NTEC Module: Water Reactor Performance and Safety

# Lecture 9: Critical flow

G. F. Hewitt

Imperial College London



## Summary





## Isentropic processes I

Ideal gas law

$$pv = RT \qquad R = \frac{R_u}{M^{'}}$$

$v =$ specific volume $\left(= \frac{1}{\rho}\right)$ (m$^3$/kg)

$p =$ pressure (Pa)

$R =$ gas constant (Nm/kg.K)

$T =$ temperature (K)

$R_u =$ Universal gas constant

$= 8314$ (Nm/kmole.K)

$M^{'} =$ Molecular weight (kg/kmole)



## Isentropic processes II

Definition of specific enthalpy

$$h = u + pv = u + RT \quad \text{(ideal gas)}$$

$u =$ specific internal energy (J/kg)

Definition of $c_v$ and $c_p$

$$dh = c_p dT$$

$$du = c_v dT$$

Thus:

$$dh = c_p dT = du + RdT = c_v dT + RdT$$

$$c_p - c_v = R$$

## Isentropic processes III

Define

$$\frac{c_p}{c_v} = k \quad (\gamma \text{ often also used})$$

$$c_p = \frac{kR}{k-1},\ c_v = \frac{R}{k-1}$$

Chenge in internal energy $du$ given by

$$du = dq - dw$$

$dq =$ heat added per kg (J/kg)

$dw =$ work done by system (J/kg)

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## Isentropic processes IV

Work done given by

$dw = pdv$

Thus

$du = dq - pdv$

Define:

$$ds = \frac{dq}{T}$$

Thus:

$Tds = du + pdv$

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## Isentropic processes V

But

$$dh = d(u + pv) = du + pdv + vdp$$

$$\therefore Tds = dh - vdp$$

Thus

$$ds = \frac{du}{T} + \frac{p}{T}dv = c_v\frac{dT}{T} + R\frac{dv}{v}$$

$$ds = \frac{dh}{T} + \frac{v}{T}dp = c_p\frac{dT}{T} + R\frac{dp}{p}$$

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## Isentropic processes VI

For an adiabatic, reversible ("isentropic") process $ds = 0$. Thus:

$$\frac{dT}{T} + \frac{R}{c_v}\frac{dv}{v} = 0 \qquad \text{--- (1)}$$

$$\frac{dT}{T} - \frac{R}{c_p}\frac{dp}{p} = 0 \qquad \text{--- (2)}$$

Subtracting (2) from (1) and dividing by $R/c_v$ :

$$\frac{dp}{p} + \frac{c_p}{c_v}\frac{dv}{v} = 0$$

Integrating

$$\ln p + \frac{c_p}{c_v}\ln v = \text{constant}$$

$$\boxed{pv^k = \text{constant}}$$

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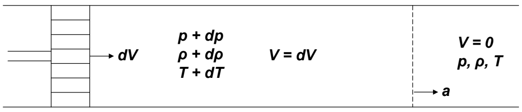
## Propagation of small amplitude pressure wave I

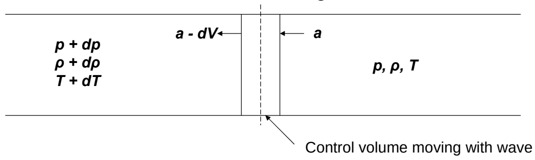



- Long cylinder filled initially with motionless fluid
- Piston at one end given infinitesmal velocity *dV*
- Pressure wave moves at velocity ***a*** separating fluid moving at velocity *dV* and fluid which remains stationary



## Propagation of small amplitude pressure wave II

Consider frame of reference moving with wave




Continuity equation on control volume

$$a\rho = (a - dV)(\rho + d\rho)$$
$$\cancel{a\rho} = \cancel{a\rho} + ad\rho - \rho dV - \cancel{dVd\rho}$$
$$ad\rho = \rho dV \qquad \text{--- (3)}$$



## Propagation of small amplitude pressure wave III

Momentum equation for control volume:

$$pA - (p + dp)A = -\rho a^2 A + (\rho + d\rho)(a - dV)^2 A$$

Dividing through by $A$ and expanding:

$$\cancel{p} - \cancel{p} - dp = -\rho a^2 + (\rho + d\rho)(a^2 - 2adV + (dV)^2)$$
$$= -\cancel{\rho a^2} + \cancel{\rho a^2} - 2\rho adV + \cancel{\rho (dV)^2} + a^2 d\rho - \cancel{2adVd\rho} + \cancel{d\rho (dV)^2}$$

