Turbulent Pair Diffusion

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Kinematic simulations of turbulent pair diffusion in planar turbulence with a $k^{-5/3}$ energy spectrum reproduce the laboratory results of Jullien *et al.* [Phys. Rev. Lett. **82**, 2872 (1999)], in particular the stretched exponential form of the probability density function of pair separations and their correlation functions. The root mean square separation is found to be strongly dependent on initial conditions for very long stretches of time. This dependence is consistent with the topological picture where pairs initially close enough travel together for long stretches of time and separate violently when they meet straining regions around hyperbolic points. A new argument based on the divergence of accelerations is given to support this picture.

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The rate with which pairs of points separate in phase space or in physical space is of central importance to the study of dynamical systems. Pairs of points in the phase space of a low-dimensional chaotic dynamical system and pairs of fluid elements in fully chaotic flows separate exponentially. However, in fully developed homogeneous and isotropic turbulence, Richardson's law [1] stipulates that fluid element pairs separate on average algebraically and in such a way that their separation statistics in a certain range of times are the same irrespective of initial conditions. Richardson's law is therefore a remarkable claim of universality. Specifically, it stipulates that in a range of times where the root mean square separation $\overline{\Delta^2}^{1/2}$ is larger than the Kolmogorov length scale η and smaller than the integral length scale L, $\overline{\Delta^2}$ is increasingly well approximated by

$$\overline{\Delta^2} = G_\Delta \epsilon t^3 \tag{1}$$

for increasing values of L/η, where t is time, ϵ is the kinetic energy rate of dissipation per unit mass, and G_Δ is a universal dimensionless constant.

Richardson accompanied his empirical law (1) with a prediction for the probability density function (PDF) of pair separations Δ. The effective diffusivity approach leading to this prediction was criticized by Batchelor [2] who developed a different approach leading to (1) but also to a different form of the PDF. Kraichnan [3] derived yet another expression for the PDF based on his Lagrangian history direct interaction approximation and so did Shlesinger *et al.* [4] on the assumption that turbulent pair diffusion is well described by Lévy walks.

Setting $\sigma(t) \equiv \overline{\Delta^2}^{1/2}$ and $r \equiv \Delta/\sigma$, the PDFs of Δ predicted by Richardson [1], Batchelor [2], and Kraichnan [3] are all of the form

$$P(\Delta, t) \sim \sigma^{-1} \exp(-\alpha r^\beta), \tag{2}$$

with different values of the dimensionless parameters α and β. Richardson's prediction for the exponent β is $\beta = 2/3$, Batchelor's is $\beta = 2$, and Kraichnan's is $\beta = 4/3$. The Lagrangian modeling approach of Shlesinger *et al.* [4] leads to a totally different, in fact algebraic, PDF form. More recently Jullien *et al.* [5] reported laboratory measurements of $P(\Delta, t)$ which are well fitted by (2) with $\alpha \approx 2.6$ and $\beta = 0.5 \pm 0.1$. These laboratory measurements invalidate the PDF predictions of Batchelor, Kraichnan, and Shlesinger *et al.* and might raise a question mark over the PDF prediction of Richardson even though they can be considered consistent with it if we account for experimental uncertainties. Jullien *et al.* [5] also observed that fluid element pairs stay close to each other for a long time until they separate quite suddenly, a behavior which seems qualitatively at odds with the effective diffusivity approach adopted by Richardson [1] to derive (2) with $\beta = 2/3$. In particular, they measured the Lagrangian autocorrelation function of pair separations $R(t, \tau) \equiv \langle \Delta(t)\Delta(t+\tau)\rangle$ for $-t \le \tau \le 0$ and found a Lagrangian pair correlation time $\tau_c \approx 0.6t$ which is surprisingly long. In this Letter we report that kinematic simulation (KS) [6] reproduces the experimental results of Jullien *et al.* [5]. KS is a Lagrangian model of turbulent diffusion which is distinct from Lévy walks [4] and makes no use of Markovianity assumptions so that it cannot be reduced to an effective diffusivity approach such as Richardson's [1]. Furthermore, the observation that fluid element pairs travel close together for long stretches of time until they separate quite suddenly has in fact already been made using KS [6].

