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A new approach to liquidity risk

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- What is *liquidity risk* ?
  - **Treasurer**'s answer: the risk of running short of cash
  - **Trader**'s answer: the risk of trading in *illiquid* markets, i.e. markets where exchanging assets for cash may be difficult or uncertain
  - **Central Bank**'s answer: the risk of concentration of cash among few economic agents

$\Rrightarrow$ Setting a precise mathematical framework is not easy

- The theoretical framework
  - Portfolios and Marginal Supply-Demand Curves
  - Liquidation value vs. usual mark-to-market value
  - Liquidity policies and general mark-to-market values
- Coherent/convex risk measures and liquidity risk
- Some numerical examples

It is possible to trade in

- $N \geqslant 1$ *illiquid* assets
- **cash**, which is by definition the only liquidity risk-free asset

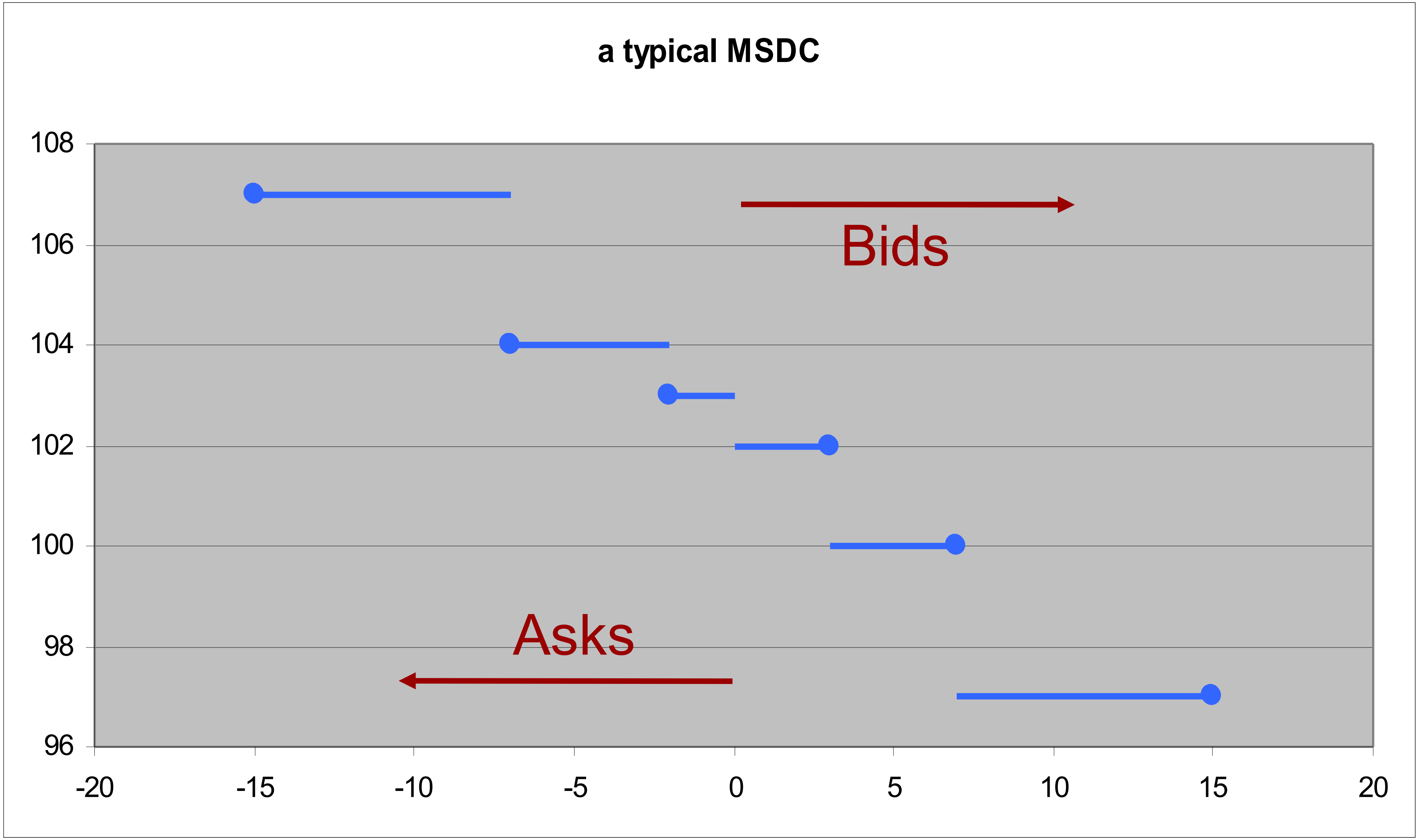
We define

- A **portfolio** is a vector $\mathbf{p} \in \mathbb{R}^{N+1}$
- p_0 is the amount of cash
