Numerical Methods for Stochastic Volatility: Fourier Methods, PDEs and Monte Carlo

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Imperial College Finance and Stochastics Seminar

Dresdner Kleinwort

Outline

- Introduction to FX options: vanillas and liquid exotics
- Heston's stochastic volatility model in FX
- Developing market intuition about Heston
- Fast semianalytic techniques: characteristic functions
- Basic calibration of the model to the market smile
- Pricing using Monte Carlo
- Pricing using numerical finite differences in 2D

Introduction: liquid FX exotics and deviation from B-S prices

- Short dated FX options (out to 3Y or so):
  - ~90% vanillas
  - ~9% binaries/barriers [continuously monitored]
  - ~1% other complex exotics
- Vanillas
  - Almost solely OTC – *not* exchange traded
  - European, not American style. Value depends only on S_T
- Binaries/Barriers:
  - Common criticism of options is that they appear quite expensive to the buyer. Leads to demand for cheaper alternatives – e.g. knock-out options.

Vanillas in FX

- B-S inadequate – a single σ will not match all vanillas in market
  - Structural deviation from B-S prices: **volatility smile**
- Benchmark FX instruments: 5 strikes per tenor
  - 10-delta put — strike K chosen so that Δ_p = -0.10
  - 25-delta put — strike K chosen so that Δ_p = -0.25
  - ATM option — either* ATMF (K=F) or D-N ($\Delta_p+\Delta_c=0$)
  - 25-delta call — strike K chosen so that Δ_c = +0.25
  - 10 delta call — strike K chosen so that Δ_c = +0.10
- Smiles are generally (JPY and EMs aside) reasonably symmetric

* depends on market convention.

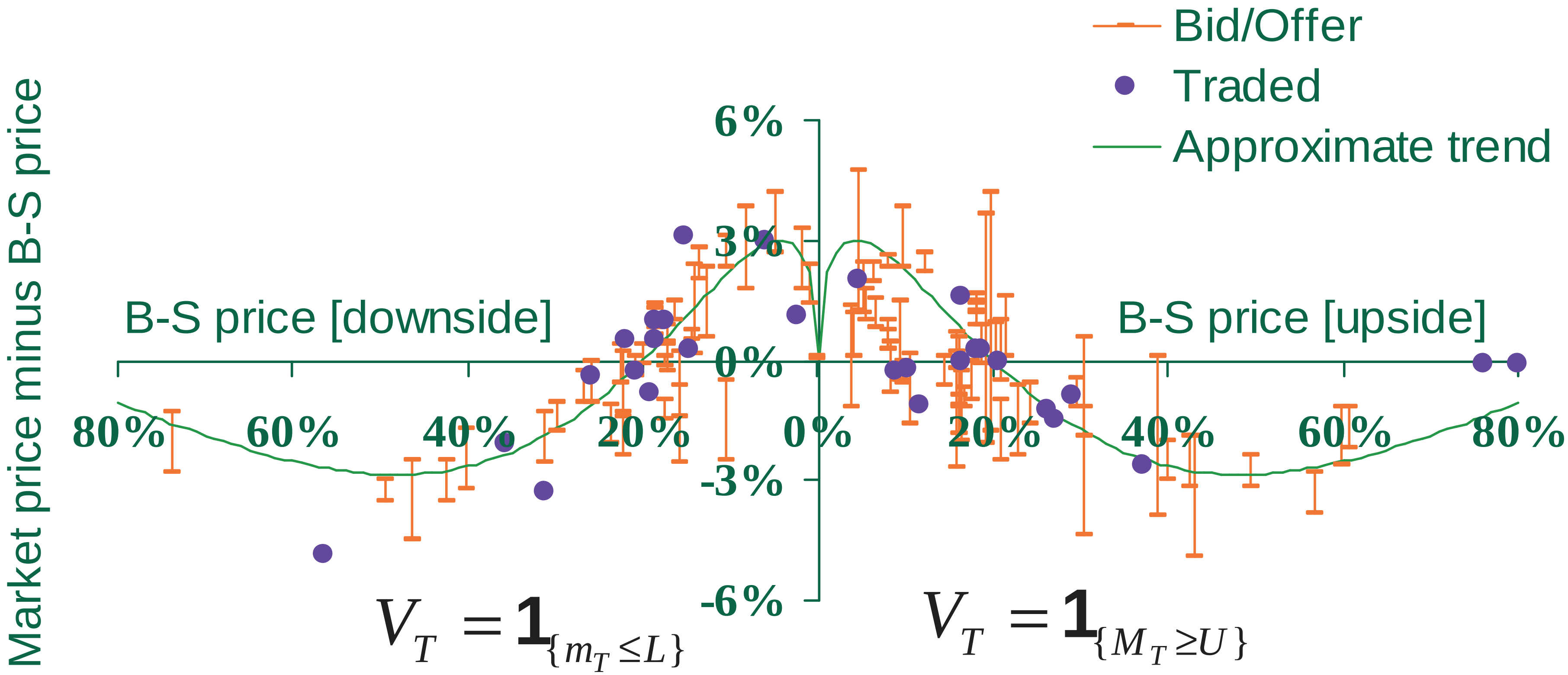
Reference: Malz, Allan M. (1997), Estimating the Probability Distribution of the Future Exchange Rate from Option Prices, *J. Derivatives*, 18-36.

Barriers in FX

- Introducing path dependency can make a vanilla option substantially cheaper
- European call: $$V_T = (S_T - K)^+ = (S_T - K)\mathbf{1}_{\{S_T \geq K\}}$$
- m_T and M_T denote the minimum and maximum (resp.) of S_t over the time interval $[0,T]$
- Cheaper alternatives:
  – Regular KO $V_T = (S_T - K)\mathbf{1}_{\{S_T \geq K\}}\mathbf{1}_{\{m_T > L\}}$
  – Reverse KO $V_T = (S_T - K)\mathbf{1}_{\{S_T \geq K\}}\mathbf{1}_{\{M_T < U\}}$
  – Double KO $V_T = (S_T - K)\mathbf{1}_{\{S_T \geq K\}}\mathbf{1}_{\{m_T > L\}}\mathbf{1}_{\{M_T < U\}}$

Binaries in FX

- Distant OTs (TV < 20%) typically trade above TV
- Nearer OTs typically trade below TV
  - Structural deviation from B-S prices: **binary moustache**



Flow exotics – depend on which processes?

- All the flow exotics have value functions V_T which depend at most on three of the following processes:

Product	S_T	m_T	M_T
European vanilla	YES		
OT or NT [downside]		YES	
OT or NT [upside]			YES
KI or KO [downside]	YES	YES	
KI or KO [upside]	YES		YES
DNT or DT		YES	YES
DKI or DKO	YES	YES	YES

- Ideally obtain market implied joint pdf of $\{S_T, m_T, M_T\}$

Heston's Stochastic Volatility Model in FX

- In practice: seek a model that accurately describes (a) volatility smile, (b) binary moustaches [i.e. marginals for S_T, m_T and M_T] and (c) DNT prices.
- The Heston model is a model for stochastic *variance*

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t^{(1)}, \quad \mu = r_d - r_f$$

$$dV_t = \kappa(m - V_t)dt + \alpha\sqrt{V_t}\,dW_t^{(2)}, \qquad \sigma_t = \sqrt{V_t}$$

$$\left\langle dW_t^{(1)}, dW_t^{(2)} \right\rangle = \rho dt$$

- What intuition should we attach to the model parameters?

Reference: Heston, S.L. (1993), A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Rev. Financ. Stud.*, **6** (2), 327-343.

Stochastic volatility and vol convexity

- All stochastic volatility/variance models generate smiles, by correctly pricing in vol convexity
- Hull/White analysis: if processes driving spot and variance are uncorrelated

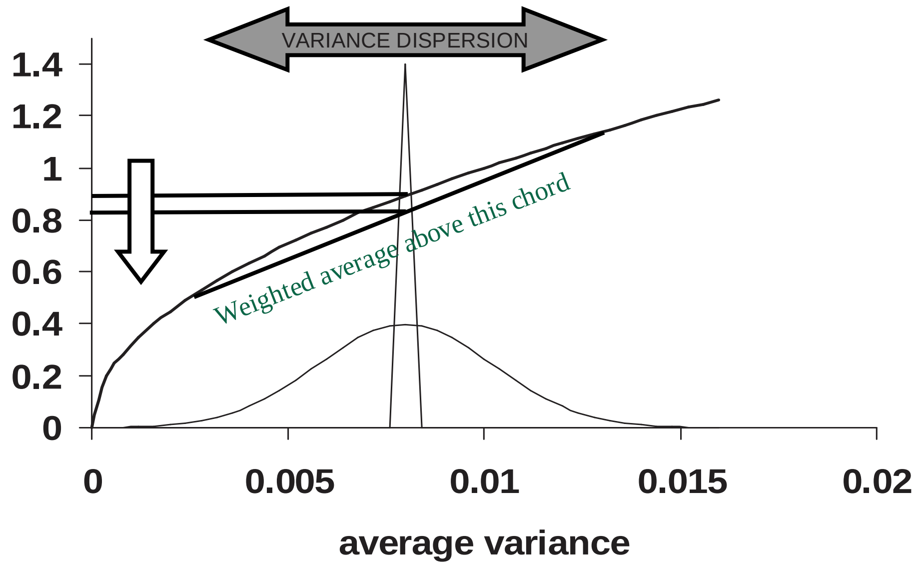
$$PV = \int_0^\infty TV\,|_{\sigma=\sqrt{v}}\; f_{\overline{V}}(v)\,dv$$

where $f_{\overline{V}}(v)$ is the pdf of average variance $\overline{V} = 1/T\int_0^T \sigma_u^2\,du$ over time interval $[0,T]$ and $TV|_\sigma$ is the Black-Scholes price with constant volatility σ.

Reference: Hull, J.C. and A. White (1987), The Pricing of Options on Assets with Stochastic Volatilities, *J. Finance*, **42**, 281-300.

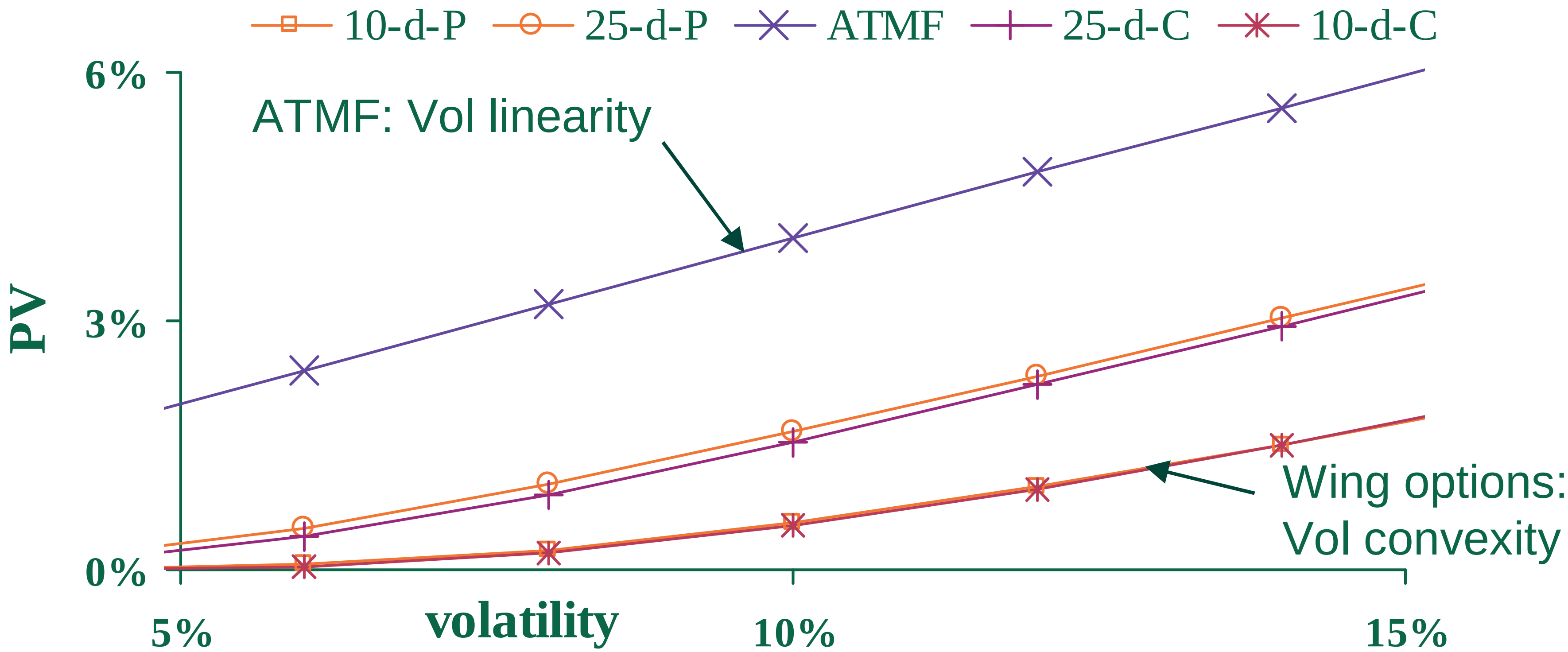
Intuition: implied ATM vol structure

- TV of the ATMF option is $TV \approx 0.4\sigma\sqrt{T} = 0.4\sqrt{VT}$
- Consider a driftless stochastic *variance* process.
- In that case the expectation of $\overline{V}$ is just V_0
- Concavity of square root function means that PV decreases as the volatility becomes increasingly dispersed around V_0
- ATMF price under a driftless stochastic *variance* process *decreases* as volatility of variance increases.
- Implied ATMF vol is $\sqrt{V_0}$ adjusted downwards for this vovariance effect



