Quantum Signature of Superfluid Turbulence

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(Received 18 July 2001; published 10 December 2001)

Using a numerical simulation backed up by physical arguments, we predict that the pressure spectrum of superfluid turbulence has a k^{-2} dependence on the wave number k, which represents a macroscopic quantum signature not to be found in the classical Kolmogorov theory of turbulence.

DOI: 10.1103/PhysRevLett.87.275302 PACS numbers: 67.40.–w, 47.37.+q

Experiments performed in the last few years showed that some turbulent flows of the superfluid phase of ^{4}He (helium II) are similar to analogous turbulent flows in a classical fluid. Typically helium II was made turbulent using a towed grid [1] or a rotating propeller [2], or pushing it at high velocity along pipes and channels [3] or around spheres [4]. The apparent classical results (in terms of energy spectrum, decay rates, pressure drops, drag crisis, etc.) are surprising because helium II is a quantum fluid. According to Landau's two-fluid theory, it consists of two copenetrating fluid components, the normal fluid and the superfluid, whose relative proportion depends on the absolute temperature T. The normal fluid, which is viscous, can form eddies of any size and strength. On the contrary, rotation in the superfluid component is constrained by quantum mechanics, and is possible only in the form of quantized vortex filaments. All superfluid vortex filaments have the same quantized circulation ($\Gamma = 9.97 \times 10^{-4}$ cm^2/sec) and the same microscopic core radius $a \approx 10^{-8}$ cm. Since the superfluid's viscosity is zero, the issue raised by the experiments is what causes the observed classical behavior, which remarkably extends to temperatures so low ($T \approx 1.4$ K) that the normal fluid fraction is only a few percent. The problem has stimulated theoretical investigations of the coupling between superfluid vortices and normal fluid [5,6], as well as studies of vortex tangles [7] and dissipation in the limit of absolute zero [8], including the effects of Kelvin waves and reconnections [9].

This Letter attempts to clarify the limits of validity of the similarities between helium II turbulence and classical turbulence by drawing attention to the spectrum of the pressure field p. Stimulated by the helium turbulence experiments and by the recent work on pressure spectra in classical turbulence [10–12], we show that the spectrum of p in turbulent helium II must be very different from the classical Kolmogorov spectrum of a classical turbulent flow. Using results of numerical simulations confirmed by theoretical arguments we show that the pressure spectrum $E_p(k)$, defined such that

$$\int_0^\infty E_p(k)\,dk = \frac{1}{\mathcal{V}} \iiint \left(\frac{p}{\rho}\right)^2 d\mathcal{V}\,, \tag{1}$$

where ρ is helium II's density, scales like $E_p(p) \sim k^{-2}$, where $k = |\mathbf{k}|$ is the magnitude of the three-dimensional wave number $\mathbf{k}$ and $\mathcal{V}$ is volume. This result is in contrast with what happens in the classical theory of Kolmogorov turbulence. Following the classical argument of Kolmogorov, assuming that in the inertial range $E_p(k)$ depends only on the wave number k and the energy dissipation rate $\epsilon = d(v^2/2)/dt$ where v is the velocity, it is found by dimensional argument that $E_p(k) \sim \epsilon^{4/3}k^{-7/3}$, a scaling [13] which corresponds to the celebrated $k^{-5/3}$ power law of the energy spectrum. The experimentalists should not have difficulty in distinguishing the two power laws: k^{-2} in turbulent helium II and $k^{-7/3} = k^{-2.33}$ in classical turbulence (e.g., helium I), hence in identifying the quantum signature of helium II turbulence.