KS Lagrangian modeling consists of integrating fluid element trajectories by solving $(d\mathbf{x}(t))/dt = \mathbf{u}(\mathbf{x}(t), t)$ in synthesized velocity fields $\mathbf{u}(\mathbf{x}, t)$. Statistically homogeneous, isotropic, and stationary KS velocity fields are superpositions of random Fourier modes [6]. KS velocity fields are Gaussian but *not* delta correlated in time [6], and this non-Markovianity is an essential ingredient in KS. The Lagrangian measurements of [5] were made in a two-dimensional inverse cascade turbulent flow. Our KS velocity field is therefore prescribed to be planar and given by

$$\mathbf{u} = \sum_{m=1}^{m=M} [\mathbf{A}_m \times \hat{\mathbf{k}}_m \cos(\mathbf{k}_m \cdot \mathbf{x} + \omega_m t) + \mathbf{B}_m \times \hat{\mathbf{k}}_m \sin(\mathbf{k}_m \cdot \mathbf{x} + \omega_m t)], \quad (3)$$

where $M = 500$ is the number of modes, $\hat{\mathbf{k}}_m$ is a random unit vector ($\mathbf{k}_m = k_m \hat{\mathbf{k}}_m$) normal to the plane of the flow, while the vectors $\mathbf{A}_m$ and $\mathbf{B}_m$ are in that plane. The random choice of directions for the mth wave mode is independent of the choices of the other wave modes. Note that the velocity field $\mathbf{u}$ is incompressible by construction. The amplitudes A_m and B_m of the vectors $\mathbf{A}_m$ and $\mathbf{B}_m$ are determined by the energy spectrum $E(k)$ via the relations $A_m^2 = B_m^2 = E(k_m)\Delta k_m$ where $\Delta k_m = (k_{m+1} - k_{m-1})/2$. Finally the unsteadiness frequencies ω_m are determined by the eddy turnover time of wave mode m, that is $\omega_m = 0.5\sqrt{k_m^3 E(k_m)}$.

The Lagrangian measurements in [5] were made when the two-dimensional flow had developed an inverse cascade with a well-defined $k^{-5/3}$ energy spectrum. The energy spectrum we have therefore chosen for this study is $E(k) \approx (2\pi)^{2/3}(2u'^2/3L^{2/3})k^{-5/3}$ in the range $(2\pi/L) \leq k \leq (2\pi/\eta)$ and equal to 0 outside this range (u'^2 is the total kinetic energy of the turbulence). The $M = 500$ wave numbers are algebraically distributed between $2\pi/L$ and $2\pi/\eta$. The eddy turnover time at the largest wave number $2\pi/\eta$ can be considered to correspond to a Kolmogorov time scale τ_η. Our motivation is to explore how much can be predicted with how little, using a model containing only a few key ingredients of the turbulence. Velocity field statistics are very close to Gaussian in the inverse energy cascade [7].

We have run simulations with $(L/\eta) = 10$, 100, 1691, 11 180, 38 748, 250 000. For initial pair separations Δ_0 smaller or equal to η our KS integrations lead to $\sigma P(\Delta, t) \sim \exp(-\alpha r^\beta)$ where $r = \Delta/\sigma(t)$ with $2.6 \leq \alpha \leq 3$ and $0.46 \leq \beta \leq 0.5$ for all the $\frac{L}{\eta}$ values that we tried, in very good agreement with [5] (see Fig. 1). (However, this PDF does seem to depend on the initial separation Δ_0 when $\Delta_0 > \eta$.) It has been noted in [8] that KS gives non-Gaussian stretched exponential PDFs of pair separations without estimating α and β. The synthetic velocity fields of [9] lead to the Richardson stretched exponential form with $\beta = (2/3)$. An approach based on asymmetric Levy walks [10] gives rise to stretched exponential forms of $\sigma P(\Delta, t)$, where β can be tuned as a function of a persistence parameter.

Following [5] we also calculate correlation functions of pair separations, i.e., $R(t, \tau) \equiv \langle\Delta(t)\Delta(t + \tau)\rangle$ for $-t \leq \tau \leq 0$ and with Δ_0 equal to η in one set of runs and 0.1η in another (the choice of Δ_0 in [5] is within this range).