- $\overrightarrow{p} = (p_1, \dots, p_N)$ is the assets' position
- p_n is the number of assets of type n

- Perfectly liquid market ($S_0(t) \equiv 1$)
  - $S_n(t)$ is the unique price, at time t, for selling/buying a unit of asset n; this price does **not** depend on the size of the trade
  - $V(\mathbf{p}, t) = \mathbf{p}_0 + \sum_{n=1}^{N} p_n S_n(t)$ is linear
- Illiquid markets ($S_0(t) \equiv 1$)
  - $S_n(t) = S_n(t, x)$ will depend on the size $x \in \mathbb{R}$ ($x > 0$ is a sale) of the trade
  - $V(\mathbf{p}, t)$ need not be linear anymore. A first idea is:

$$V(\mathbf{p}, t) = \mathbf{p}_0 + \sum_{n=1}^{N} p_n S_n(t, p_n)$$

  - But this is not the only sensible notion of value


a typical MSDC
Bids
Asks
108
106
104
102
100
98
96
-20
-15
-10
-5
0
5
10
15
20

Some basic definitions

- A Marginal Supply-Demand Curve (**msdc**) is a decreasing function

$$m : \mathbb{R} \to (0, +\infty)$$

- $m^+ = m(0+)$ and $m^- = m(0-)$ are the best bid (sell) and ask (buy) prices. Of course $m^+ \leqslant m^-$.

- Let x be the size of the transaction ($x > 0$ sale, $x < 0$ purchase). The **unit price** is