Intuition: implied wing structure

- ATMF options are linear in volatility but wing options are convex.
  - Hence increasing vovariance increases the implied smile



Intuition: effects of Heston model parameters

- Five parameters have quite different effects on the shape of implied volatility surface generated

Parameter	Effect
Initial variance V_0	Fixes overall level of implied ATM vol
Vovariance α	Generates volatility smile as α increases
Spot/Variance correlation ρ	Generates volatility skew for nonzero ρ
Mean reversion rate κ	Combined effect: increasing κ, term structure of implied ATM vol shifts in direction of $m^{1/2}$ & smile flattens
Mean reversion level m	

Risk neutral pricing

- Let asset have price process $dS_t = (r_d - r_f)S_t dt + \sigma_t S_t dW_t$
- Black-Scholes formula for European call option
  - take discounted expectation of payout under domestic RN measure
  - use Girsanov to change from domestic RN to foreign RN measure

$$
\begin{aligned}
C(S,T) &= e^{-r_d T}\mathbf{E}^d[(S_T - K)^+] \\
&= e^{-r_d T}\mathbf{E}^d[(S_T - K)\mathbf{1}_{\{S_T \geq K\}}] \\
&= e^{-r_d T}\mathbf{E}^d[S_T \mathbf{1}_{\{S_T \geq K\}}] - Ke^{-r_d T}\mathbf{E}^d[\mathbf{1}_{\{S_T \geq K\}}] \\
&= S_0 e^{-r_f T}\mathbf{E}^f[\mathbf{1}_{\{S_T \geq K\}}] - Ke^{-r_d T}\mathbf{E}^d[\mathbf{1}_{\{S_T \geq K\}}] \\
&= S_0 e^{-r_f T}\mathbf{P}^f[S_T \geq K] - Ke^{-r_d T}\mathbf{P}^d[S_T \geq K] \\
&= S_0 e^{-r_f T} N(d_1) - Ke^{-r_d T} N(d_2)
\end{aligned}
$$

Risk neutral pricing

- Express in terms of asset log-returns $X_t = \ln(S_t)$
  - S_t constrained to [0, ∞) but X_t defined on (–∞, ∞)
  - X_t in BS world follows an ABM and has normally distributed marginals (easier to compute characteristic functions)

- Call price is given by

$$C(S,T) = e^{-r_d T} \mathbf{E}^d[(\exp(X_T) - K)\mathbf{1}_{\{X_T \geq K\}}]$$

$$= S_0 e^{-r_f T} \mathbf{P}^f[X_T \geq \ln K] - K e^{-r_d T} \mathbf{P}^d[X_T \geq \ln K]$$

Pricing in Fourier space

- Denote pdfs in foreign and domestic risk-neutral measures by $f_{X_T}^{d}(x)$ and $f_{X_T}^{f}(x)$ respectively. $h(x)$ is the payout function.

$$\mathbf{E}^{d;f}[h(X_T)] = \underbrace{\int_{-\infty}^{\infty} h(x)\, f_{X_T}^{d;f}(x)\, dx}_{\text{an inner product in } L^2(-\infty,\infty)}$$

- Parseval's theorem: inner products preserved under Fourier transforms

$$\underbrace{\hat{h}(\phi)}_{\substack{\text{Fourier}\\ \text{transform}\\ \text{of payout}\\ \text{function}}} = \int_{-\infty}^{\infty} e^{i\phi x}\, \underbrace{h(x)}_{\substack{\text{payout}\\ \text{function}}}\, dx$$

$$\overbrace{\hat{f}_{X_T}^{d;f}(\phi)}^{\substack{\text{characteristic}\\ \text{function of } X_T}} = \mathbf{E}^{d;f}\left[e^{i\phi X_T}\right] = \int_{-\infty}^{\infty} e^{i\phi x}\, \underbrace{f_{X_T}^{d;f}(x)}_{\text{pdf of } X_T} dx$$

$$2\pi \int_{-\infty}^{\infty} h(x)\, f_{X_T}^{d;f}(x) dx = \int_{-\infty}^{\infty} \hat{h}(\phi)\, \hat{f}_{X_T}^{d;f}(\phi)\, d\phi$$

Pricing in Fourier space

- Clearly, expectations can be computed in $\phi -$ space

$$\mathbf{E}^{d;f}[h(X_T)] = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(\phi)\, \hat{f}_{X_T}^{d;f}(\phi)\, d\phi$$

- Now we can calculate the cdf's – which take the form

$$\mathbf{P}^{d;f}[X_T \geq \ln K] = \mathbf{E}^{d;f}[1_{\{X_T \geq \ln K\}}]$$

- The Fourier transform of $h(x) = 1_{\{x \geq \ln K\}}$ is given by

$$\hat{h}(\phi) = \int_{-\infty}^{\infty} 1_{\{x \geq \ln K\}} e^{ix\phi}\, dx = \int_{\ln K}^{\infty} e^{ix\phi}\, dx = \left.\frac{e^{ix\phi}}{i\phi}\right|_{\ln K}^{\infty}$$

- Issues:
  - The limit as $x \to \infty$ of $e^{ix\phi}$ isn't formally defined
  - Complex pole at the origin ($\phi = 0$)
  - No major impediment

Pricing in Fourier space

- Fourier inversion formula results (Heston; Bates; Bakshi et al.)

$$\mathbf{P}^{d;f}[X_T \geq \ln K] = \tfrac{1}{2} + \tfrac{1}{\pi}\int_0^{\infty} \mathrm{Re}[\,\hat{f}_{X_T}^{d;f}(\phi)\,\tfrac{\exp(-i\phi \ln K)}{i\phi}\,]d\phi$$

- Need to compute the c.f. $\hat{f}_{X_T}^{d;f}(\phi)$ of the log-return asset process.
  - This is where things get interesting because it can be calculated analytically – e.g. for spot processes driven by Heston stochastic volatility process.
- First, we work through Black-Scholes case, then look at Heston.

References: Bates, D.S. (1996), Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options, *Rev. Financ. Stud.*, **9**, 69-107.

Bakshi, G., Cao, C. and Z. Chen (1997), Empirical Performance of Alternative Option Pricing Models, *J. Finance*, **52**, 2003-2049.

Pricing in Fourier space – Black-Scholes case

- Assume volatility constant. Spot follows $dS_t = (r_d - r_f)S_t dt + \sigma S_t dW_t$ and the log-returns therefore follow

$$dX_t = (r_d - r_f - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t$$

with solution

$$X_t = X_0 + (r_d - r_f - \tfrac{1}{2}\sigma^2)t + \sigma W_t$$

- This is normal with mean $X_0 + (r_d - r_f - \tfrac{1}{2}\sigma^2)t$ and variance $\sigma\sqrt{t}$

- For a $N(\mu,\sigma^2)$ r.v., with pdf $f_X(x) = \frac{1}{\sqrt{2\pi\sigma}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$ the c.f. is given by (complete the square)

$$\hat{f}_X(\phi) = \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} e^{i\phi x}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)dx = \exp(i\mu\phi - \tfrac{1}{2}\sigma^2\phi^2)$$

Pricing in Fourier space – Black-Scholes case

- We now have all we need to price any European option using Fourier integration in Black-Scholes. For example, a call is priced at

$$C(S,T)=S_0e^{-r_fT}\mathbf{P}^f[X_T\geq\ln K]-Ke^{-r_dT}\mathbf{P}^d[X_T\geq\ln K]$$

- where

$$\mathbf{P}^{d;f}[X_T\geq\ln K]=\tfrac{1}{2}+\tfrac{1}{\pi}\int_0^\infty \underbrace{\mathrm{Re}[\hat{f}_{X_T}^{d;f}(\phi)\tfrac{\exp(-i\phi\ln K)}{i\phi}]}_{\text{real integrand}}d\phi$$

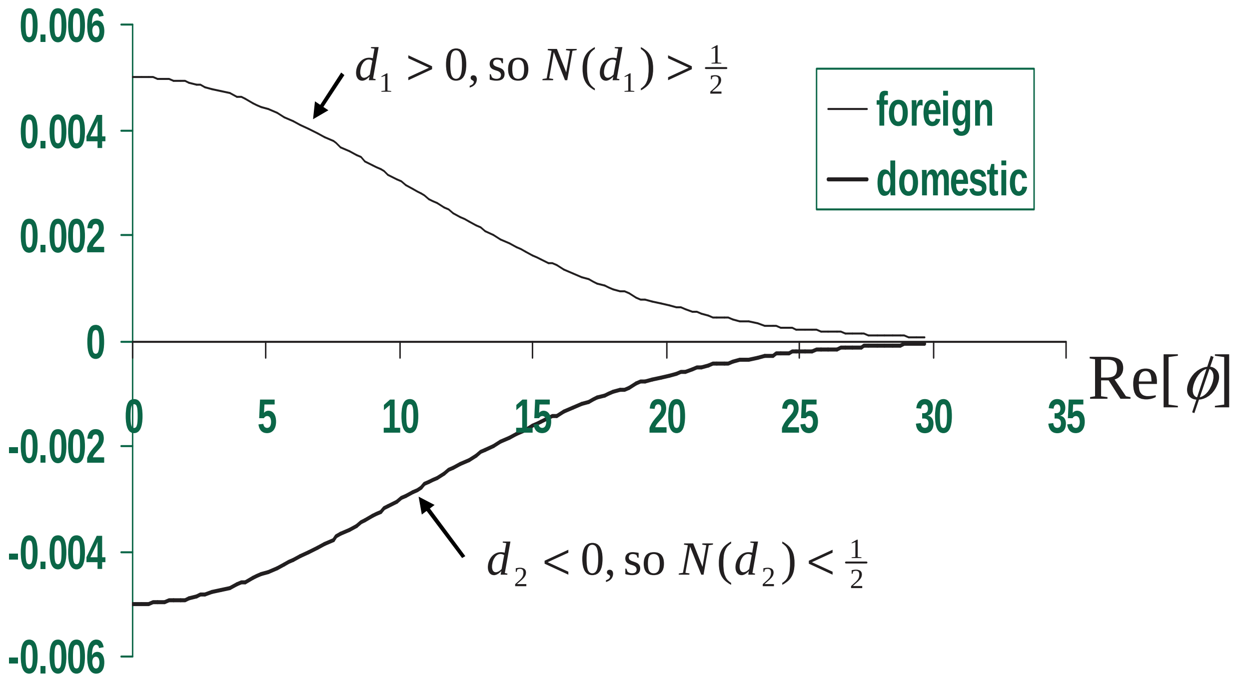
- For Black-Scholes:

$$\hat{f}_{X_T}^{d;f}(\phi)=\exp(i[(r_d-r_f\pm\tfrac{1}{2}\sigma^2)T+X_0]\phi-\tfrac{1}{2}\sigma^2T\phi^2)$$

- **Questions:** What do the integrands look like?
  For stochastic volatility models, how might we obtain the required characteristic functions?

Examination of integrands – Black-Scholes case

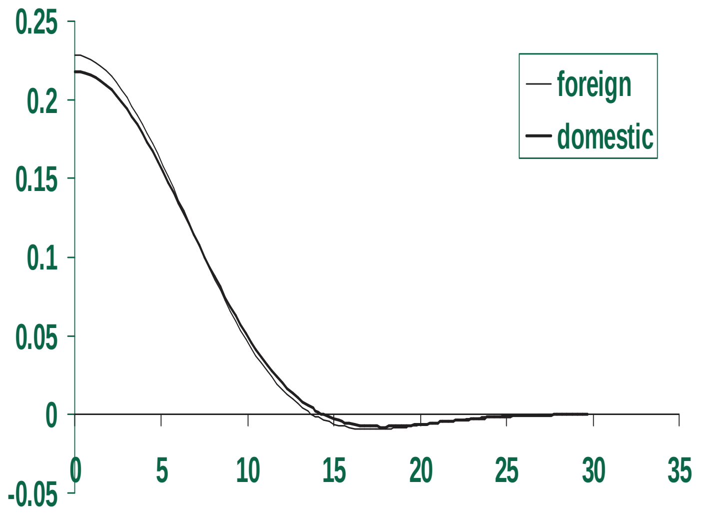
- ATMF: S_0=1, σ= 10%, r_d=0, r_f=0, K=1, T=1

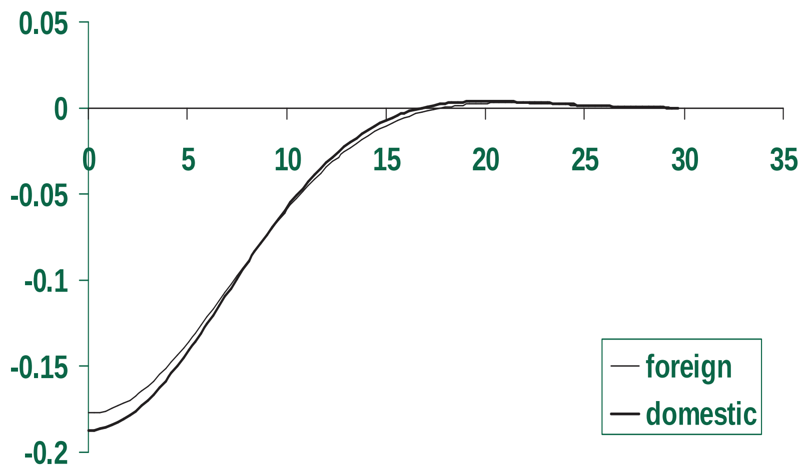


- The symmetry reminds us of $d_1 = -d_2$ for the ATMF case

Examination of integrands – Black-Scholes case

- Market: $S_0=1$, $\sigma=10\%$, $r_d=0$, $r_f=0$
- ITM: $K=0.8$, $T=1$
- OTM: $K=1.2$, $T=1$