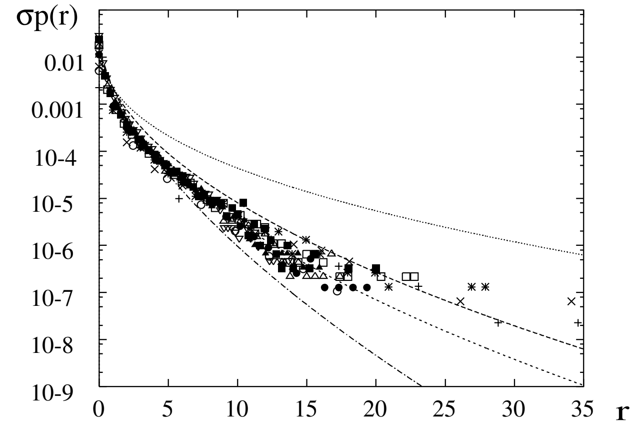

FIG. 1. Semilog plot of $\sigma P(r)$ as a function of $r = (\Delta/\sigma)$ in the case $(L/\eta) = 1691$. $(\Delta_0/\eta) = 0.1$ and $(tu'/L) = +0.10$, $\times 0.20$, $*0.30$, $\square 0.40$, $\blacksquare 0.5$. $(\Delta_0/\eta) = 1$ and $(tu'/L) = \odot 0.10$, $\bullet 0.20$, $\triangle 0.30$, $\blacktriangle 0.4$, $\triangledown 0.5$. Lines are from top to bottom: $\sigma P(r) = 0.03e^{-2.6r^{0.4}}$, $\sigma P(r) = 0.03e^{-2.6r^{0.5}}$, $\sigma P(r) = 0.03e^{-2.9r^{0.5}}$, and $\sigma P(r) = 0.03e^{-2.6r^{0.6}}$.

The laboratory results [5] show that $R(t, \tau)/\sigma^2(t)$ is a function of τ/t and exactly the same collapse is found here with KS (Fig. 2). We calculate a Lagrangian correlation time from $R(t, \tau)$ as done in [5] and obtain $\tau_c \approx 0.45t$ from Fig. 2. This value 0.45 is the constant asymptotic value that we obtain for all large enough scale ratios $\frac{L}{\eta}$, i.e., $\frac{L}{\eta} \geq 10$, and it compares sufficiently well with $\tau_c \approx 0.6t$ in the laboratory experiment [5]. The agreement is good and KS leads to the conclusion, in agreement with the laboratory experiment, that pair separations remember about half of their history.

The last set of statistics measured in [5] are Lagrangian correlations of pair velocity differences, i.e., $D_{ij} \equiv \langle V_i^L(t)V_j^L(t + \tau)\rangle$ with $-t \leq \tau \leq 0$, where $V_i^L(t)$ denotes the ith component of the Lagrangian relative velocity

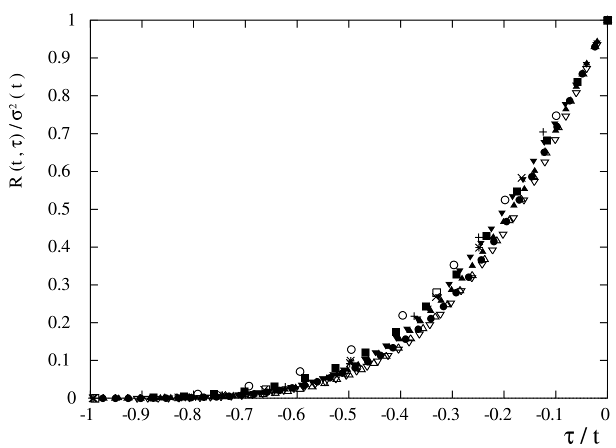

FIG. 2. Lagrangian separation correlation factor $R(t, \tau)/\sigma^2(t)$ as a function of τ/t in the case $(L/\eta) = 1691$. $(\Delta_0/\eta) = 1$ and $(tu'/L) = +2.01$, $\times 1.51$, $*1.01$, $\square 0.76$, $\blacksquare 0.11$, $\odot 0.06$. $(\Delta_0/\eta) = 0.1$ and $(tu'/L) = \bullet 1.25$, $\triangle 0.62$, $\blacktriangle 0.16$, $\triangledown 0.08$, $\blacktriangledown 0.04$. This figure and Fig. 3 were found to be effectively insensitive to variations of t and $(L/\eta) > 10$.

between a pair of fluid elements. We calculate these same statistics using our KS model and find that D_{ij} remains close to 0 for $i \neq j$, that $D_{11}(t, \tau)/D_{11}(t, 0)$ and $D_{22}(t, \tau)/D_{22}(t, 0)$ are functions of τ/t (see Fig. 3), and that this collapse is the same for D_{11} and D_{22} again in agreement with the laboratory results of [5].