$$S(x) = \frac{1}{x} \int_0^x m(y)\, dy > 0$$

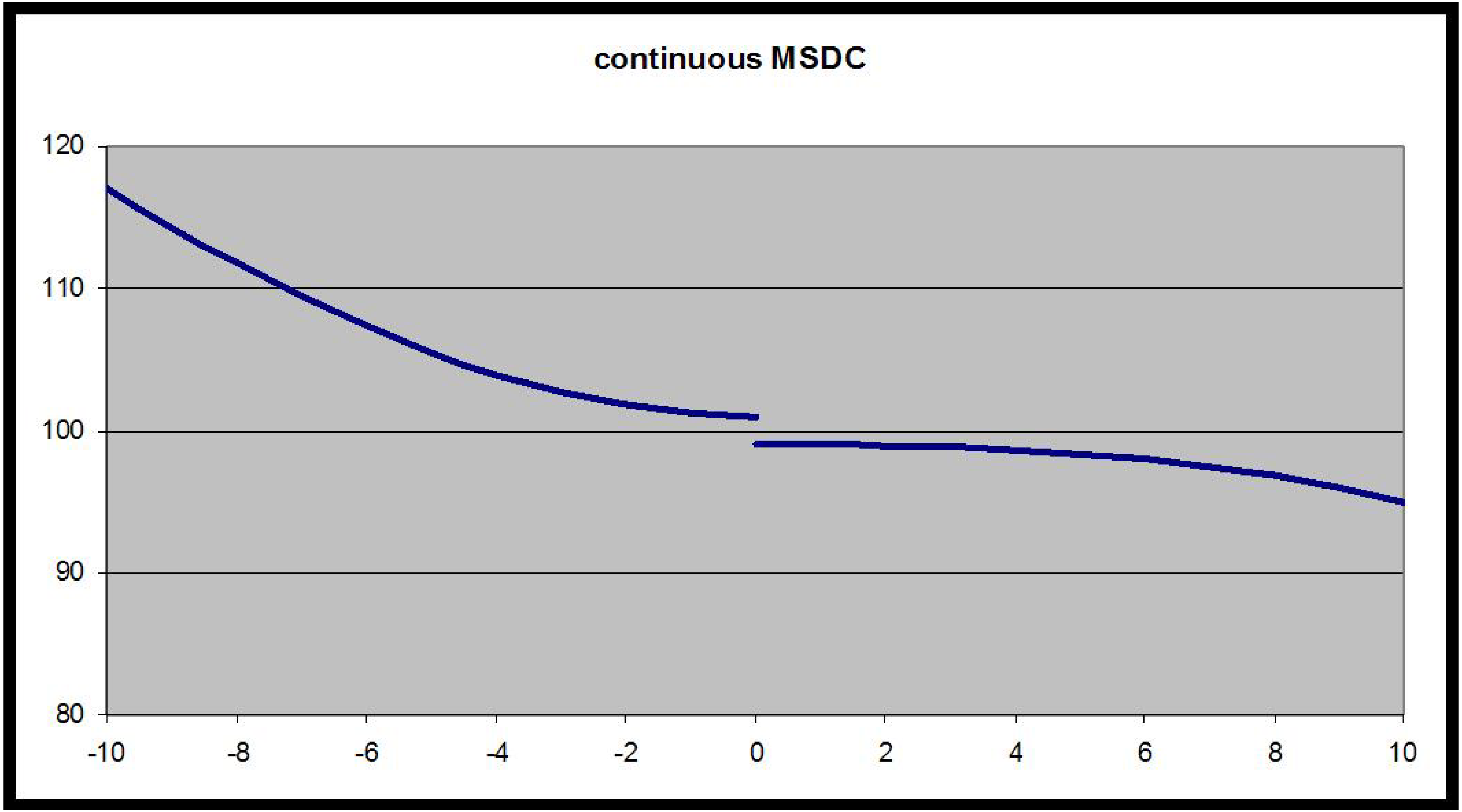
and the **proceeds** are

$$P(x) = xS(x) = \int_0^x m(y)\, dy \gtrless 0$$

"Same" setting as in Cetin-Jarrow-Protter 2005, but our focus is on m.

We can also allow for (care is needed with the details):

- Assets which are not securities (e.g. swaps) and can display negative (marginal) prices: $m : \mathbb{R} \to \mathbb{R}$
- Securities with finite depth market: $m(x) = +\infty$ for $x << 0$ and/or $m(x) = 0$ for $x >> 0$
- Swaps with finite depth market: $m(x) = +\infty$ for $x << 0$ and/or $m(x) = -\infty$ for $x >> 0$


continuous MSDC
120
110
100
90
80
-10
-8
-6
-4
-2
0
2
4
6
8
10

Given m $= (m_1, \ldots, m_N)$ a vector of msdc. Let $\mathbf{p} \in \mathbb{R}^{N+1}$ be a portfolio.

- The **Liquidation Value** of $\mathbf{p}$ is

$$L(\mathbf{p}) = p_0 + \sum_{n=1}^{N} p_n S_n(p_n) = p_0 + \sum_{n=1}^{N} \int_0^{p_n} m_n(x)\, dx$$

- The **Usual Mark-to-Market Value** of $\mathbf{p}$ is

$$U(\mathbf{p}) = p_0 + \sum_{p_n > 0} p_n m_n^+ + \sum_{p_n < 0} p_n m_n^-$$

  as if only the best bid and ask would matter.