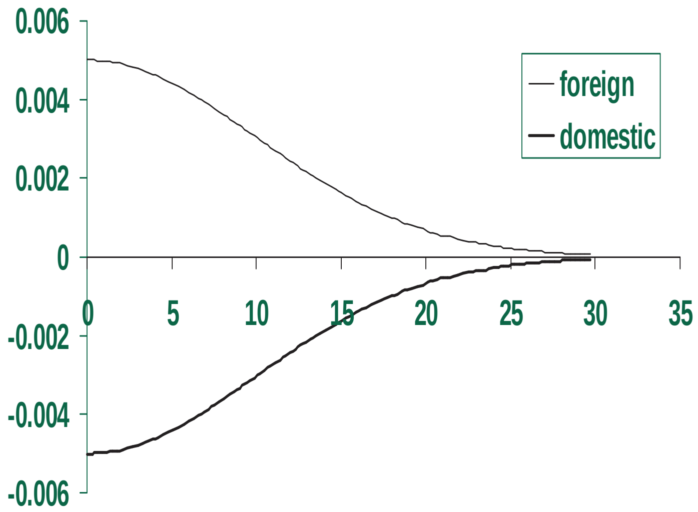




- RN probabilities of exceeding 0.8 at expiry > 0.5 : integrals are positive
- RN probabilities of exceeding 1.2 at expiry < 0.5 : integrals are negative

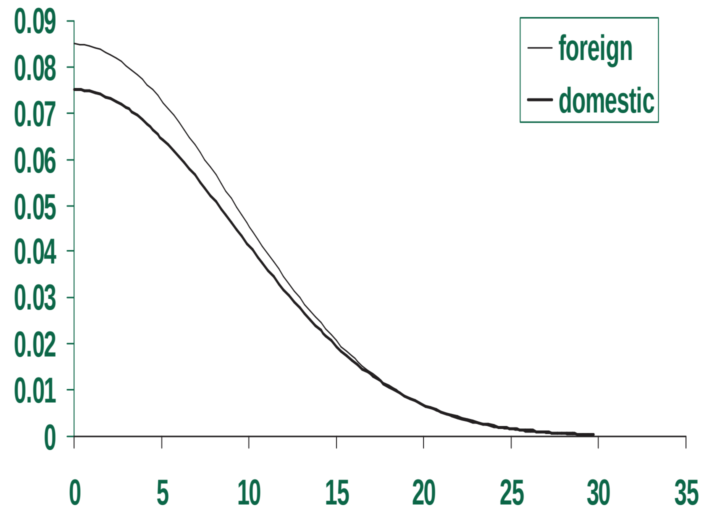
Examination of integrands – Black-Scholes case

- ATMS: K=1, T=1, S_0=1
- Market: σ=10%, r_d=8%, r_f=8%



- Unaffected by changes in rates which maintain same IR differential

- Market: σ=10%, r_d=8%, r_f=0%



- Affected by changes in IR differential
  – forward rate moves, affects RN probs.

Examination of integrands – Black-Scholes case

- Following is a sensible choice: $\phi_{\max} = Q/(\sigma\sqrt{T})$
  Makes sense as $\sigma\sqrt{T}$ is dimensionless.
- In fact you can see this analytically from

$$\mathbf{P}^{d;f}[X_T \geq \ln K] = \tfrac{1}{2} + \tfrac{1}{\pi}\int_0^\infty \underbrace{\mathrm{Re}[\hat{f}_{X_T}^{d;f}(\phi)\frac{\exp(-i\phi\ln K)}{i\phi}]}_{\text{real integrand}} d\phi$$

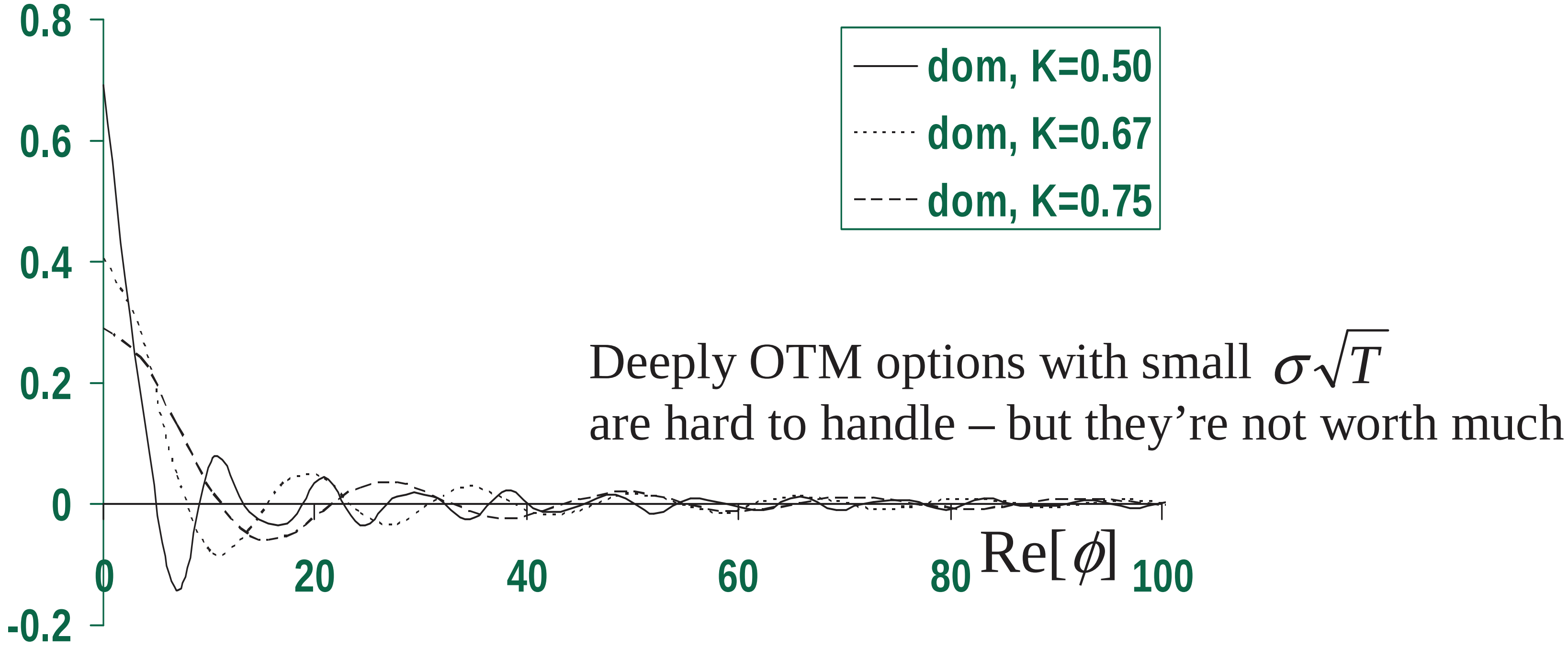
$$\hat{f}_{X_T}^{d;f}(\phi) = \exp(i[(r_d - r_f \pm \tfrac{1}{2}\sigma^2)T + X_0]\phi - \tfrac{1}{2}\sigma^2 T\phi^2)$$

$$\mathrm{Re}[\hat{f}_{X_T}^{d;f}(\phi)\frac{\exp(-i\phi\ln K)}{i\phi}] = \underbrace{\frac{\exp(-\tfrac{1}{2}\sigma^2 T\phi^2)}{\phi}}_{\text{envelope}} \times \underbrace{\sin([(r_d - r_f \pm \tfrac{1}{2}\sigma^2)T + X_0 - \ln K]\phi)}_{\text{oscillatory term}}$$

- Choice of Q – somewhere between 2 and 5 is generally sufficient
- Simple trapezoidal integration on [0, $\phi_{\max}$] is OK in practice.
- Backtest against exact Black-Scholes price to make sure integration is OK

Examination of integrands – Black-Scholes case

- How oscillatory can these integrals get?
- Difficult cases: K={0.5, 0.66, 0.75}, S_0=1, σ= 3%, r_d=0, r_f=0, T=1/12



A member of Allianz

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Computing price with just one Fourier integral

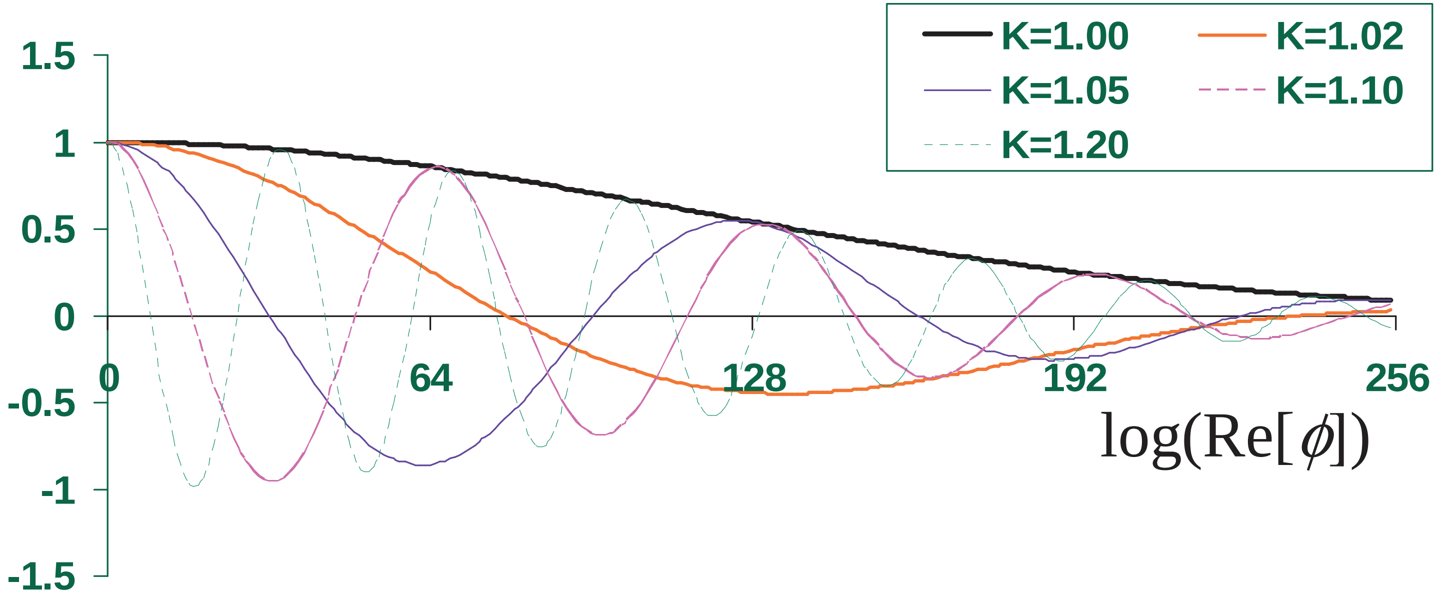
- Wanted to show how the domestic and foreign risk-neutral probabilities can be calculated using Fourier methods and related back to $N(d_1)$ and $N(d_2)$. Easy to visualise.
- In fact, for European calls and puts, the computation can be performed using a single Fourier integral along a contour in the complex plane – see Lewis, p.37 for details
- This is more efficient as only one integral to compute.
- Need to use inversion formula (2.5) in Lewis.
- Recommend starting with the 2 integral technique, then implement the single integral technique as a companion scheme.
  Ensure results agree.

Reference: Lewis, A.L. (2000), Option Valuation Under Stochastic Volatility with Mathematica Code, Finance Press.

Out of print but Chapters 1 and 2 available at www.optioncity.net

Computing price with just one Fourier integral

- Integrals are less singular for Europeans using the 1-integral technique
- Difficult cases again: $S_0=1$, $\sigma = 3\%$, $r_d=0$, $r_f=0$, $T=1/12$.
  Horizontal axis is log(ϕ) – hence integral on positive half-line OK



- Single integration should recover Black-Scholes price accurately

Pricing in Fourier space – Heston stochastic volatility model

- This seems like a lot of extra work when we can just go directly to Black-Scholes closed form formulae. What's the point?
- **Key point:** this method *extends* to stochastic volatility models.
- How? Go back to the definition of characteristic function

$$\hat{f}^{d;f}_{X_T}(\phi) = \mathbf{E}^{d;f}\left[e^{i\phi X_T}\right] = \int_{-\infty}^{\infty} e^{i\phi x} f^{d;f}_{X_T}(x)dx$$

- Following Section 2.2.2 of Zhu (2000), compute in risk neutral measures

$$\hat{f}^{d}_{X_T}(\phi) = \mathbf{E}^{d}\left[e^{i\phi X_T}\right]$$

$$\hat{f}^{f}_{X_T}(\phi) = e^{(r_f - r_d)T}\mathbf{E}^{d}\left[\frac{S_T}{S_0}e^{i\phi X_T}\right] = e^{(r_f - r_d)T}e^{-X_0}\mathbf{E}^{d}\left[e^{(i\phi+1)X_T}\right]$$

References: Zhu, J. (2000), Modular Pricing of Options: An Application of Fourier Analysis, Springer, Berlin-Heidelberg.

Schöbel, R. & Zhu, J. (1999), Stochastic Volatility with an Ornstein-Uhlenbeck Process: An Extension, *Eur. Fin. Rev.* **3**, 23-46.

Pricing in Fourier space – Heston stochastic volatility model

- Clearly we need to compute $\mathbf{E}^d[e^{(i\phi+j)X_T}],\ \ j \in \{0,1\}$

- Consider by way of example the Heston model.