Having shown how KS reproduces the laboratory results of [5], we now turn our attention to Richardson's law (1) and the claim of universality on which it is based. We do indeed observe this law over the entire inertial range of times $\tau_\eta < t < L/u'$, but only for initial separations Δ_0 between η and 0.1η, and this for all the ratios L/η that we tried (see Fig. 4(a)). Of course this ratio should be large enough, otherwise Richardson's law is not observed for any Δ_0, but it is surprising that Richardson's law is so Δ_0 specific even at enormous values of L/η reaching more than 10^5. Note that the laboratory verification of Richardson's law in [5] has been made only for Δ_0 close to 0.1η.

When $\Delta_0 \leq \eta$, Richardson's law (1) is observed over the limited large scale range $0.2(L/u')$ to L/u' (Fig. 4), and the coefficient 0.2 seems to have no dependence on L/η in our simulations as long as L/η is 10^3 or larger, so if it has one it must be weak. In the remainder of the inertial range between τ_η and $0.2(L/u')$, the time dependencies of $\overline{(\Delta - \Delta_0)^2}$ and $\overline{\Delta^2}$ are different from Richardson's (1) and different for different values of $\Delta_0 \leq \eta$ (Fig. 4), even at extremely high L/η. We tried to replace t by $t - t_0(\Delta_0)$, where $t_0(\Delta_0)$ is a virtual origin significantly smaller than $0.2(L/u')$ but did not recover Richardson's law (1), particularly since the discrepancies we observe are over such wide time ranges. When Δ_0 is significantly larger than η there is no clear indication of a Richardson law at all (Fig. 4).

We have carefully studied the time dependence of $\overline{(\Delta - \Delta_0)^2}$ in the range between τ_η to L/u' for different values of $\Delta_0 \leq \eta$ and have found the following collapse of the data in that range (see Fig. 4(b)):

$$\overline{(\Delta - \Delta_0)^2} = G_\Delta \frac{u'^3}{L} t^3 f(t, \Delta_0), \tag{4}$$

where the dimensionless function f is given by [using $T \equiv 0.2(L/u')$]

$$f(t, \Delta_0) = \exp\left\{\frac{\ln[\frac{5}{G_\Delta}(\Delta_0/\eta)^2]}{2\ln(\tau_\eta/T)}\left[\ln(t/T) - \sqrt{\ln^2(t/T) + 2}\right]\right\}. \tag{5}$$

Note that $f(t, \Delta_0)$ tends to 1 when t is between T and L/u' and $L/\eta \to \infty$ [i.e., $(T/\tau_\eta) \to \infty$], that it is equal to 1 for $\Delta_0 = \sqrt{G_\Delta/5}\,\eta$, and that, for any finite value of L/η, it significantly differs from 1 at all times $t < T$. The Richardson constant G_Δ is determined from the value of $[\overline{(\Delta - \Delta_0)^2}]/[(u'^3/L)t^3]$ in the range $0.2(L/u')$ to L/u' and we find $G_\Delta \approx 0.03$ for large enough scale ratio L/η (of order 10^3 and larger). We should stress that in KS, G_Δ effectively contains both the original Richardson constant as in $G_\Delta \epsilon t^3$ but also the constant of proportionality relating the kinetic energy dissipation rate to u'^3/L. We therefore retain the orders of magnitude of G_Δ obtained by KS but not the actual values.