Our numerical simulations are based on the approach of Schwarz [14]. We represent a quantized vortex filament as a space curve $\mathbf{s} = \mathbf{s}(\xi, t)$ where ξ is arclength and t is time, which moves with velocity $\mathbf{v}_L = d\mathbf{s}/dt$ given by

$$\mathbf{v}_L = \mathbf{v}_s + \alpha \mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_s) + \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_s)]\,, \tag{2}$$

where $\mathbf{s}' = d\mathbf{s}/d\xi$, α and α' are mutual friction coefficients [15], $\mathbf{v}_n$ is the prescribed normal fluid velocity, and $\mathbf{v}_s$ is the self-induced velocity at the point $\mathbf{s}$ given by

$$\mathbf{v}_s = \frac{\Gamma}{4\pi} \int \frac{(\mathbf{r} - \mathbf{s}) \times \mathbf{dr}}{|\mathbf{r} - \mathbf{s}|^3}\,. \tag{3}$$

The calculation is performed in a periodic box of volume $\mathcal{V} = d^3 = (0.1)^3$ cm^3. The numerical technique is standard [14] and the details of our algorithm, including how to perform vortex reconnections, have been published elsewhere [9]. The time step is chosen so that we can resolve Kelvin waves corresponding to the minimum spatial scale, $\delta = d/128 = 0.78 \times 10^{-3}$ cm.

In the current experiments helium II is made turbulent by standard classical techniques (imposed pressure gradients, grids, propellers, etc.), so it is natural to assume that the normal fluid $\mathbf{v}_n$ consists of a uniform mean flow and superimposed turbulent fluctuations, $\mathbf{v}_n = \mathbf{U}_n + \mathbf{u}_n$. Following Vassilicos and co-workers [16], we choose the fluctuations $\mathbf{u}_n$ so that they have a Kolmogorov energy spectrum $E_n(k) \sim k^{-5/3}$, which is defined such that

$$\int_0^{k_{\max}} E_n(k)\,dk = \frac{1}{\mathcal{V}} \iiint \frac{1}{2} |\mathbf{v}_n^2|\, d\mathcal{V}\,, \qquad (4)$$

where $k_{\max} = 1/(2\eta)$ where η represents the Kolmogorov length. Specifically we have

$$\mathbf{u}_n = \sum_{m=1}^{m=M} [\mathbf{A}_m \times \hat{\mathbf{k}}_m \cos(\mathbf{k}_m \cdot \mathbf{x} + \omega_m t) + \mathbf{B}_m \times \hat{\mathbf{k}}_m \sin(\mathbf{k}_m \cdot \mathbf{x} + \omega_m t)]\,, \qquad (5)$$

where $M = 64$ is the number of modes, $\hat{\mathbf{k}}_m$ is a random unit vector ($\mathbf{k}_m = k_m \hat{\mathbf{k}}_m$), and the directions and orientations of $\mathbf{A}_m$ and $\mathbf{B}_m$ are chosen randomly under the assumption (with no loss of generality) that they are normal to $\hat{\mathbf{k}}_m$, the random choice of directions for the mth wave mode being independent of the choices of the other wave modes. Note that the velocity field $\mathbf{u}_n$ is incompressible by construction. The amplitudes A_m and B_m of the vectors $\mathbf{A}_m$ and $\mathbf{B}_m$ are determined by the Kolmogorov energy spectrum $E_n(k) \sim k^{-5/3}$ via the relations $(3/2)A_m^2 = (3/2)B_m^2 = E_n(k_m)\Delta k_m$ where $\Delta k_m = (k_{m+1} - k_{m-1})/2$. Finally, the unsteadiness frequencies ω_m are determined by the eddy turnover time of wave mode m, that is, $\omega_m = \sqrt{k_m^3 E_n(k_m)}$.

The results presented here correspond to the following choice of parameters: $T = 1.3$ K, $U_n = 2.36$ cm/sec, and $u_{n,\mathrm{rms}} = 11.927$ cm/sec. The Kolmogorov length $\eta \approx 0.78 \times 10^{-3}$ cm is consistent with the minimum scale δ used; η is estimated from $\eta/\ell = \mathcal{R}^{-3/4}$ where $\ell = d/2$ is the integral scale, $\mathcal{R} = u_{n,\mathrm{rms}}\ell/\nu_n$, $\nu_n = \mu/\rho_n = 2.34 \times 10^{-3}$ cm^2/sec is the kinematic viscosity of the normal fluid, ρ_n is the normal fluid density, and μ is the viscosity of helium II. The calculation starts with some initial superfluid vortex rings [14] (the result is independent of the arbitrary initial configuration), which immediately interact with each other and with the normal fluid, soon forming a tangle of vortex lines. We stop the calculation at time $t = 0.093$ sec when the tangle is so intense that the vortex line density is $L_0 = \Lambda/\mathcal{V} = 14\,180$ cm^{-2} where Λ is the total vortex length; this value is large enough to be typical of experiments [17].