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)} \qquad \left\langle dW_t^{(1)}, dW_t^{(2)} \right\rangle = \rho dt$$

$$dV_t = \kappa(m - V_t)dt + \alpha\sqrt{V_t} dW_t^{(2)}$$

- Log-returns:

$$dX_t = (\mu - \tfrac{1}{2}V_t)dt + \sqrt{V_t} dW_t^{(1)}$$

- Integrate the log-return process to get

$$X_T = X_0 + \mu T - \tfrac{1}{2}\int_0^T V_t dt + \int_0^T \sqrt{V_t} dW_t^{(1)}$$

Pricing in Fourier space – Heston stochastic volatility model

- A few pages to show how Girsanov works here. Substituting X_T in:

$$\mathbf{E}^d[e^{(i\phi+j)X_T}] = e^{(i\phi+j)(X_0+\mu T)}\mathbf{E}^d[\exp(-\tfrac{1}{2}\int_0^T V_t dt + \int_0^T \sqrt{V_t}\, dW_t^{(1)})]$$

- Several terms cancel out, leaving the c.f.s (j=0 for d, j=1 for f)

$$\hat{f}_{X_T}^{d;f}(\phi) = e^{i\phi[X_0+\mu T]}\mathbf{E}^d\Big[\exp\Big\{\big(i\phi+j\big)\big(-\tfrac{1}{2}\int_0^T V_t dt + \underbrace{\int_0^T \sqrt{V_t}\, dW_t^{(1)}}_{\text{correlation...}}\big)\Big\}\Big]$$

- Apply Cholesky decomposition

$$dW_t^{(1)} = \rho dW_t^{(2)} + \bar{\rho} dW_t^{(\sim 2)} \qquad \bar{\rho} = \sqrt{1-\rho^2}$$

to obtain

$$\mathbf{E}^d\Big[\exp\Big\{\big(i\phi+j\big)\big(-\tfrac{1}{2}\int_0^T V_t dt + \rho\int_0^T \sqrt{V_t}\, dW_t^{(2)} + \bar{\rho}\int_0^T \sqrt{V_t}\, dW_t^{(\sim 2)}\big)\Big\}\Big]$$

Pricing in Fourier space – Heston stochastic volatility model

- This can be simplified by some Girsanov sleight of hand

$$\mathbf{E}^d\Big[\exp\{(i\phi+j)(-\tfrac{1}{2}\int_0^T V_t dt+\rho\int_0^T \sqrt{V_t}\, dW_t^{(2)}+\bar{\rho}\int_0^T \sqrt{V_t}\, dW_t^{(\sim 2)})\}\Big]$$

$$=\mathbf{E}^d\big[e^{\int_0^T (i\phi+j)\bar{\rho}\sqrt{V_t}\,dW_t^{(\sim 2)}}\, e^{(i\phi+j)(-\frac{1}{2}\int_0^T V_t dt+\rho\int_0^T \sqrt{V_t}\,dW_t^{(2)})}\big]$$

- We can find a Radon-Nikodym derivative $\frac{dQ^{d*}}{dQ^d}=e^{\int_0^T \lambda_t dW_t^{(\sim 2)}-\frac{1}{2}\int_0^T \lambda_t^2 dt}$
  so put $\lambda_t=(i\phi+j)\bar{\rho}\sqrt{V_t}$

$$=\mathbf{E}^d\big[e^{\int_0^T \lambda_t dW_t^{(\sim 2)}-\frac{1}{2}\int_0^T \lambda_t^2 dt}\times e^{\frac{1}{2}\int_0^T \lambda_t^2 dt}\, e^{(i\phi+j)(-\frac{1}{2}\int_0^T V_t dt+\rho\int_0^T \sqrt{V_t}\,dW_t^{(2)})}\big]$$

$$=\mathbf{E}^{d*}\big[e^{\frac{1}{2}(i\phi+j)^2\bar{\rho}^2\int_0^T V_t dt}\, e^{(i\phi+j)(-\frac{1}{2}\int_0^T V_t dt+\rho\int_0^T \sqrt{V_t}\,dW_t^{(2)})}\big]$$

$$=\mathbf{E}^d\big[e^{[\frac{1}{2}(i\phi+j)^2\bar{\rho}^2-\frac{1}{2}(i\phi+j)]\int_0^T V_t dt}\, e^{\rho(i\phi+j)\int_0^T \sqrt{V_t}\,dW_t^{(2)}}\big]$$

Pricing in Fourier space – Heston stochastic volatility model

- Integrating the Heston process we obtain

$$V_T = V_0 + \kappa m T - \kappa \int_0^T V_t dt + \alpha \int_0^T \sqrt{V_t}\, dW_t^{(2)}$$

- So obviously

$$\int_0^T \sqrt{V_t}\, dW_t^{(2)} = \frac{V_T - V_0 - \kappa m T}{\alpha} + \frac{\kappa}{\alpha} \int_0^T V_t dt$$

- We obtain

$$\mathbf{E}^d \left[e^{\left[\frac{1}{2}(i\phi + j)^2 \bar{\rho}^2 - \frac{1}{2}(i\phi + j)\right] \int_0^T V_t dt} \; e^{\rho(i\phi + j) \int_0^T \sqrt{V_t}\, dW_t^{(2)}} \right]$$

$$= \mathbf{E}^d \left[e^{\left[\frac{1}{2}(i\phi + j)^2 \bar{\rho}^2 - \frac{1}{2}(i\phi + j) + \rho\kappa/\alpha\right] \int_0^T V_t dt} \; e^{\rho(i\phi + j)(V_T - V_0 - \kappa m T)/\alpha} \right]$$

and by suitably defining terms $s_1^{(j)}, s_2^{(j)}$ (given on next slide) we obtain

$$\hat{f}_{X_T}^{d;f}(\phi) = e^{i\phi[X_0 + \mu T]} \exp\left(-(V_0 + \kappa m T) s_2^{(j)}\right) \mathbf{E}_d \left[\exp\left(-s_1^{(j)} \int_0^T V_t dt + s_2^{(j)} V_T \right) \right]$$

Pricing in Fourier space – Heston stochastic volatility model

- Zhu (2000) computes this expectation, obtaining the following characteristic functions (j=0 for d, j=1 for f) – note I use $\mu = r_d - r_f$

$$\hat{f}_{X_T}^{d,f}(\phi) = \exp\left(i\phi[X_0 + \mu T] - (V_0 + \kappa m T)s_2^{(j)}\right)\exp\left(A^{(j)}V_0 + C^{(j)}\right)$$

where

$$s_1^{(j)} = -(i\phi + j)\left(-\tfrac{1}{2} + \frac{\rho\kappa}{\alpha} + \tfrac{1}{2}(i\phi + j)(1-\rho^2)\right)$$

$$s_2^{(j)} = (i\phi + j)\rho / \alpha$$

$$\gamma_1^{(j)} = \sqrt{\kappa^2 + 2\alpha^2 s_1^{(j)}}$$

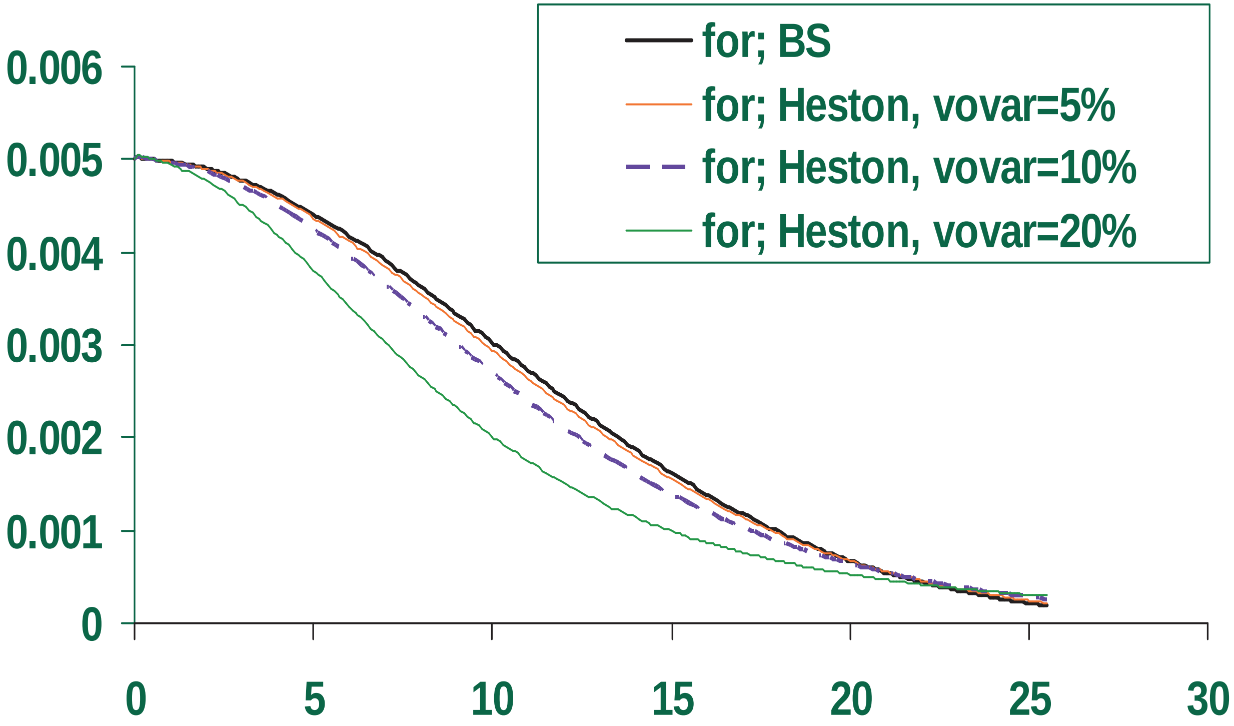
$$\gamma_2^{(j)} = 2\gamma_1^{(j)}\exp\left(-\gamma_1^{(j)}T\right) + \left(\kappa + \gamma_1^{(j)} - \alpha^2 s_2^{(j)}\right)\left(1 - \exp(-\gamma_1^{(j)}T)\right)$$

$$A^{(j)} = \left[\gamma_1^{(j)}s_2^{(j)}\left(1 + \exp(-\gamma_1^{(j)}T)\right) - \left(1 - \exp(-\gamma_1^{(j)}T)\right)\left(2s_1^{(j)} + \kappa s_2^{(j)}\right)\right] / \gamma_2^{(j)}$$

$$C^{(j)} = 2\kappa m \alpha^{-2}\ln\left[2\gamma_1^{(j)}\exp\left(\tfrac{1}{2}(\kappa - \gamma_1^{(j)})T\right) / \gamma_2^{(j)}\right]$$

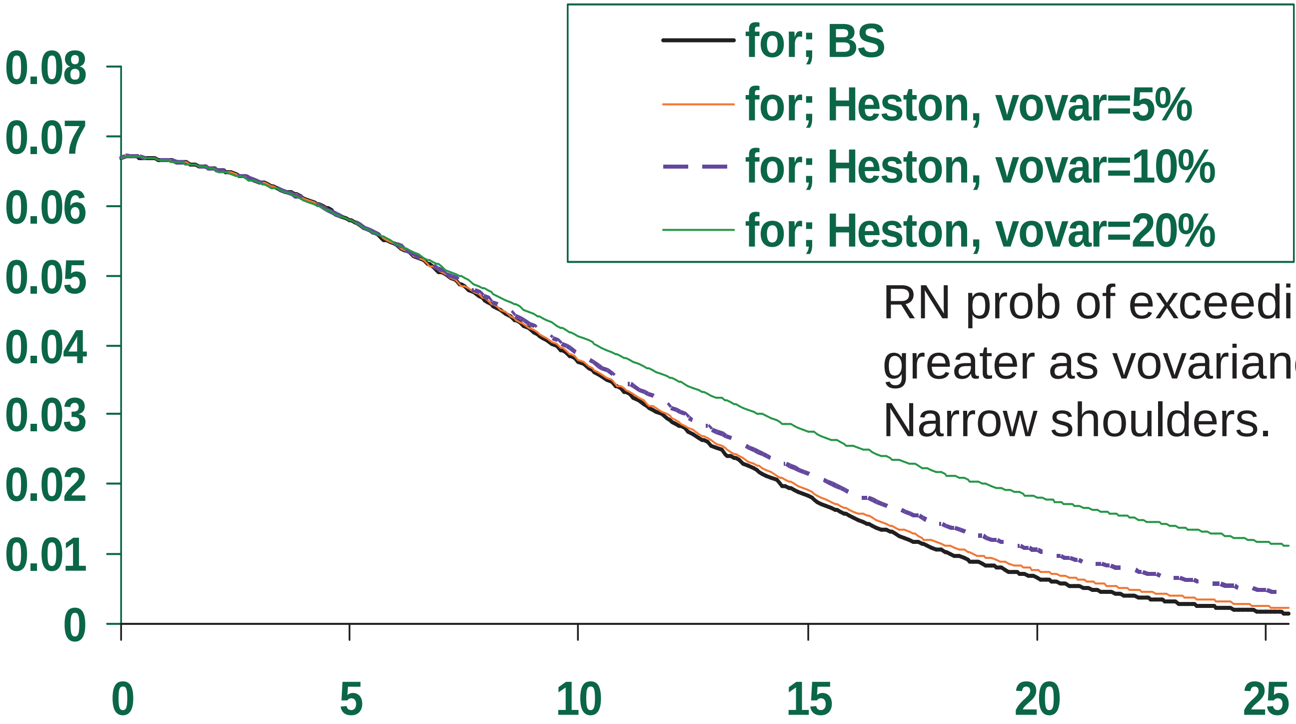
Examination of integrands – stochastic volatility models

- ATMF: K=1.0, S_0=1, σ= 10%, r_d=0, r_f=0, T=1
  Heston with vovol α= 5%, 10% or 20%