Note that $U(\mathbf{p}) \geqslant L(\mathbf{p})$ for any $\mathbf{p}$.

Some properties of L and U:

concavity. both L and U are *concave* (but not linear)

additivity. L is *subadditive* $(L(\mathbf{p}+\mathbf{q}) \leqslant L(\mathbf{p}) + L(\mathbf{q}))$ whenever $\mathbf{p}$ and $\mathbf{q}$ are concordant ($p_n q_n \geqslant 0$ for $n \geqslant 1$); it is *superadditive* for discordant portfolios
U is always *superadditive*, and it is additive for concordant portfolios

scaling if $\lambda \geqslant 1$

$$L(\lambda\mathbf{p}) \leqslant \lambda L(\mathbf{p}) \qquad U(\lambda\mathbf{p}) = \lambda U(\mathbf{p})$$

- Liquidation Mark-to-Market Value (L):
  measure of the portfolio value as if we are forced to entirely liquidate it (so, liquidity risk is a big concern)

- Usual MtM Value (U):
  measure of the portfolio value as if we don't have to liquidate even a small part of it (so, liquidity risk is not a concern)

Our aim is to introduce notions of value between the two extreme cases. Whether and what to liquidate is a need that may vary.

First we give a notion of acceptability for a portfolio:

A **liquidity policy** is a convex and closed subset $\mathcal{L} \subseteq \mathbb{R}^{N+1}$ such that

1. $\mathbf{p} \in \mathcal{L}$ implies $\mathbf{p} + a \in \mathcal{L}$ for any $a \geqslant 0$ (adding cash cannot worsen the liquidity properties of a portfolio)

2. $(p_0, \overrightarrow{p}) \in \mathcal{L}$ implies $(p_0, \overrightarrow{0}) \in \mathcal{L}$ (if a portfolio is acceptable, its cash component is acceptable as well)

$\mathcal{L}$ collects the portfolios whose liquidity risk is not a concern and thus may be valued through U

Examples of liquidity policies:

1. $\mathcal{L} = \mathbb{R}^{N+1}$ Every portfolio is acceptable: no need to liquidate (this will lead to U)

2. $\mathcal{L} = \{\mathbf{p} : \overrightarrow{p} = \overrightarrow{0}\}$ Only pure-cash portfolios are acceptable: need to entirely liquidate $\mathbf{p}$ (this will lead to L)

3. $\mathcal{L} = \{\mathbf{p} : p_0 \geqslant a\}$ ($a \geqslant 0$ fixed) This is a typical requirement imposed by the ALM of an institution

4. other examples may be based on bounds on concentration...

1. Start with a portfolio $\mathbf{p}$, which need not be acceptable

2. Make it acceptable by liquidating the assets' (sub)position $\overrightarrow{q} \in \mathbb{R}^N$

$$\mathbf{r} = \mathbf{p} - \overrightarrow{q} + L(0, \overrightarrow{q}) = (p_0 + L(0, \overrightarrow{q}), \overrightarrow{p} - \overrightarrow{q}) \in \mathcal{L}$$

3. Find the best way to do this, maximizing the Usual MtM value $U(\mathbf{r})$

4. Note that L is used in 2. and U in 3.:
   in 2. we care about liquidity risk, in 3. we don't as $\mathbf{r} \in \mathcal{L}$

— Having fixed a liquidity policy $\mathcal{L}$ we can define the **associated MtM Value** ($\sup\emptyset = -\infty$)

$$V_{\mathcal{L}}(\mathbf{p}) = \sup\{U(\mathbf{r}) \,:\, \mathbf{r} = \mathbf{p} - \overrightarrow{q} + L(0, \overrightarrow{q}) \in \mathcal{L},\ \overrightarrow{q} \in \mathbb{R}^N\}$$