Thus

$$-dp = -2\rho adV + a^2 d\rho$$



## Propagation of small amplitude pressure wave IV

But from equation 3 (continuity), $\rho dV = ad\rho$

$$-dp = -2a^2 d\rho + a^2 d\rho$$
$$a^2 = \frac{dp}{d\rho}$$

For thin region around wave with a disturbance of infinitesmal amplitude, the process is essentially ADIABATIC AND REVERSIBLE, hence ISENTROPIC

$$a^2 = \left(\frac{dp}{d\rho}\right)_S$$

## Propagation of small amplitude pressure wave V

For ideal gas, isentropic process gives

$$pv^k = \frac{p}{\rho^k} = \text{constant } (X)$$

$$dp = Xk\rho^{k-1}d\rho = \frac{p}{\rho^k}\cdot k\rho^{k-1}\cdot d\rho$$

$$= k\frac{p}{\rho}d\rho$$

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \frac{kp}{\rho} = kRT$$

$$\boxed{a = \sqrt{kRT}}$$

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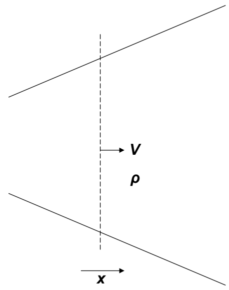
# Isentropic flow I




Procedure: Derive continuity and momentum equations for element of length dx.

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# Isentropic flow II

Continuity equation (steady flow)

$$\frac{d}{dx}(\rho VA) = 0$$

or $\rho VA = \dot{M} = \text{const}$

Differentiating

$$VAd\rho + \rho VdA + \rho AdV = 0$$

or $\dfrac{d\rho}{\rho} + \dfrac{dA}{A} + \dfrac{dV}{V} = 0$

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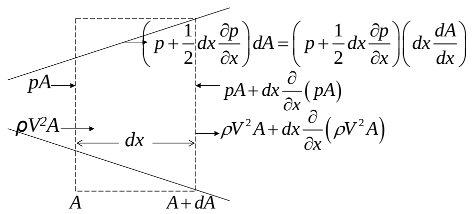
# Isentropic flow III

Momentum equation (steady flow without body or shear forces)




Momentum flux out – momentum flux in
= sum of (pressure) forces on element

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## Isentropic flow IV

$$\rho V^2 A + dx\frac{\partial}{\partial x}\left(\rho V^2 A\right) - \rho V^2 A$$
$$= pA + \left[p + \frac{1}{2}dx\frac{\partial p}{\partial x}\right]dx\frac{\partial A}{\partial x} - \left[pA + dx\frac{\partial}{\partial x}(pA)\right]$$
$$\rho V^2 A + dx\frac{\partial}{\partial x}\left(\rho V^2 A\right) - \rho V^2 A$$
$$= pA + pdx\frac{\partial A}{\partial x} + \frac{1}{2}(dx)^2\frac{\partial p}{\partial x}\cdot\frac{\partial A}{\partial x} - pA - pdx\frac{\partial A}{\partial x} - Adx\frac{\partial p}{\partial x}$$
$$dx\frac{\partial}{\partial x}\left(\rho V^2 A\right) = -Adx\frac{\partial p}{\partial x}$$

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## Isentropic flow V

$$dx\frac{\partial}{\partial x}\left(\rho V^2 A\right) = -Adx\frac{\partial p}{\partial x}$$

or $\frac{\partial}{\partial x}\left(\rho V^2 A\right) + A\frac{\partial p}{\partial x} = 0$

or $\frac{\partial}{\partial x}\left(V\cdot\rho VA\right) + A\frac{\partial p}{\partial x} = 0$

$$\rho VA\frac{\partial V}{\partial x} + V\frac{\partial}{\partial x}\left(\rho VA\right) + A\frac{\partial p}{\partial x} = 0 \qquad \text{--- (4)}$$

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## Isentropic flow VI

But from the continuity equation

$$\frac{\partial}{\partial x}\left(\rho VA\right) = 0$$

Thus equation 4 becomes (eliminating *A*)

$$\rho V\frac{\partial V}{\partial x} + \frac{\partial p}{\partial x} = 0$$

or $dp = -\rho VdV$

or $\frac{dV}{V} = -\frac{dp}{\rho V^2}$ --- (5)

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## Isentropic flow VII

$\frac{dV}{V}$ also follows from the continuity expression

$$\frac{dV}{V} = -\frac{dA}{A} - \frac{d\rho}{\rho}$$

and introducing this into equation 5 we have:

$$-\frac{dA}{A} - \frac{d\rho}{\rho} = -\frac{dp}{pV^2}$$
$$\frac{dA}{A} = -\frac{d\rho}{\rho} + \frac{dp}{pV^2} = \frac{dp}{pV^2}\left[1 - \frac{V^2}{(dp/d\rho)}\right]$$

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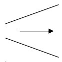
# Isentropic flow VIII

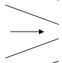
But $a^2 = \left(dp / d\rho\right)$ and $V^2 / a^2 = M^2$ where