Examination of integrands – stochastic volatility models

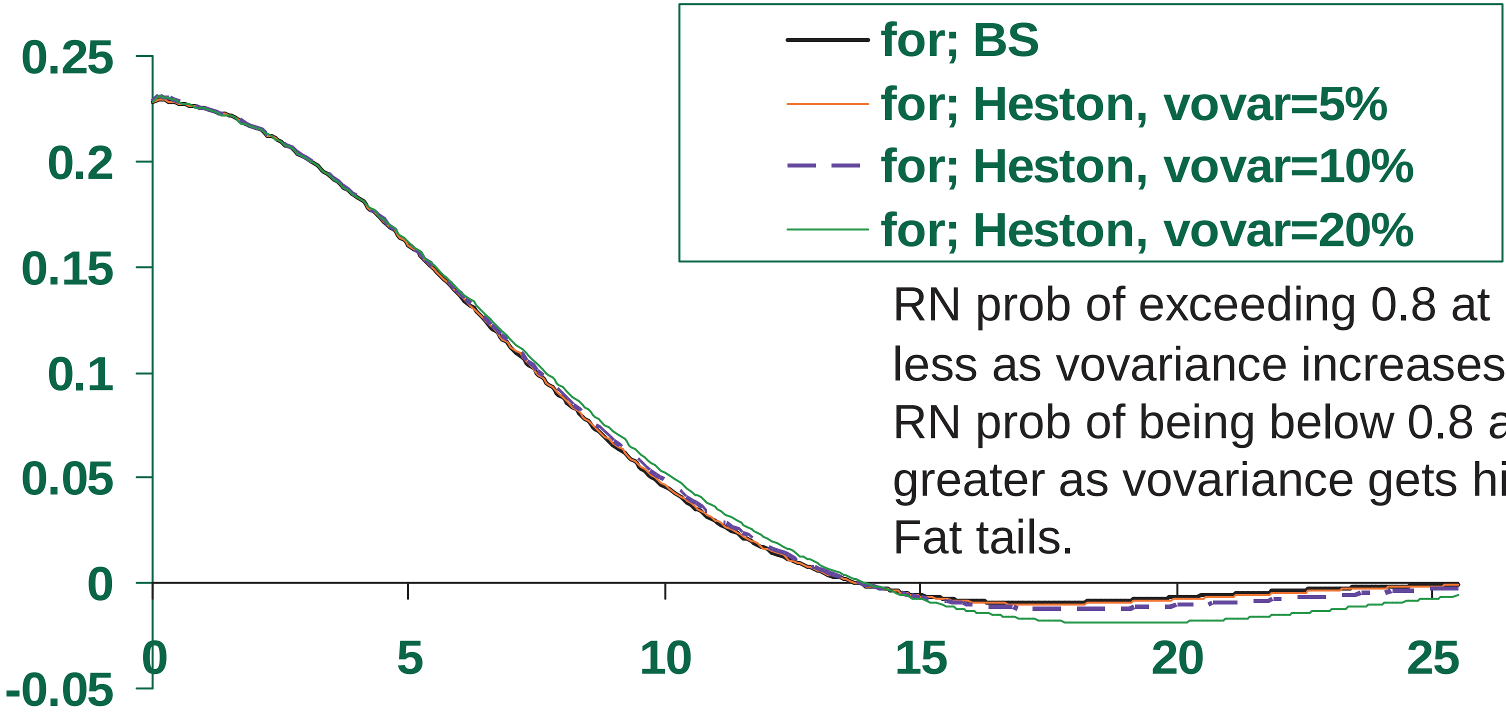
- Wings: K=0.94, S_0=1, σ= 10%, r_d=0, r_f=0, T=1
  Heston with vovol α= 5%, 10% or 20%



RN prob of exceeding 0.94 at T is greater as vovariance increases. Narrow shoulders.

Examination of integrands – stochastic volatility models

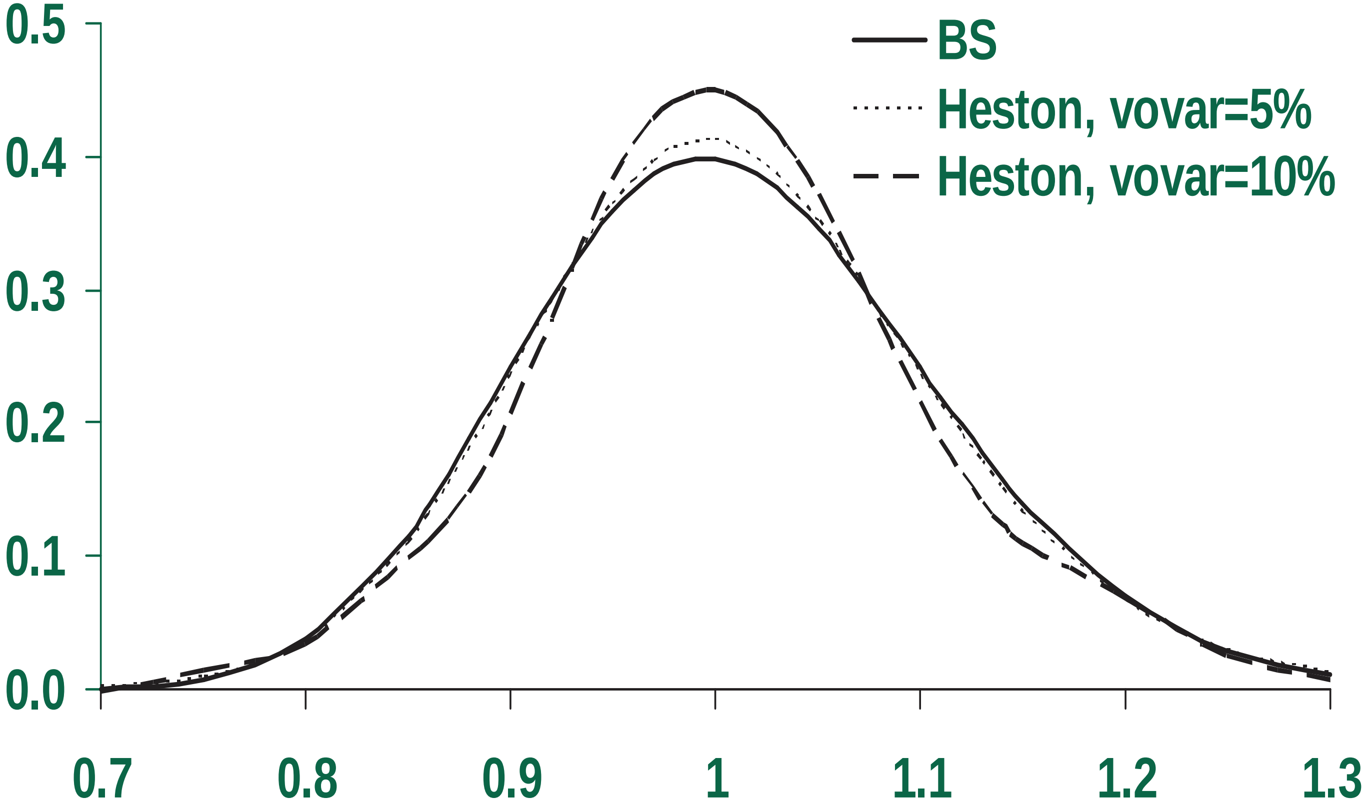
- Distant wings: K=0.80, S_0=1, σ= 10%, r_d=0, r_f=0, T=1
  Heston with vovol α= 5%, 10% or 20%



RN prob of exceeding 0.8 at T is less as vovariance increases.
RN prob of being below 0.8 at expiry greater as vovariance gets higher.
Fat tails.

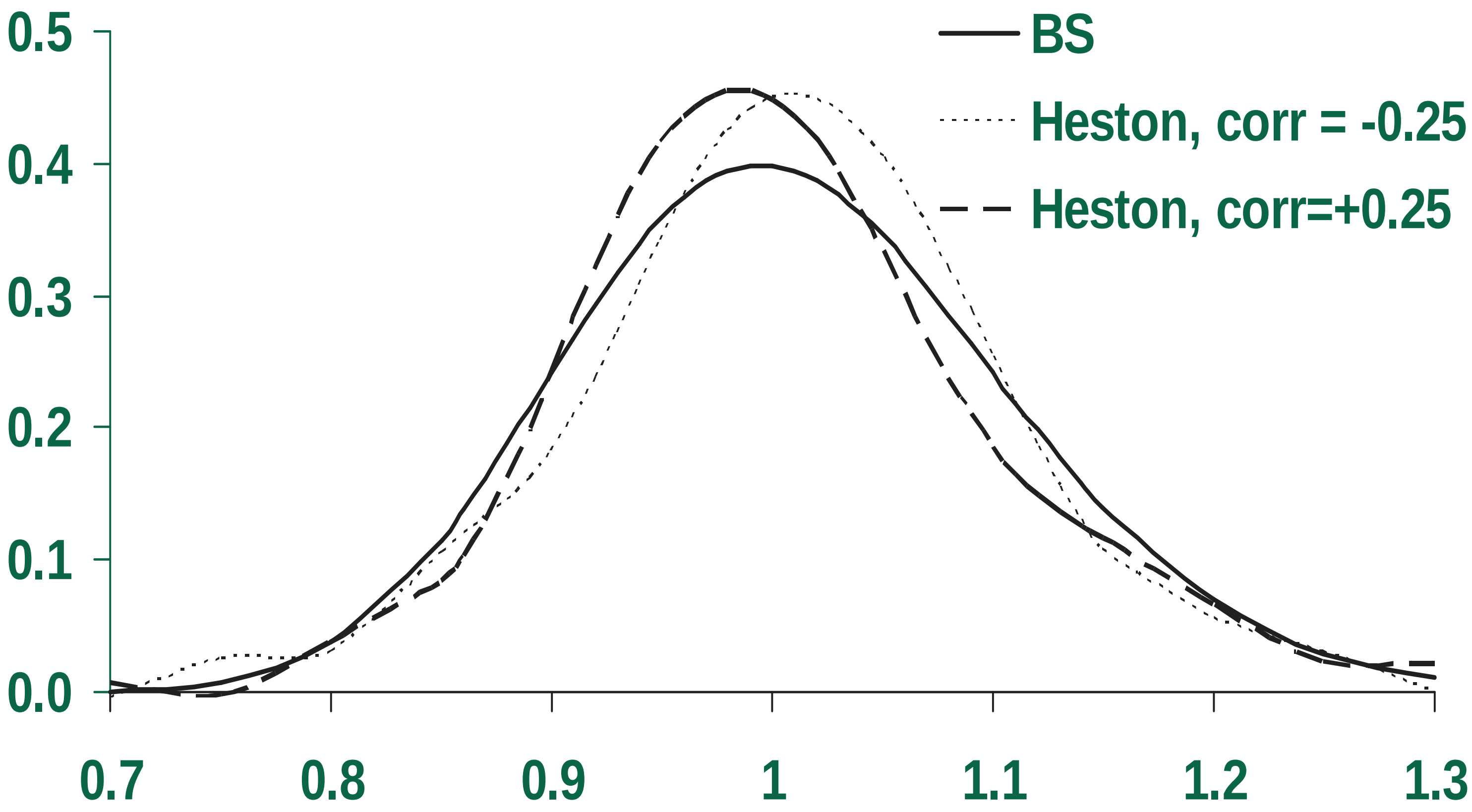
Examination of implied pdfs – stochastic volatility models

- Implied pdfs behave as expected – chart generated by pricing up a strip of Arrow-Debrue securities – see Lewis, p. 37 for details
- $S_0=1$, $\sigma= 10\%$, $r_d=0$, $r_f=0$, $T=1$



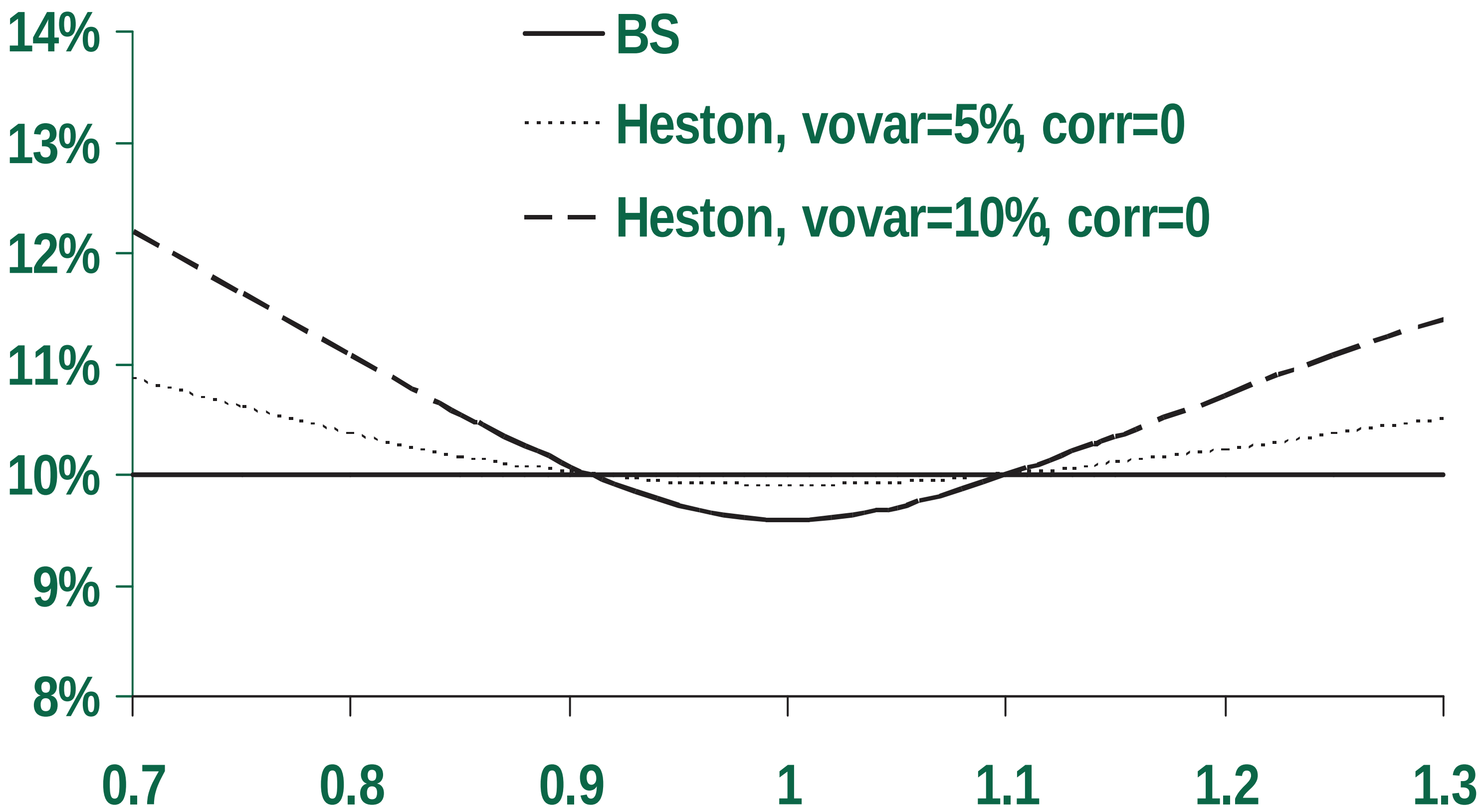
Examination of integrands – stochastic volatility models