— $\mathbf{r}^* \in \mathbb{R}^{N+1}$ is optimal if $V_{\mathcal{L}}(\mathbf{p}) = U(\mathbf{r}^*)$.
It is immediate to see that $V_{\mathcal{L}}(\mathbf{r}^*) = U(\mathbf{r}^*) = V_{\mathcal{L}}(\mathbf{p})$ (there is no change in value passing from $\mathbf{p}$ to $\mathbf{r}^*$)

— The set over which U (concave) is maximized is convex. Thus

The optimization program defining $V_{\mathcal{L}}$ is always **convex**.

1. If $\mathcal{L} = \mathbb{R}^{N+1}$, then
$$V_{\mathcal{L}}(\mathbf{p}) = U(\mathbf{p})$$

2. If $\mathcal{L} = \{\mathbf{p} : \overrightarrow{p} = \overrightarrow{0}\}$, then
$$V_{\mathcal{L}}(\mathbf{p}) = L(\mathbf{p})$$

3. If $\mathcal{L} = \{\mathbf{p} : p_0 \geqslant a\}$, then
$$V_{\mathcal{L}}(\mathbf{p}) = \sup\{U(\mathbf{p} - \overrightarrow{q}) + L(0, \overrightarrow{q}) : L(0, \overrightarrow{q}) \geqslant a - p_0, \ \overrightarrow{q} \in \mathbb{R}^{N+1}\}$$
which is not trivial (and non-linear)

- If $\mathcal{L} \subset \mathcal{L}'$, then $V_{\mathcal{L}} \leqslant V_{\mathcal{L}'}$.
- Thus,

$$V_{\mathcal{L}}(\mathbf{p}) \leqslant U(\mathbf{p}) \qquad \forall \mathcal{L}$$

- For any $\mathcal{L}$, $V_{\mathcal{L}}$ is *concave* and *translational supervariant*

$$V_{\mathcal{L}}(\mathbf{p} + a) \geqslant V_{\mathcal{L}}(\mathbf{p}) + a \qquad \forall a \geqslant 0$$

- As the problem defining $V_{\mathcal{L}}$ is convex, many *fast algorithms* are available

- An *analytical solution* is sometime easy. Assume:
  - $\{\mathbf{p} : p_0 \geqslant a\}$
  - m_i continuous and strictly decreasing $\forall i$

  Then
  - if $p_0 \geqslant a$ ($\mathbf{p} \in \mathcal{L}$) then $\mathbf{r}^* = \mathbf{p}$ and $V_{\mathcal{L}}(\mathbf{p}) = U(\mathbf{p})$
  - if $p_0 < 0$ then

$$r_i^* = m_i^{-1}\left(\frac{m_i(0)}{1+\lambda}\right)$$

    where λ is determined by $L(\mathbf{r}^*) = p_0 - a$.

Coherent risk measures (CRM) $\rho : L \to \mathbb{R}$ (L space of r.v.) are characterized by (Artzner-Delbaen-Eber-Heath-98)

1. *Translation invariance*: $\rho(X + c) = \rho(X) - c \quad \forall c \in \mathbb{R}$;
2. *Monotonicity*: $\rho(X) \leqslant \rho(Y)$ whenever $X \geqslant Y$
3. *Positive homogeneity*: $\rho(\lambda X) = \lambda\rho(X) \quad \forall\lambda \geqslant 0$
4. *Subadditivity*: $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$.

Axioms 3 and 4 do not seem to take into account liquidity risk:
if I double my portfolio, its risk should more than double in many cases.

They were replaced (Follmer-Schied02, Frittelli-Rosazza02) by the weaker axiom of convexity.

In our opinion, CRM are appropriate to deal with liquidity risk. The key point is that:

If I double my portfolio... means $\mathbf{p} \longrightarrow 2\mathbf{p},$ **not** $X \longrightarrow 2X$

The relation between $\mathbf{p}$ and its value X is **not linear.**

We define risk measures defined directly on portfolios $R = R(\mathbf{p})$ that are not necessarily positively homogeneous or subadditive.