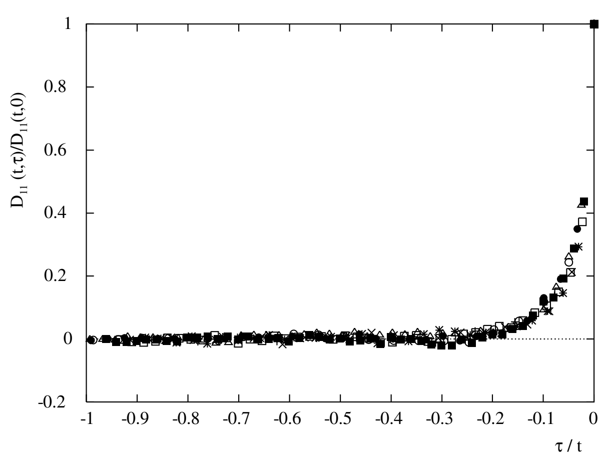

FIG. 3. Nondimensional diagonal Lagrangian velocity correlation $D_{11}(t, \tau)/D_{11}(t, 0)$ as a function of τ/t. Same case as Fig. 2. $(tu'/L) =$ +0.03, ×0.06, ∗0.08, □0.11, ■0.25, ⊙0.50, ●0.76, △1.

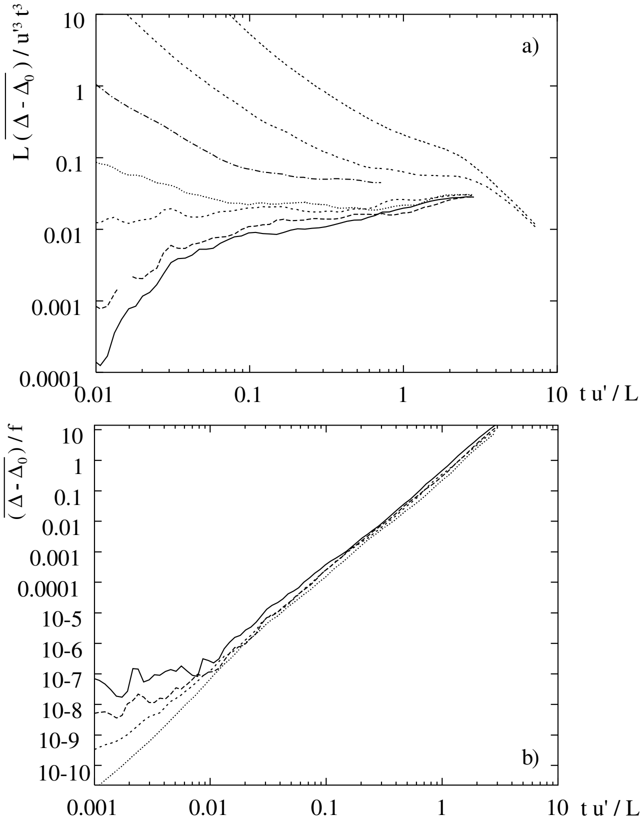

FIG. 4. Pair diffusion as a function of time for $(L/\eta) = 38748$ and $\tau_\eta = 0.0027(L/u')$ and different initial separations. (a) $\overline{(\Delta - \Delta_0)^2}/(u'^3 t^3/L)$ as a function of $t(u'/L)$, from top to bottom $(\Delta_0/\eta) = 1000, 100, 10, 1, 0.1, 0.01, 0.001$. (b) $\overline{(\Delta - \Delta_0)^2}/f$ as a function of $t(u'/L)$ for $(\Delta_0/\eta) = 1, 0.1, 0.01, 0.001$.

The deviations from Richardson's law (1) observed when L/η is about 100 might perhaps be due to edge effects, but can this also be the case in our KS where L/η reaches values above 10^5? Clearly one cannot answer this question with numerical simulations rigorously except if future runs with even higher L/η eventually converge to Richardson's law without Δ_0 dependencies.

Nevertheless, the success of our KS to reproduce the laboratory observations of [5] and its failure to retrieve Richardson's law without Δ_0 dependencies even at extremely high L/η does raise the question of the validity of Richardson's universality and of the locality assumption on which it is based [6], even asymptotically for arbitrarily high L/η. In general, $\overline{\Delta^2}$ is a function of t, L, η, Δ_0, and u' in KS, and the Richardson locality assumption adapted for KS states that, for large enough L/η, $(d/dt)\overline{\Delta^2}$ should depend only on $\overline{\Delta^2}$ and $E(k)$ at $k = 2\pi/\sqrt{\overline{\Delta^2}}$ when $\max(\eta, \Delta_0) \ll \sqrt{\overline{\Delta^2}} \ll L$. Fung and Vassilicos (1998) [6] found this assumption to be valid in planar KS for different spectral exponents p between 1 and 2 $[E(k) \sim k^{-p}]$ but specifically for $\Delta_0 = \eta/2$ and unsteadiness parameter λ is about 1 or smaller in $\omega_m = \lambda\sqrt{k_m^3 E_n(k_m)}$. The direct consequence of this assumption is that $\overline{\Delta^2} \sim t^\gamma$ with $\gamma = [4/(3-p)]$ which is indeed observed in KS for different values of p but only for Δ_0 close to η [6]. What could invalidate locality and Richardson's law for Δ_0 very different from η?