We now calculate the pressure spectrum $E_p(k)$. Figure 1 shows that the pressure spectrum is visibly different from the Kolmogorov $k^{-7/3}$ pressure spectrum over a range of wave numbers smaller than $1/\eta$. It exhibits a k^{-2} shape even though the energy of the normal fluid follows the Kolmogorov $k^{-5/3}$ spectrum over the same range of wave numbers. In fact, as shown by Kivotides *et al.* [18], even the energy of the total fluid follows the Kolmogorov $k^{-5/3}$ spectrum over this wave number range. The pressure spectrum of the superfluid turbulence stands out because of its nonclassical behavior over inertial range wave numbers.

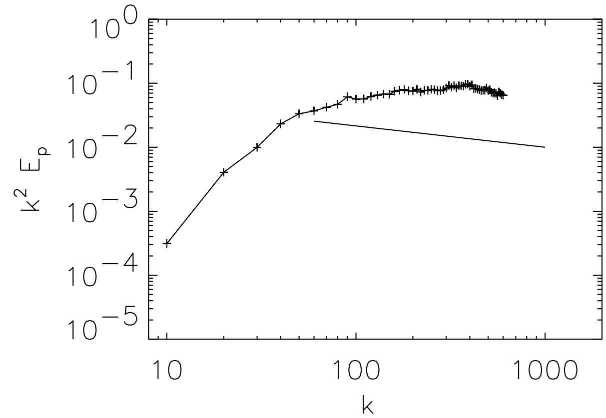

FIG. 1. Compensated pressure spectrum $E_p(k)$ (in units of cm^5/sec^4) as a function of wave number k (in units of cm^{-1}). $E_p(k)$ is multiplied times k^2 to make the scaling $E_p(k) \sim k^{-2}$ apparent. The solid line with crosses shows the result of our calculation for a vortex tangle of density $L_0 = 14\,180$ cm^{-2}. The solid line without crosses shows the classical Kolmogorov pressure scaling $k^{-7/3}$. The difference between the quantum (k^{-2}) and classical ($k^{-7/3}$) dependence is clearly noticeable.

It is interesting to notice that we find that the scaling $E_p \sim k^{-2}$ is valid in the region $1/(2\eta) \geq k \geq 1/\ell_0 \approx 119$ cm^{-1} where $\ell_0 = L_0^{-1/2} = 8.4 \times 10^{-3}$ cm is the average spacing between the vortex lines. A similar k^{-2} dependence of the spectrum is found in runs with smaller values of $u_{n,\mathrm{rms}}$.

Why the k^{-2} dependence? The k^{-2} pressure spectrum is a consequence of the locality in physical space of the vorticity filaments in the superfluid turbulence, as the following argument shows. Consider the equations of motion of the normal fluid and the superfluid [19]:

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_n(\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = -\frac{\rho_n}{\rho}\nabla p - \rho_s S \nabla T + \mu \nabla^2 \mathbf{u}_n + \mathbf{F}_{ns}\,, \qquad (6)$$

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_s(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = -\frac{\rho_s}{\rho}\nabla p + \rho_s S \nabla T - \mathbf{F}_{ns}\,, \qquad (7)$$

where S is the entropy per unit mass, $\rho_s = \rho - \rho_n$ the superfluid density, and $\mathbf{F}_{ns}$ the mutual friction force. By taking the divergence of the sum of (6) and (7) and assuming incompressibility ($\nabla \cdot \mathbf{v}_s = 0$ and $\nabla \cdot \mathbf{v}_n = 0$) we obtain the governing Poisson equation for the pressure field which we have used to calculate Fig. 1

$$\nabla^2\left(\frac{p}{\rho}\right) = \frac{\rho_s}{\rho}(\omega_s^2 - s_s^2) + \frac{\rho_n}{\rho}(\omega_n^2 - s_n^2)\,, \qquad (8)$$