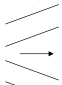
$M$ is the MACH NUMBER. Thus

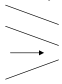
$$\frac{\partial p}{\partial V^2} = \frac{dA / A}{1 - M^2}$$

Thus:

Subsonic flow $M < 1$: $dp$ positive if $dA$ positive 

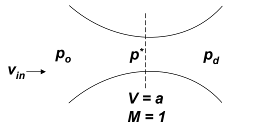
$dp$ negative if $dA$ negative 

Supersonic flow $M > 1$: $dp$ negative if $dA$ positive 

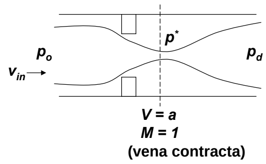
$dp$ positive if $dA$ negative 



# Critical flow I



(a) Nozzle



(b) Orifice

$p_o > p^* > p_d$

When $V = M$ at throat of nozzle, further decrease in downstream pressure does not change velocity (or mass flux) at throat or vena contracta or $p^*$.

Hence MAXIMUM or CRITICAL flow observed in these conditions.



# Critical flow II

Kinetic energy per unit mass of fluid $= V^2 / 2$.

First law of thermodynamics:

$$h_o = \text{constant} = h + \frac{V^2}{2}$$

$h_o$ = STAGNATION ENTHALPY

= enthalpy of fluid brought to rest ($\approx$ upstream enthalpy if $V_{in} \ll a$)

$h = c_p T = \dfrac{kRT}{(k-1)}$ (Slide 5)

Thus:

$$\frac{kRT}{(k-1)} + \frac{V^2}{2} = \frac{kRT_o}{(k-1)}$$



# Critical flow III

$$\frac{kRT}{(k-1)} + \frac{V^2}{2} = \frac{kRT_o}{(k-1)}$$

If $V = a = \sqrt{kRT^*}$ then $V^2 = kRT^*$. Thus:

$$\frac{kRT^*}{(k-1)} + \frac{kRT^*}{2} = \frac{kRT_o}{(k-1)}$$

Dividing through by $kRT^*$ and multiplying through by $(k-1)$

$$\boxed{1 + \frac{k-1}{2} = \frac{T_o}{T^*} = \frac{1+k}{2}}$$

# Critical flow IV

To obtain $p_o / p^*$ we use

$$pv^k = \text{constant} = c_1 \quad \text{(isentropic expansion)}$$

$$pv = RT \quad \text{(ideal gas law)}$$

$$\frac{RT}{v} \cdot v^k = \text{constant} = c_1$$

$$Tv^{k-1} = \text{constant} = \frac{c_1}{R} = c_2$$

$$v = \left(\frac{c_1}{p}\right)^{1/k} \qquad v^{k-1} = \left(\frac{c_1}{p}\right)^{(k-1)/k}$$

$$\frac{T}{p^{(k-1)/k}} = \text{constant}$$

Thus $$\boxed{\frac{p_o}{p^*} = \left(\frac{T_o}{T^*}\right)^{k/(k-1)} = \left(\frac{1+k}{2}\right)^{k/(k-1)}}$$

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# Critical flow V

Maximum (critical) mass rate of flow

$$\dot{M}_{\max} = \frac{aA^*}{v^*} = aA^*\rho^* = A^*\rho^*\sqrt{kRT^*}$$

$A^*$ is flow area at which $V = a$ ($A^* = CA,\ C =$ coefficient of discharge)

We express $T^*$ and $\rho^*$ in terms of (known) $p_o$ and $T_o$

$$T^* = \frac{2T_o}{k+1}$$

$$\rho^* = \frac{p^*}{RT^*} \quad \text{(ideal gas law)}$$

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# Critical flow VI

$$T^* = \frac{2T_o}{k+1}$$

$$\rho^* = \frac{p^*}{RT^*} \quad \text{(ideal gas law)}$$

$$p^* = \frac{p_o}{\left(\frac{1+k}{2}\right)^{k/(k-1)}}$$

$$\rho^* = \frac{p_o}{\left(\frac{1+k}{2}\right)^{k/(k-1)}} \cdot \frac{1}{RT_o\left(\frac{1+k}{2}\right)^{-1}}$$

$$= \frac{p_o}{RT_o} \cdot \frac{1}{\left(\frac{1+k}{2}\right)^{1/(k-1)}}$$

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# Critical flow VII

$$\dot{M}_{crit} = A^* \frac{p_o}{RT_o} \cdot \frac{1}{\left(\frac{k+1}{2}\right)^{1/(k-1)}} \sqrt{kRT_o\left(\frac{k+1}{2}\right)^{-1}}$$

$$= A^* p_o \left(\frac{k}{RT_o}\right)^{1/2} \left[\left(\frac{k+1}{2}\right)^{-\frac{1}{k-1}-\frac{1}{2}}\right]$$

$$= A^* p_o \left(\frac{k}{RT_o}\right)^{1/2} \left[\frac{k+1}{2}\right]^{-\frac{k+1}{2(k-1)}}$$