Given

- a liquidity policy $\mathcal{L}$
- a probability space $(\Omega, \mathcal{F}, P)$ describing randomness up to $T > 0$
- a coherent risk measure defined on some $L \subset L^0(\Omega, \mathcal{F}, P)$
- the random future msdc: $(m_i(x, T))$ for any i, where
  - for any x, $m_i(x, T)$ is a r.v.
  - for any ω, $x \mapsto m_i(x, T)(\omega)$ is decreasing

We compute $V_{\mathcal{L}}(\mathbf{p}) = V_{\mathcal{L}}(\mathbf{p}, T)(\omega)$ for any ω (it is a r.v.) and set

$$R_{\mathcal{L}}(\mathbf{p}) = \rho(V_{\mathcal{L}}(\mathbf{p}))$$

Some properties (for general $\mathcal{L}$ and ρ)

1. $R_{\mathcal{L}}$ is *convex*

2. $R_{\mathcal{L}}$ is *translational subvariant*: $R_{\mathcal{L}}(\mathbf{p} + c) \leqslant R_{\mathcal{L}}(\mathbf{p}) - c$

3. $R_{\mathcal{L}}$ is in general *not homogeneous*, *nor subadditive*

4. specific properties for $R_{\mathcal{L}}$ may be derived from properties of $V_{\mathcal{L}}$ (and coherency of ρ)

5. no monotonicity property can be introduced for R

Consider (T is fixed)

$$m_i(x) = \alpha_i \exp\{-\beta_i x\},$$

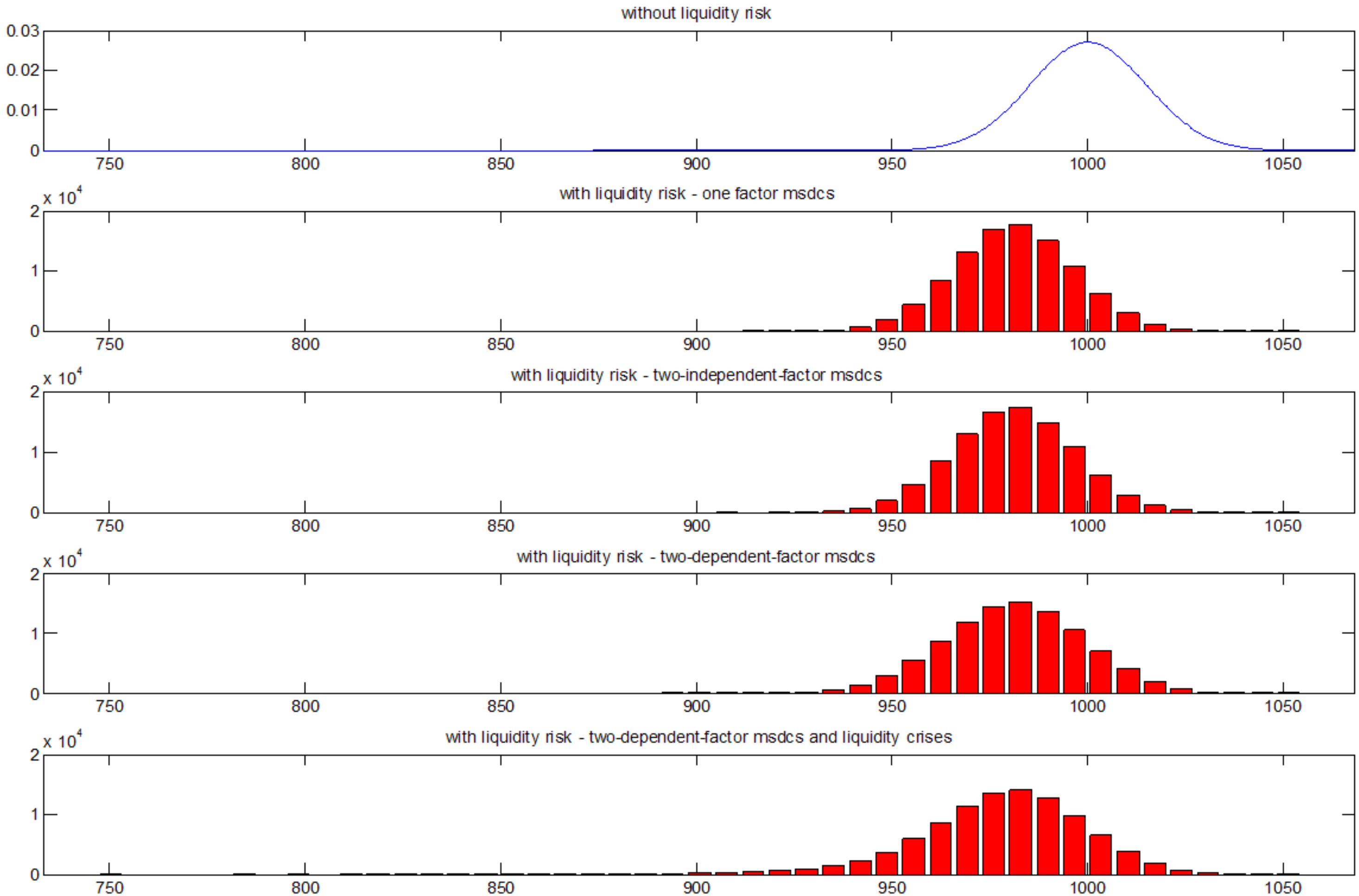
where, $A_i > 0$ and $\beta_i \geqslant 0$ are r.v. There can be