The low values of G_Δ and the very large Lagrangian flatness factors of V_i^L also observed in KS [6] are consistent with the observation that fluid element pairs travel close to each other for long stretches of time and separate in sudden bursts [5,6]. Fluid element accelerations $\mathbf{a} \equiv (D/Dt)\mathbf{u}$ [where $(D/Dt) \equiv (\partial/\partial t) + \mathbf{u} \cdot \boldsymbol{\nabla}$] are such that $\boldsymbol{\nabla} \cdot \mathbf{a} = \mathbf{s}^2 - (\omega^2/2)$ where $\mathbf{s}$ is the strain rate matrix and $\boldsymbol{\omega}$ the vorticity vector. Hence, $\boldsymbol{\nabla} \cdot \mathbf{a}$ is large and positive most often in straining regions around hyperbolic points of the flow where $\mathbf{s}^2$ is large and $\boldsymbol{\omega}^2$ close to 0 (regions where $\boldsymbol{\omega}^2$ is large and $\mathbf{s}^2$ much larger are extremely rare by comparison). Close fluid element pairs can separate violently where $\boldsymbol{\nabla} \cdot \mathbf{a}$ is large and positive, and the separation is effective if the streamline structure of the turbulence is persistent enough in time (in which case fluid element trajectories will closely follow streamlines at least for a while). Hence, such violent separation events will most often occur where close fluid element pairs meet hyperbolic points that are persistent enough.

Based on their KS results which were limited to $\Delta_0 = \eta/2$, Fung and Vassilicos (1998) [6] rephrased Richardson's locality assumption as follows: "in the inertial range, the dominant contribution to the turbulent diffusivity $(d/dt)\overline{\Delta^2}$ comes from straining regions of size $\sqrt{\overline{\Delta^2}}$; these straining regions are embedded in a fractal-eddy structure of cat's eyes within cat's eyes and therefore straining regions exist with a variety of length scales over the entire inertial range." Davila and Vassilicos [11] have related γ to the fractal dimension D of this fractal-eddy streamline structure of straining regions when Δ_0 is close to η ($\gamma = 4/D$). These results suggest that when Δ_0 is between η and 0.1η, the evolution of fluid element pairs by bursts when they meet straining regions somehow tunes into the straining fractal structure of the flow and gives rise to Richardson's law. This requires some persistence of the streamline structure, and indeed Richardson's law is lost when the unsteadiness parameter λ is made significantly larger than 1 [6].

This topological picture of turbulent pair diffusion suggested by results in previous papers and our argument concerning $\boldsymbol{\nabla} \cdot \mathbf{a}$ could also explain the strong Δ_0 dependence of $\overline{\Delta^2}$. As Δ_0 decreases well below η, the probability for fluid element pairs to encounter a hyperbolic point and be separated by it also decreases and can become so small for $\Delta_0 \ll \eta$ that pairs may travel close to each other for very long times. Eventually, at times nearing L/u', the eddy turnover time of the turbulence, the two fluid elements will be separated by the unsteadiness of the flow rather than by its streamline structure as they will have to become independent at times $t \gg L/u'$. They therefore largely bypass the relatively persistent straining fractal streamline structure of the turbulence and also Richardson's law as a result. For initial conditions $\Delta_0 \gg \eta$, the argument based on $\boldsymbol{\nabla} \cdot \mathbf{a}$ does not apply and the separation of fluid element pairs cannot be considered to be dominated by straining events in the vicinity of hyperbolic regions. In the framework of the topological turbulent pair diffusion picture, this is consistent with the absence of a Richardson law for $\Delta_0 \gg \eta$.

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