- Heston implied pdfs [vovar=10%] become skewed with correlation
- $S_0=1, \ \sigma = 10\%, \ r_d=0, \ r_f=0, T=1$



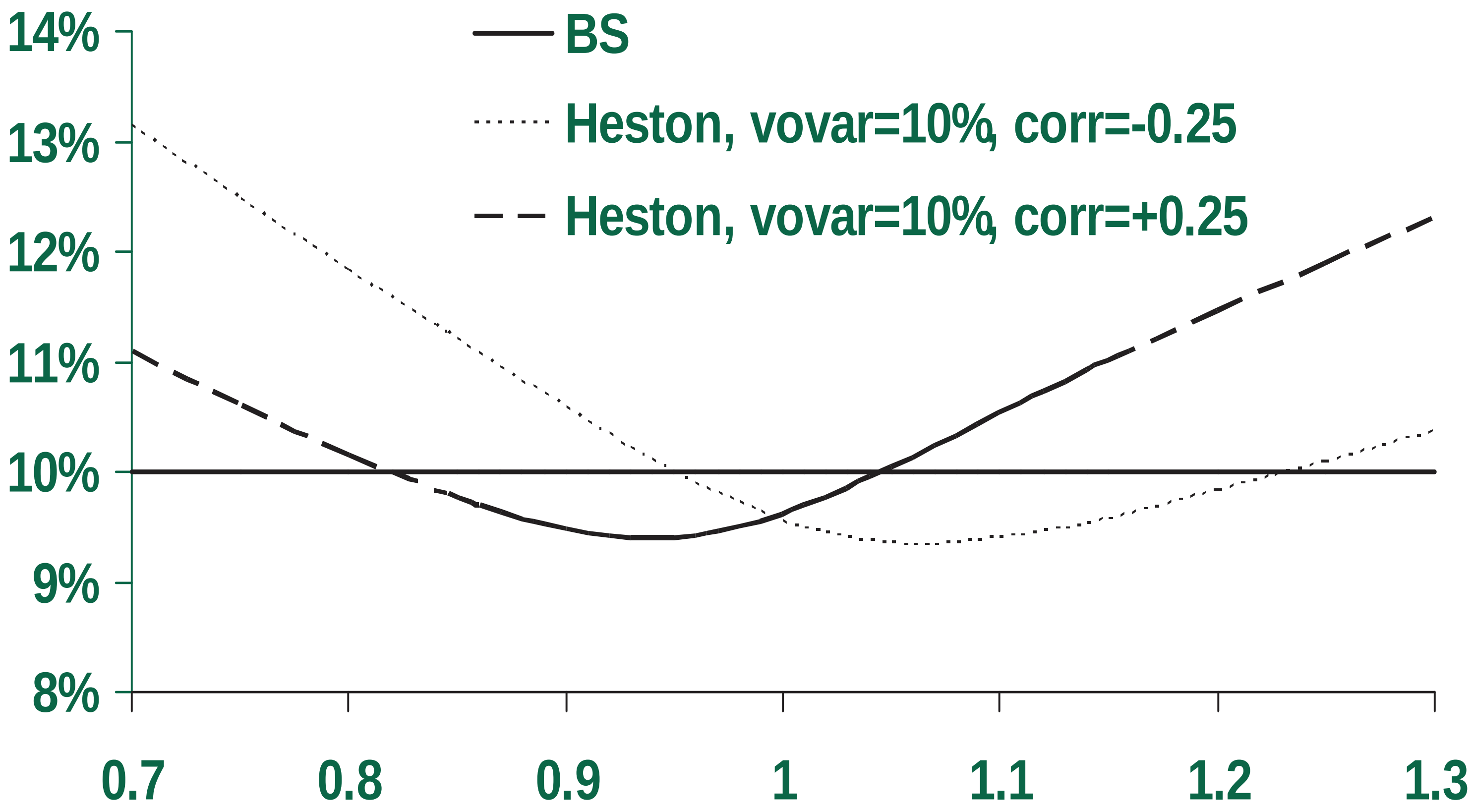
Examination of implied smiles – stochastic volatility models

- It is quite clear that smiles are generated by increasing vovariance
- $S_0=1,\ \ \sigma = 10\%,\ \ r_d=0,\ \ r_f=0,\ T=1$



Examination of implied smiles – stochastic volatility models

- Skews are generated by nonzero values for the correlation between spot and variance



Basic calibration of the model to market smile

- Heston model has no problem generating smiles and skews
- SV calibration is a fairly simple optimisation exercise using semianalytic methods discussed in this talk.
- **Terminal calibration:** take as inputs the volatilities at three strikes (25-d-P, ATM, 25-d-C), at one expiry time T. Lock down κ and m. Attempt to minimise objective function which measures the sum of squares of the errors in the vol by varying V_0, ρ, α. The objective function calculates Heston prices using the characteristic function method and backs out implied volatilities.
- **Term structure calibration:** With suitably chosen mean reversion parameters κ and m, possible to generate upward sloping or downward sloping ATM volatility surfaces. Increasing mean reversion causes smiles to flatten and diminish as the mean reversion of variance takes effect.

Pricing in Heston using Monte Carlo

- Monte Carlo is always a useful check for testing other algorithms against.
  - Draw samples $\Delta \boldsymbol{W}$ from a standardised 2D bivariate normal distribution at timepoints $\{0, \Delta t, 2\Delta t, \dots T\text{-}\Delta t\}$
  - Compute drift vector μ_t and volatility vector Σ_t at time t_i (see below)
  - Integrate the factor from its initial value $\boldsymbol{X}_0 = (\log(S_0), V_0)$ out to time T

$$\boldsymbol{X}_{t+\Delta t} = \boldsymbol{X}_t + \mu_t \Delta t + (\Sigma_t \Delta \boldsymbol{W})\sqrt{\Delta t}$$

$$\mu_t = (r_d - r_f - \tfrac{1}{2}V_t, \kappa[m - V_t])$$

$$\Sigma_t = \begin{bmatrix} V_t^{1/2} & 0 \\ \alpha\rho V_t^{1/2} & \alpha\sqrt{1-\rho^2}V_t^{1/2} \end{bmatrix}$$

  - Evaluate payoff (function of X_T) at time T. Integrate over all simulations.

Numerical solution of the Heston PDE by finite differences

- Characteristic function technique can be used for any option that has value at T as a function of S_T. Europeans, digitals, etc…
- Rules out all path dependent options (barriers and binaries) and early exercisable options.
- Heston model can be solved using 2D PDE for these products with suitable boundary conditions
  - Approximate spatial & temporal differences with mesh differences

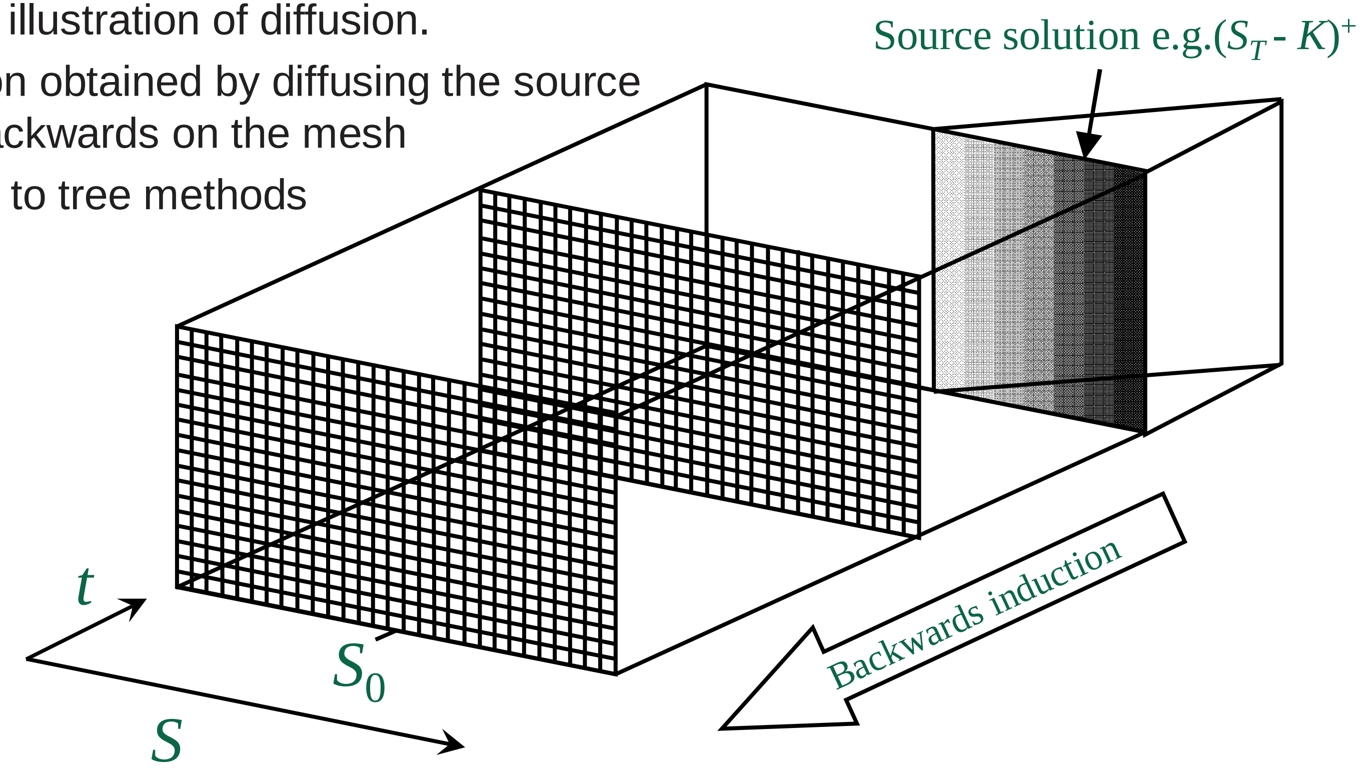
$$\frac{1}{2}VS^2\frac{\partial^2 U}{\partial S^2}+\rho\sigma VS\frac{\partial^2 U}{\partial S\partial V}+\frac{1}{2}\alpha^2 V\frac{\partial^2 U}{\partial V^2}$$

$$+(r_d-r_f)S\frac{\partial U}{\partial S}+\{\kappa[m-V]-\lambda\}\frac{\partial U}{\partial V}-r_dU+\frac{\partial U}{\partial t}=0$$