where the source term on the right hand side is a function of the vorticities $\omega = \nabla \times \mathbf{v}$ and the strain rate tensors, $s_{ij} = (1/2)(\partial v_i/\partial x_j + \partial v_j/\partial v_i)$, of the normal fluid and the superfluid velocity fields (indicated by the subscripts n and s, respectively). Unlike the normal fluid contribution, the superfluid part of this source term is extremely well localized in physical space because the superfluid vorticity is concentrated in thin vortex filaments. Fourier transformation of (8) shows that $k^2\hat{p}(\mathbf{k})/\rho$ is equal to a function that is very slowly varying in wave number space for $k \ll 1/a$ plus contributions from the normal fluid which can be expected to decay with increasing wave number k because the normal fluid vorticity and strain rate fields are not so localized in physical space. This property is independent of the exact nature of the core size. Hence

$$\frac{|\hat{p}(\mathbf{k})|^2}{\rho^2} \sim k^{-4} \tag{9}$$

for $k \ll 1/a$. The pressure spectrum $E_p(k)$ of the pressure field is defined such that

$$\int_0^\infty E_p(k)\,dk = \iiint \frac{|\hat{p}(\mathbf{k})|^2}{\rho^2}\,d^3\mathbf{k}\,, \tag{10}$$

and therefore $E_p(k) \sim k^2|\hat{p}(\mathbf{k})|^2$. The conclusion is that

$$E_p(k) \sim k^{-2} \tag{11}$$

for $k \ll 1/a$, which is confirmed by the numerical simulation. Note that although we used spherical coordinates in Fourier space, we did not make any assumption about symmetry. The result (11) is a consequence of the extreme localization of superfluid vorticity in physical space, that is to say of the quantum nature of helium II. The effect is increasingly sharp when the temperature is made so small that ρ_s is much larger than ρ_n [see Poisson equation (8) above]. In the absence of superfluid vorticity the pressure spectrum is dominated by the normal fluid velocity field (5) and has a well defined Kolmogorov $k^{-7/3}$ scaling [20].

It must be said, however, that the exact Kolmogorov scaling for the pressure field has not been verified in recent experiments [12] and in direct numerical simulations of classical fluid turbulence either: Gotoh and Fukayama [10] claimed evidence of a $k^{-7/3}$ range in the spectrum, but also of a different dependence at higher wave numbers. More work is clearly required. It is, however, encouraging to remark that Abry *et al.* [12] found that the removal of the signal arising from strong pressure drops (intense vortices) affects the spectrum at small wave number k, not at high k. This observation, that it is the pressure spectrum at the large scale which is mostly affected by localized vortices, is not inconsistent with our suggestion that there should be a macroscale effect of superfluid filaments.

Finally, it must be noticed that there is nothing comparable in classical fluid turbulence to the extreme localization of superfluid vortex filaments in superfluid turbulence. There is small-scale organized vorticity in classical fluid turbulence [21] which does indeed seem to affect the pressure spectrum [10,22] but only at the smallest scales and without the generation of a clear k^{-2} pressure spectrum.

In conclusion, numerical simulations backed up by physical arguments predict a k^{-2} pressure spectrum for superfluid turbulence which is a macroscale quantum signature not to be found in the classical theory of fluid turbulence. Experimental attempts to measure spectra in turbulent helium II concentrated the attention on the velocity field [2]. However, accurate pressure gauges for cryogenics fluid now exist [23], and work is in progress to build devices suitable for helium turbulence [24]. There are clearly experimental problems to solve regarding the sensitivity and the size of the probe, as well as the frequency response, but they do not seem impossible to overcome, more so if MEME technology is used. We hope therefore that our prediction will add to the motivations behind this development of instrumentation and stimulate the measurement of pressure spectra in turbulent helium II. Ideally one should measure turbulent pressure spectra in the same apparatus at two different temperatures, just above and below the lambda point, to compare predictions in the classical (above the lambda point) and superfluid (below the lambda point) regimes. On the theoretical side, our results call for the investigation of the fully coupled motion of turbulent normal fluid and superfluid vortices, something which, until now, has been attempted only for very simple vortex configurations [6].

This research is supported by the Leverhulme Trust. J. C. V. acknowledges support from the Royal Society and M. A. I. K. from the Cambridge Commonwealth Trust.

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