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## Example of critical flow of steam I

- A 1100 MW(e) reactor operating at 150 bar is shut down due to a smqll break corresponding to an area of 0.011m2. A short time after the shutdown, the rate of power generation has reduced to 200MW(t).
- The reactor has remained pressurised and the steam generators are not operating. Heat removal is by “feed and bleed” with water being injected by the HPIS and the heat being removed in the form of steam which flows through the break.
- Assuming that the specific heat ratio k (= cp/cv) is 1.3 for steam and that the steam is saturated at 343 C, calculate the maximum flow rate of steam which could occur. Also, assuming a latent heat of vaporisation of 1000 kJ/kg, calculate the maximum amount of heat which could be removed. Is this sufficient?
- The molecular weight of water should be taken as 18 kg/kmole and the break treated as a nozzle with a discharge coefficient of unity. The Universal Gas Constant is $R_u$ = 8314 J/kmole K

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## Example of critical flow of steam II

For steam

$$R = \frac{R_u}{M'} = \frac{8314}{18} = 461.9$$

$$p_o = 150 \text{ bar} = 1.5\times10^7 \text{ Pa}$$

$$T_o = 342 + 273 = 615 \text{ K}$$

$$A^* = CA = 1\times 0.011 = 0.011$$

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## Example of critical flow of steam III

$$\dot{M}_{crit} = A^* p_o \left(\frac{k}{RT_o}\right)^{1/2} \left[\frac{k+1}{2}\right]^{-\frac{k+1}{2(k-1)}}$$

$$= 0.011\times1.5\times10^7 \times \left(\frac{1.3}{461.9\times615}\right)^{1/2} \times \left(\frac{1.3+1}{2}\right)^{-(1.3+1)/2(1.3-1)}$$

$$= 0.011\times1.5\times10^7 \times 0.002139\times(1.15)^{-3.833}$$

$$= 206 \text{ kg/s}$$

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## Example of critical flow of steam IV

Maximum rate of heat removal

$$\equiv \dot{M}_{\max} \times h_{LG} = 206\times1000\times10^3$$

$$= 206 \text{ MW}$$

(i.e. just sufficient to remove decay heat)

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## Homogeneous equilibrium model (HEM) for two phase flow I

Gas  Temperature $T_g$

Velocity $u_g$

Composition $\tilde{y}_i$ = Mole fraction of $i$th component

Liquid  Temperature $T_l$

Velocity $u_l$

Composition $\tilde{x}_i$ = Mole fraction of $i$th component

Assumptions:

$T_g = T_l$

$u_g = u_l$

$\tilde{y}_i$'s in equilibrium with $\tilde{x}_i$'s

Applied along path of flow from upstream to throat

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## Homogeneous equilibrium model (HEM) for two phase flow II

Density of fluid ($\overline{\rho}$) given by

$$\overline{\rho} = \left[ \left( x / \rho_g \right) + \left( 1 - x \right) / \rho_l \right]^{-1}$$

($\rho_l$ and $\rho_g$ and liquid and gas densities)

$x$ = quality (vapour mass flow as a fraction of total flow)

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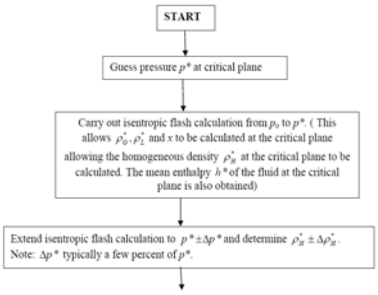
## Homogeneous equilibrium model (HEM) for two phase flow III




Method general and applies to both single component and multi-component fluids

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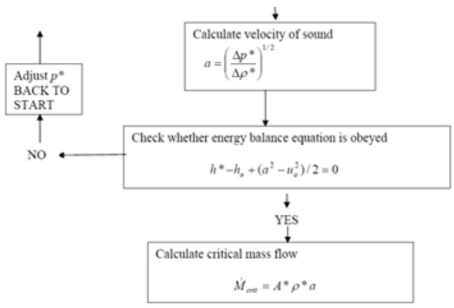
## Homogeneous equilibrium model (HEM) for two phase flow IV

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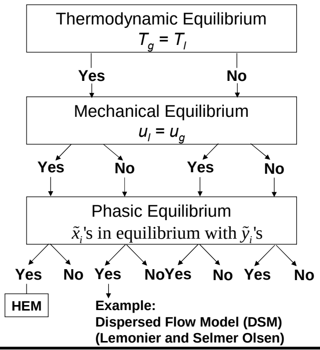
# Homogeneous equilibrium model (HEM) for two phase flow V

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