- *Market risk only*:
  α_i jointly lognormal, $\beta_i = 0$
- *Market and "non-random" liquidity risk*:
  α_i jointly lognormal, $\beta_i > 0$ non-random
- *Market and independent random liquidity risk*:
  $(\alpha_i, \beta_i)_i$ jointly lognormal, with $\alpha_i \perp \beta_i$
- *Market and correlated random liquidity risk*:
  $(\alpha_i, \beta_i)_i$ jointly lognormal, with α_i and β_i negatively correlated
- *Market and correlated random liquidity risk with shocks*:
  $(\alpha_i, \widetilde{\beta}_i)_i$ jointly lognormal, with α_i and $\widetilde{\beta}_i$ negatively correlated, $\beta_i = \widetilde{\beta}_i + \varepsilon_i$

For a given portfolio $\mathbf{p}$ and $\mathcal{L} = \{\mathbf{q} \,:\, q_0 \geqslant a\}$, in any of the 5 previous situations we:

- set $I = 10$, α_i and β_i id. distr. for different i
- we perform 100k simulations of $(m_i(x))_i$
- for any outcome of the simulation we compute $V_{\mathcal{L}}(\mathbf{p})$
- we repeat for different inputs ($\mathbf{p}$, a, mean, variances and correlations of α_i and β_i)

A typical outcome is:


without liquidity risk
0.03
0.02
0.01
0
750
800
850
900
950
1000
1050
with liquidity risk - one factor msdcs
x 10^4
2
1
0
with liquidity risk - two-independent-factor msdcs
with liquidity risk - two-dependent-factor msdcs
with liquidity risk - two-dependent-factor msdcs and liquidity crises

Messages:

- Liquidity risk arises when msdc are ignored
- Liquidity risk can be captured by a redefinition of the concept of value, which depends on a liquidity policy
- Coherent risk measures are perfectly adequate to deal with liquidity risk

To do:

- study possible realistic (yet analytically tractable) stochastic models for a msdc
- portfolio optimization with liquidity risk