* λ denotes the market price of volatility risk

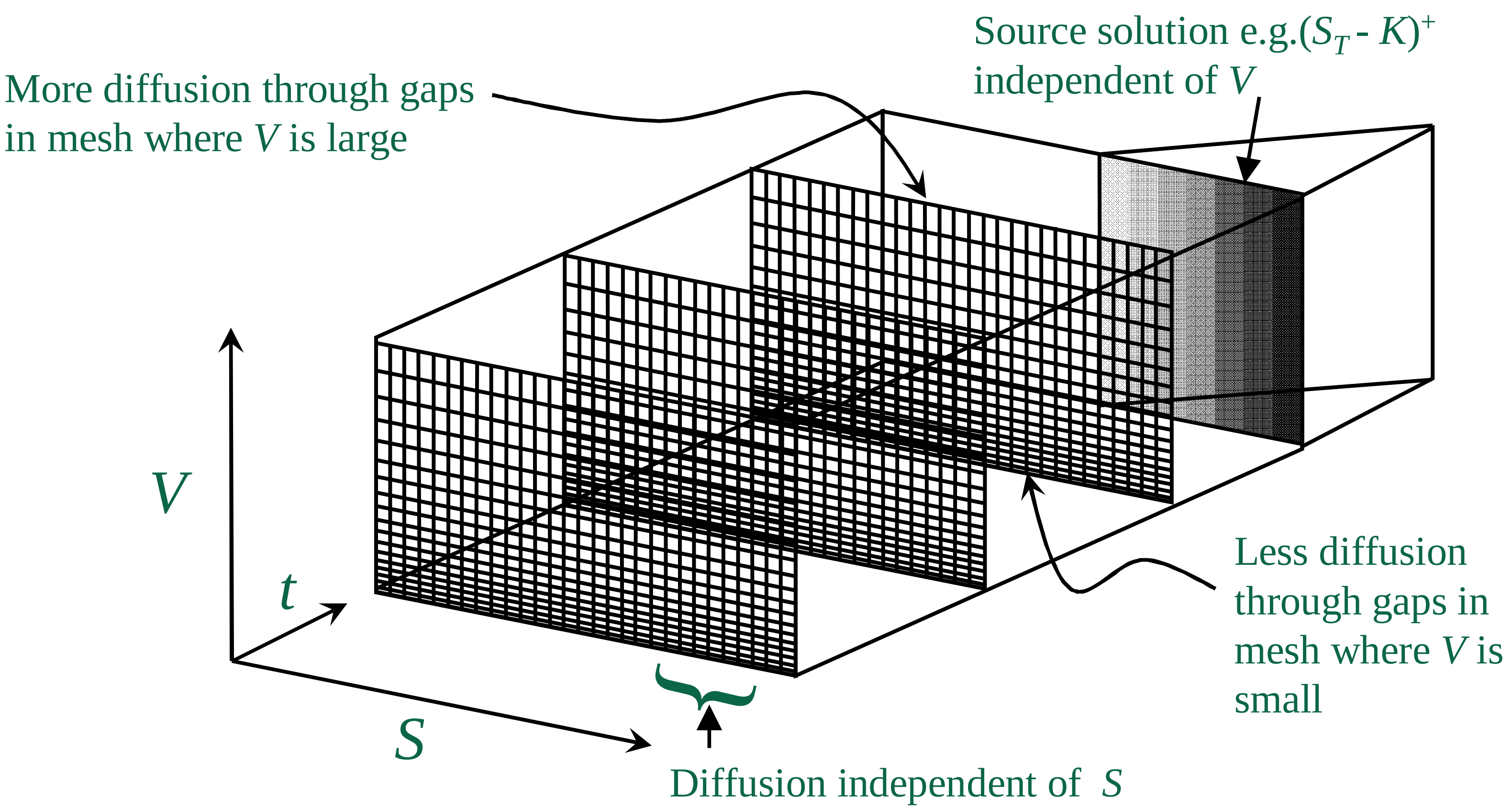
Black-Scholes volatility (constant)

- Imagine a solution diffusing through the **gaps** in the following uniform mesh
  Note: *not* a representation of a finite difference mesh.
  Schematic illustration of diffusion.
- B-S solution obtained by diffusing the source solution backwards on the mesh
- Analogous to tree methods



Stochastic volatility (or variance)

- Stochastic volatility: extend from one “spatial” to two “spatial” dimensions
- Source solution diffuses more rapidly where volatility/variance is larger



Solution of 2D Heston PDE using finite differences

- Easiest to start with a 2D explicit PDE scheme. Simple to code up.
  - However this will be far too slow for anything except development
  - 2D EFD prices should converge (slowly) to Fourier & MC prices
- The standard method for these problems is the ADI [alternating direction implicit] scheme. References given below.
  - Quite useful to set up PDE engines so that the mesh can be output to files – makes it quite easy to see when there are problems with boundary conditions, or stability.
- Also helps to compare with output from 1D PDE engines (B-S)

References: Clewlow, L. and C. Strickland (1996), Implementing Derivatives Models, Wiley, Berlin-Heidelberg.

Craig, I.J.D. & A. D. Sneyd (1988), An Alternating Direction Implicit Scheme for Parabolic Equations with Mixed Derivatives, *Comput. Math. App.* **16** (4), 341-350.

PDE implementation

- Standard PDE schemes:
  - 1D: (i) fully explicit, (ii) fully implicit, (iii) Crank-Nicolson
  - 2D: (i) fully explicit, (ii) ADI
- Consider dimensionless pde:

$$\frac{\partial U}{\partial \tau} = \sum_{i,j} A_{ij} \frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial U}{\partial x_i} + fU$$

  - transform Heston pde (slide 43) to log-spot $x=\log(S)$ and read off convection & diffusion coefficients

- In 1D: $U_\tau = AU_{xx} + bU_x + fU$
- Apply x and τ discretisation $U_i^j = U(x_i, \tau_j)$

PDE implementation

- Time derivative is given by: $\dfrac{\partial U}{\partial \tau} = \dfrac{U_i^{j+1} - U_i^j}{\Delta \tau}$
- (Central) spatial derivatives can either be taken at τ_j

$$\frac{\partial U}{\partial x} = \frac{U_{i+1}^j - U_{i-1}^j}{2\Delta x} \qquad \frac{\partial^2 U}{\partial x^2} = \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{\Delta x^2}$$

- …or at τ_{j+1}

$$\frac{\partial U}{\partial x} = \frac{U_{i+1}^{j+1} - U_{i-1}^{j+1}}{2\Delta x} \qquad \frac{\partial^2 U}{\partial x^2} = \frac{U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}}{\Delta x^2}$$

- …or in between

$$\frac{\partial U}{\partial x} = \frac{\theta[U_{i+1}^{j+1} - U_{i-1}^{j+1}] + (1-\theta)[U_{i+1}^j - U_{i-1}^j]}{2\Delta x}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\theta[U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}] + (1-\theta)[U_{i+1}^j - 2U_i^j + U_{i-1}^j]}{\Delta x^2}$$

PDE implementation

- …leading to the fully explicit scheme

$$U_i^{j+1} = U_i^j + \frac{A\Delta\tau}{\Delta x^2}\left[U_{i+1}^j - 2U_i^j + U_{i-1}^j\right] + \frac{b\Delta\tau}{2\Delta x}\left[U_{i+1}^j - U_{i-1}^j\right] + f\Delta\tau U_i^j$$

- … the fully implicit scheme

$$U_i^{j+1} - \frac{A\Delta\tau}{\Delta x^2}\left[U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}\right] - \frac{b\Delta\tau}{2\Delta x}\left[U_{i+1}^{j+1} - U_{i-1}^{j+1}\right] - f\Delta\tau U_i^{j+1} = U_i^j$$

- …or the Crank-Nicolson scheme ($\theta = 1/2$)

$$U_i^{j+1} - \frac{A\Delta\tau}{2\Delta x^2}\left[U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}\right] - \frac{b\Delta\tau}{4\Delta x}\left[U_{i+1}^{j+1} - U_{i-1}^{j+1}\right] - \frac{f}{2}\Delta\tau U_i^{j+1}$$

$$= U_i^j + \frac{A\Delta\tau}{2\Delta x^2}\left[U_{i+1}^j - 2U_i^j + U_{i-1}^j\right] + \frac{b\Delta\tau}{4\Delta x}\left[U_{i+1}^j - U_{i-1}^j\right] + \frac{f}{2}\Delta\tau U_i^j$$

Handling boundary conditions

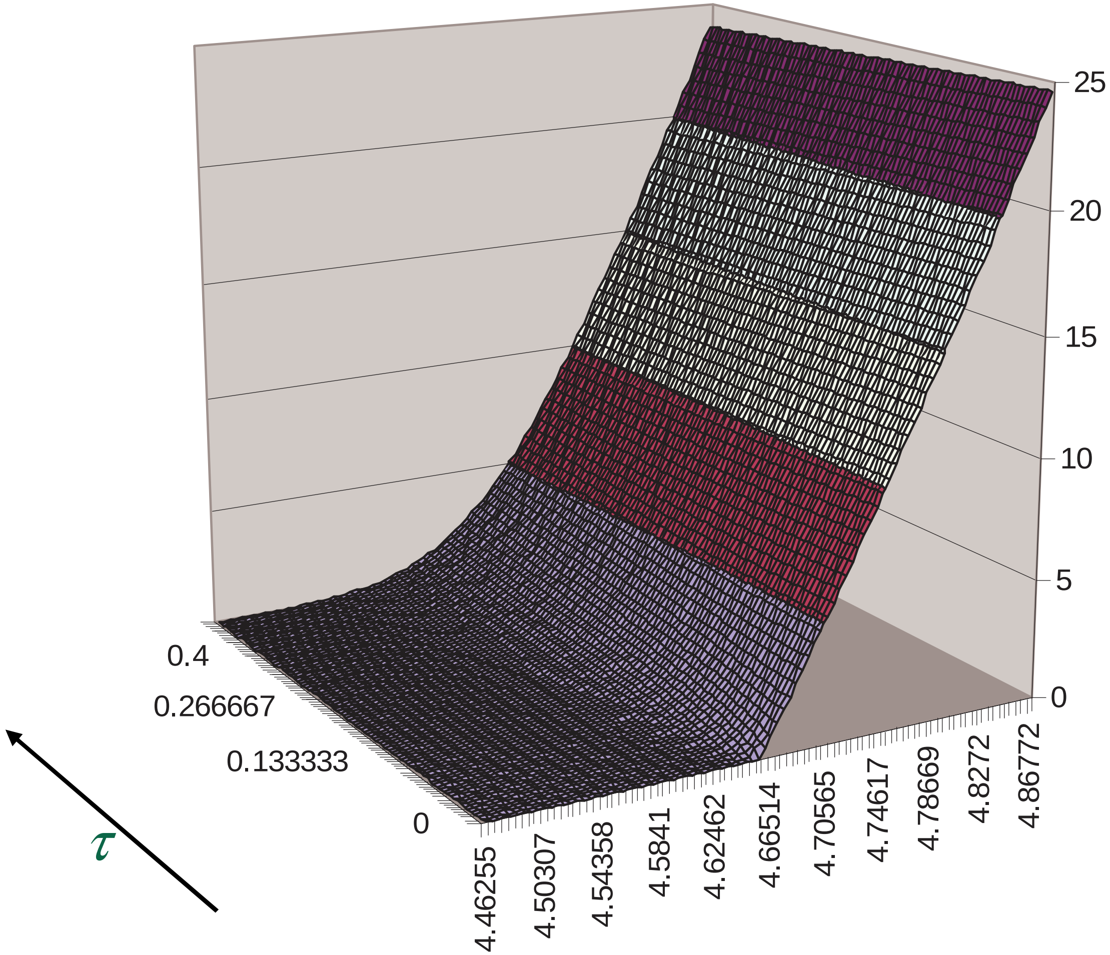
- Extinguishing options (NT, DNT, KO, DKO) are easily handled by placing a Dirichlet boundary condition at the barrier level.
- Without KO barriers (e.g. Europeans without barriers), common technique is to assume 2nd derivative vanishes on the boundaries. Hence solution is linear.
- Suffices therefore to use one-sided differences (neglect diffusion terms), to time-step the solution on the boundary
  - This is for explicit finite differences; use $j+1$ for IFD

$$\left.\frac{\partial U}{\partial x}\right|_{x=x_0,\tau=\tau_j} = \frac{U_1^j - U_0^j}{2\Delta x} \qquad \left.\frac{\partial U}{\partial x}\right|_{x=x_N,\tau=\tau_j} = \frac{U_N^j - U_{N-1}^j}{2\Delta x}$$

Reference: Tavella, D. and C. Randall. (2000) Pricing Financial Instruments: The Finite Difference Method, Wiley, New York.

Example: 1D PDE scheme

- Black-Scholes (Crank-Nicolson)



PDE implementation

- Algebra of Crank-Nicolson can be simplified introducing a node at half-time

$$U_i^{j+1} - \frac{A\Delta\tau}{2\Delta x^2}\left[U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}\right] - \frac{b\Delta\tau}{4\Delta x}\left[U_{i+1}^{j+1} - U_{i-1}^{j+1}\right] - \frac{f}{2}\Delta\tau U_i^{j+1} = U_i^{j+1/2}$$

$$U_i^{j+1/2} = U_i^j + \frac{A\Delta\tau}{2\Delta x^2}\left[U_{i+1}^j - 2U_i^j + U_{i-1}^j\right] + \frac{b\Delta\tau}{4\Delta x}\left[U_{i+1}^j - U_{i-1}^j\right] + \frac{f}{2}\Delta\tau U_i^j$$

- Can be seen as equivalent to an explicit step over time interval $(\tau_j, \tau_{j+1/2})$ followed by an implicit step over time interval $(\tau_{j+1/2}, \tau_j)$.
- The ADI [alternating direction implicit] scheme, which we use for problems with <u>**two**</u> spatial variables, works similarly by applying an explicit step in one spatial direction, followed by an implicit step in the ***other*** spatial direction.
- Since each diffusion & convection term is only applied over half of the time stepping, we have to double the effective contribution of these terms when they are in fact applied. Correlation handled in the explicit steps.
- Boundary conditions handled similarly to1D PDEs (no variance barriers).

