{p-1}$ for small enough values of r is equivalent to $E(k)\sim k^{-p}$ for large enough values of k. Hence (3) and (4) follow from $\langle \delta u^2(r)\rangle\sim r^{p-1}$ and $\langle \delta v^2(\tau)\rangle\sim\langle \delta u^2(r)\rangle$ for $r\sim v'\tau$. The validity of the latter condition (including $r\sim v'\tau$) might be due to the fact that the flow is nearly frozen when λ is small and therefore fluid elements are advected past frozen small scale eddies. For values of λ much larger than 1, α is a monotonically increasing nonlinear function of p, in fact the same function of p for all large enough λ, and is such that $\alpha>p-1$ and $\alpha=1$ at $p=5/3$. What is this function of p?

To answer this question we proceed by analogy with Richardson's locality hypothesis for two-particle diffusion in statistically homogeneous turbulence. According to this hypothesis, the rate of change of the mean square separation between two particles, i.e., $d/dt\langle\Delta^2\rangle$, is a function only of $\langle\Delta^2\rangle$ and of the variance of Eulerian velocity variations across that length-scale (see Ref. 14). That is

$$\frac{d}{dt}\langle\Delta^2\rangle=f(\langle\Delta^2\rangle,\langle\delta u^2(\sqrt{\langle\Delta^2\rangle})\rangle). \tag{5}$$

The analogy we are trying here, which we call the inverse locality hypothesis, is to say that the rate of change of $\langle \delta v^2(\tau) \rangle$ with respect to τ is a function only of $\langle \delta v^2(\tau) \rangle$ and of the length-scale across which the variance of Eulerian velocity variations is equal to $\langle \delta v^2(\tau) \rangle$. That is

$$\frac{d}{d\tau}\langle \delta v^2(\tau) \rangle = f(\langle \delta v^2(\tau) \rangle, l_e(\tau)), \tag{6}$$

where $l_e(\tau)$ is an Eulerian–Lagrangian length-scale defined as follows:

$$\langle \delta u^2[l_e(\tau)] \rangle = \langle \delta v^2(\tau) \rangle. \tag{7}$$

We call $l_e(\tau)$ the equivalent length-scale. Dimensional analysis applied to (6) and the use of (7) and $\langle \delta u^2(r) \rangle \sim r^{p-1}$ leads to (3) with

$$\alpha = \frac{2(p-1)}{3-p}. \tag{8}$$

This dependence of α on p fits surprisingly well the results obtained by KS for large values of λ (see Fig. 2). These results reveal, therefore, the existence of an Eulerian–Lagrangian length-scale $l_e(\tau)$ which controls the second order Lagrangian structure function and Lagrangian frequency spectrum provided that the sweeping effects are taken into account by high values of λ. Our results [Figs. 1 and 2 and Eqs. (4) and (8)] also reveal the impact that the two-point spatial flow structure (as reflected in the scaling of the energy spectrum) has on one-particle Lagrangian statistics whether with or without sweeping. In particular, these results suggest that an Eulerian intermittency correction on the scaling of the energy spectrum might impose a correction on the scaling of Lagrangian structure functions even when, as in the second order case, they are linear in the dissipation rate.

Changes in $\langle \delta v^2(\tau) \rangle$ imply changes in $R_L(\tau)$ and therefore in turbulent diffusion. Taylor's (1921) formula for the mean square one-particle displacement $\langle x^2(t) \rangle$ is

$$\frac{d}{dt}\langle x^2(t) \rangle = v'^2 \int_0^t R_L(\tau) d\tau \tag{9}$$

which implies a turbulent diffusivity equal to $v'^2 T_L$. Of course this relation remains valid in our KS for any value of p but the Lagrangian correlation time scale T_L is found to be an increasing function of p (see Fig. 3), thus implying a turbulent diffusivity that is also an increasing function of p. This conclusion can be explained qualitatively by the relation $p + 2D_s/3 = 3$ between p and the fractal dimension D_s of the set of straining stagnation points that has recently been obtained by Davila and Vassilicos[8] for KS turbulence. As p decreases, D_s increases implying an increase in the number density of straining stagnation points and thereby an increased probability for particle trajectories to come in the vicinity of such points and experience sudden changes of tack. As a consequence, the Lagrangian correlation time scale is smaller/larger for smaller/larger values of p. In terms of D_s, $\alpha = 2[(3-D_s)/D_s]$ when λ is large and $\alpha = 2/3(3-D_s)$ when λ is small; α is therefore a decreasing function of D_s thus reflecting the fact that a larger number of changes of tack produces more irregular particle trajectories. In terms

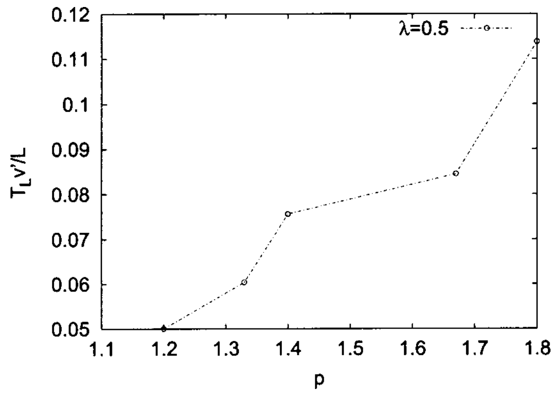

FIG. 3. Plot of Lagrangian correlation time T_L as a function of p for $\lambda = 0.5$ in KS with $L/\eta = 100$. Similar results are found for other values of λ and L/η.

of the equivalent length-scale, which from (7), (8), and $\langle \delta u^2(r) \rangle \sim r^{p-1}$ has the power-law time dependence $l_e(\tau) \sim \tau^{2/(3-p)} \sim \tau^{3/D_s}$, an increased irregularity of particle trajectories is reflected in a faster growth of the equivalent length-scale $l_e(\tau)$ for τ bounded from above by L/v'. For $p = 5/3$, $l_e(\tau) \sim \tau^{3/2}$.

ACKNOWLEDGMENTS

M.A.I.K. and J.C.V. acknowledge support from EPSRC Grant No. GR/K50320, from EC TMR Research network on intermittency in turbulent systems, and from the Hong Kong Research Grant Council, China (Project No. HKUST6121/00P). M.A.I.K. also wishes to thank the Cambridge Commonwealth Trust, Wolfson College, Cambridge and DAMTP for financial support while this work was being completed. J.C.V. is grateful to the Royal Society of London for its unflinching support over the past eight years.

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