PDE implementation

- In 2D (correlation neglected) with discretisation $U_{i,j}^{k} = U(x_i, y_j, \tau_k)$

$$U_\tau = A_{11}U_{xx} + A_{22}U_{yy} + b_1 U_x + b_2 U_y + fU$$

- Explicit in X, then implicit in Y

$$U_{i,j}^{k+1} - \frac{A_{22}\Delta\tau}{\Delta y^2}\left[U_{i,j+1}^{k+1} - 2U_{i,j}^{k+1} + U_{i,j-1}^{k+1}\right] - \frac{b_2\Delta\tau}{2\Delta y}\left[U_{i,j+1}^{k+1} - U_{i,j-1}^{k+1}\right] - \frac{f}{2}\Delta\tau U_{i,j}^{k+1} = U_{i,j}^{k+1/2}$$

$$U_{i,j}^{k+1/2} = U_{i,j}^{k} + \frac{A_{11}\Delta\tau}{\Delta x^2}\left[U_{i+1,j}^{k} - 2U_{i,j}^{k} + U_{i-1,j}^{k}\right] + \frac{b_1\Delta\tau}{2\Delta x}\left[U_{i+1,j}^{k} - U_{i-1,j}^{k}\right] + \frac{f}{2}\Delta\tau U_{i,j}^{k}$$

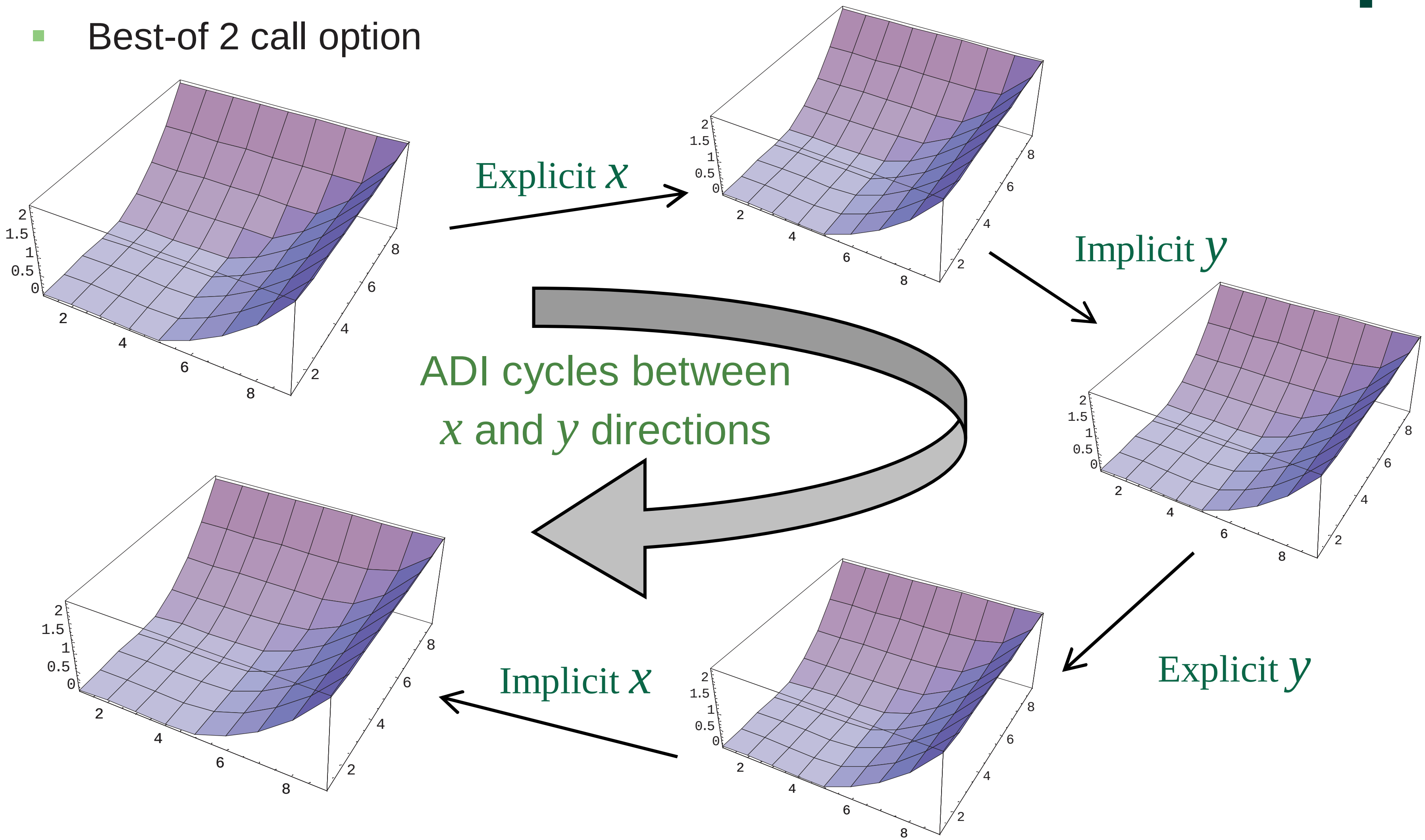
- …then explicit in Y, and implicit in X

$$U_{i,j}^{k+2} - \frac{A_{11}\Delta\tau}{\Delta x^2}\left[U_{i+1,j}^{k+2} - 2U_{i,j}^{k+2} + U_{i-1,j}^{k+2}\right] - \frac{b_1\Delta\tau}{2\Delta x}\left[U_{i+1,j}^{k+2} - U_{i-1,j}^{k+2}\right] - \frac{f}{2}\Delta\tau U_{i,j}^{k+2} = U_{i,j}^{k+3/2}$$

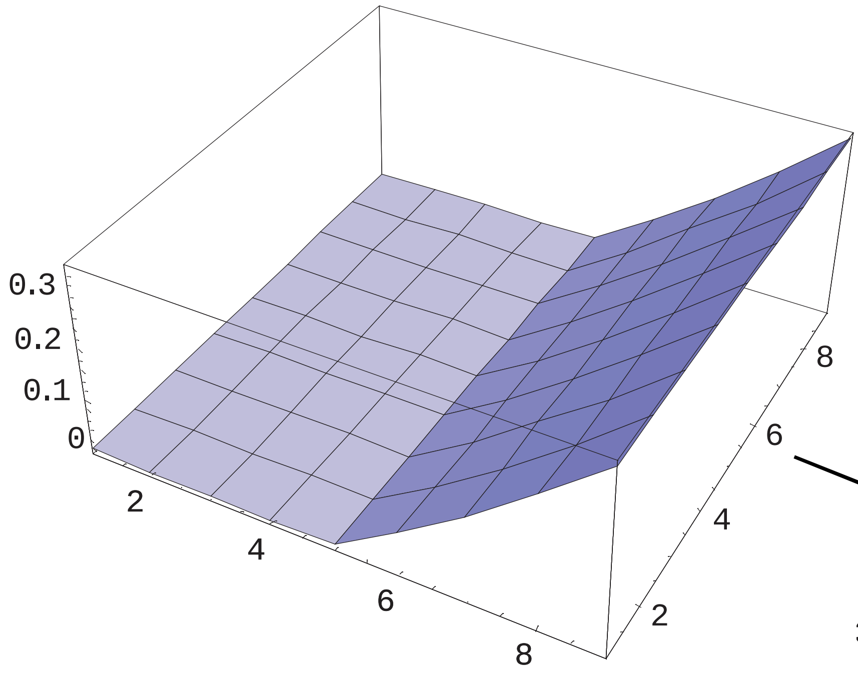
$$U_{i,j}^{k+3/2} = U_{i,j}^{k+1} + \frac{A_{22}\Delta\tau}{\Delta y^2}\left[U_{i,j+1}^{k+1} - 2U_{i,j}^{k+1} + U_{i,j-1}^{k+1}\right] + \frac{b_2\Delta\tau}{2\Delta y}\left[U_{i,j+1}^{k+1} - U_{i,j-1}^{k+1}\right] + \frac{f}{2}\Delta\tau U_{i,j}^{k+1}$$

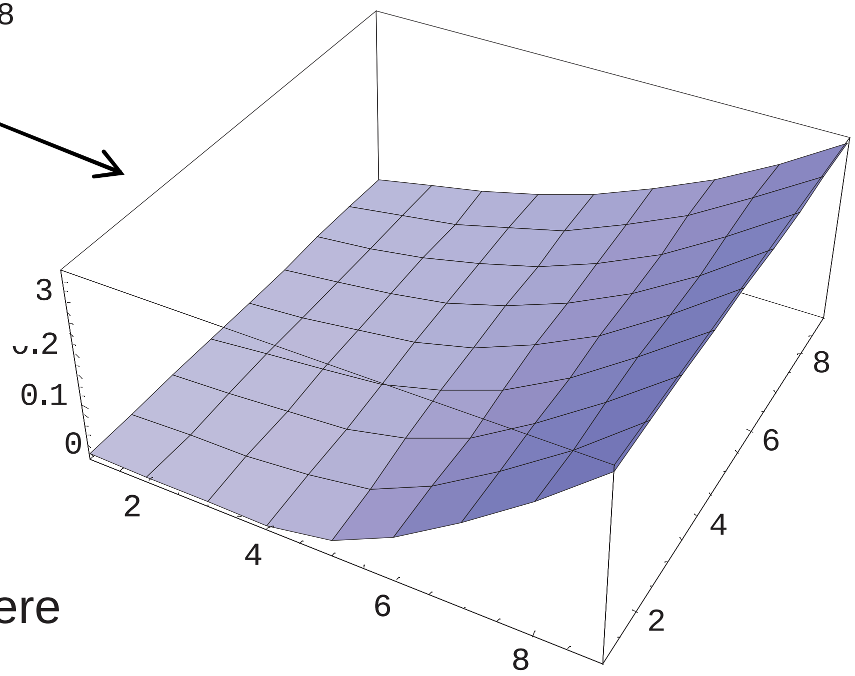
2D algorithm: best-of-call & BlackScholes

- Best-of 2 call option



2D algorithm: European call under Heston dynamics

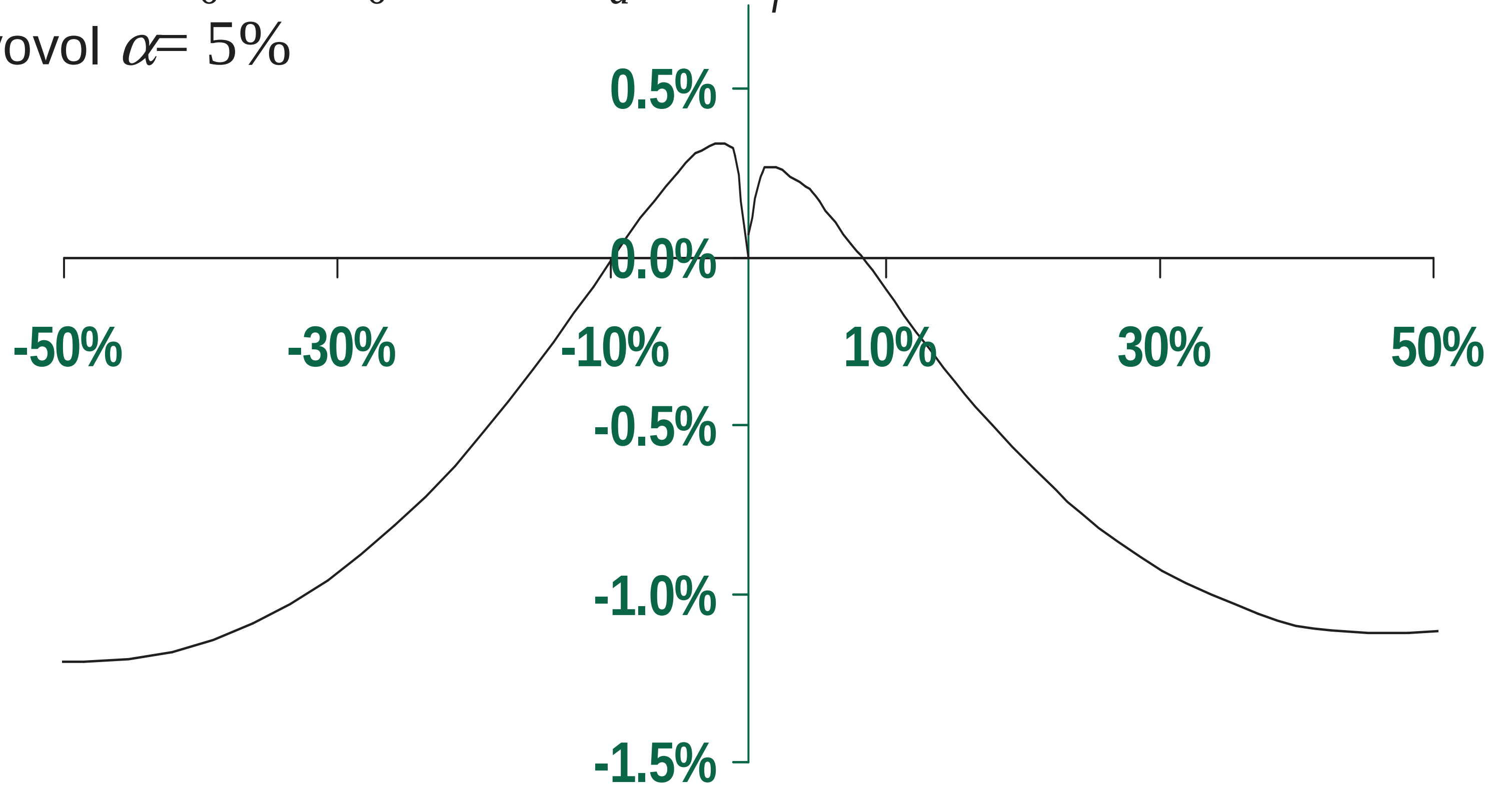




- x direction is log-spot
- y direction is variance
- Solution has diffused out most where variance is large

Examination of binary moustache – Heston model

- The binary moustache generated by Heston model broadly exhibits correct qualitative features (priced using 2D ADI)
- Model: S_0=1, V_0=0.01, r_d=0, r_f=0, strip of binaries with T=1 vovol α= 5%



Summary

- Heston stochastic volatility model:
  - capable of generating realistic smiles and skews for vanillas
  - generates sensible deviations from B-S prices for binaries
  - able to admit very fast calibration scheme via semianalytic pricing
- Monte Carlo is easy to implement and provides useful “reality check” for other pricing algorithms
- When 2D finite difference engine such as ADI implemented, fast pricing of flow exotics in FX is quite straightforward
  - pricing requres solution of 2D convection-diffusion problem where diffusion is anisotropic in variance direction
  - compare with 2-factor Black-Scholes: isotropic in both log-spots