Stochastic Calculus

Rama Cont

Note Title

1/15/2013

TUESDAYS 15 Jan - 19 MARCH 2013, 5-8 PM.

No course on Feb 12, 2013.

Course objectives:

This course is an introduction to the theory of stochastic integration and the Ito calculus, a calculus applicable to functions of stochastic processes with irregular paths, which has many applications in finance, engineering and physics. The course shall focus on the mathematical foundations of stochastic calculus, motivated in particular by applications to stochastic models in finance. We will develop the theory in the setting of semimartingales, which covers all examples of interest in applications - including jump processes.

Prerequisites:

Students are expected to have graduate-level knowledge of probability theory, stochastic processes, real analysis and measure theory.

Outline:

Stochastic processes. Right continuous processes with left limits.
Filtrations and sigma-fields.
Non-anticipative processes. Optional and predictable processes.
Riemann-Stieltjes integration with respect to finite variation processes.
Obstructions to the extension of the Riemann Stieltjes integral to infinite variation paths.
Martingales and local martingales. Properties of martingales. Doob's inequality.
The Ito stochastic integral: definition and fundamental properties.
Semimartingales: definition, examples.
Quadratic variation. Ito isometry formula. Lévy's theorem. BDG inequality.
Brownian integrals. Ito processes.
Poisson random measures. Poisson stochastic integrals. Ito-Lévy processes.
The Ito formula: pathwise and probabilistic versions.
Stochastic exponentials. Stochastic exponential of a martingale.
Change of probability measure. Girsanov's Theorem.
Stochastic differential equations. Strong solutions and weak solutions.
Representation of martingales as stochastic integrals.

The Tanaka formula. Semimartingale local time. (*)
Functional Ito calculus (*).

Some References:

Philip Protter: Stochastic Integration and Stochastic Differential Equations, 2nd Edition, Springer.

Semimartingales, Ito integral, Ito formula, local time
Girsanov theorem, strong solutions of SDEs

Ikeda & Watanabe : Stochastic differential equations, 2nd Ed.

Poisson integrals, Lévy-Ito processes, L^2 construction
of Ito integral, stochastic differential equations
Weak solutions, Martingale representation theorem

J Jacod & A Shiryaev : Limit Theorems for Stochastic Processes, 2nd Ed, Springer.

Semimartingales, integration with respect to
integer-valued random measures

Stochastic processes

Intuitively, a stochastic process is a family $(X_t)_{t\geq 0}$ of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and indexed by time: $X_t : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

However this definition does not say anything about

✓ the regularity of the sample paths $t \mapsto X_t(\omega)$

✓ the relation between "randomness" and time: when is the value of X_t "observed"/revealed?

Cadlag functions

Right-continuous with left limits (RCLL)

Def: $f : [0,\infty) \to \mathbb{R}^d$ is cadlag if for each $t \geq 0$

$$\begin{cases} \cdot\ f \text{ is right continuous at } t: \quad f(t) = \lim\limits_{s \downarrow t,\, s > t} f(s) = f(t+) \\ \cdot\ f(t-) = \lim\limits_{s \uparrow t} f(s) \text{ exists} \end{cases}$$

Notations:

$D([0,\infty), \mathbb{R}^d) :=$ set of $\mathbb{R}^d$-valued cadlag functions

- For $f \in D([0,\infty), \mathbb{R}^d)$: $\Delta f(t) = f(t) - f(t-)$ Discontinuity at t

Property: Define $g(t) = f(t-)$ for $f \in D([0,\infty), \mathbb{R}^d)$

Then g is left-continuous.

Properties: ✓ for each $\varepsilon > 0$, a cadlag function has at most a finite number of discontinuities of magnitude $> \varepsilon$ in any bounded time interval:

$\forall \varepsilon > 0$, $\forall T > 0$, $\{ t \in [0,T], |\Delta f(t)| > \varepsilon \}$ is finite.

Consequences:

- a cadlag function has at most a countable number of discontinuities: $\{ t \geq 0, \Delta f(t) \neq 0 \}$ is countable.

$$\forall f, g \in D, \quad [f(t) = g(t) \ \forall t \in \underbrace{\mathbb{Q}}_{\text{rational numbers}}] \Rightarrow f = g$$

Filtrations

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered space

i.e. $\mathcal{F}$ is a σ-algebra of subsets of Ω.

- $(\mathcal{F}_t)_{t\geq 0}$ is a filtration on $(\Omega, \mathcal{F})$:
  - ✓ $\mathcal{F}_t \subset \mathcal{F}$ is a σ-algebra for each $t \geq 0$
  - ✓ $t_2 \geq t_1 \Rightarrow \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$

We will denote $\mathcal{F}_{t-} = \bigvee_{0\leq s<t} \mathcal{F}_s$, $\mathcal{F}_{t+} = \bigwedge_{t\leq s<\infty} \mathcal{F}_s$

$(\mathcal{F}_t)_{t\geq 0}$ is said to be right-continuous if

$$\mathcal{F}_{t+} = \mathcal{F}_t \quad \forall t \geq 0$$

Stochastic processes

Def : a cadlag stochastic process

on a filtered space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}\right)$

is a map $X : \Omega \times [0, \infty) \longrightarrow \mathbb{R}^d$

Such that

✓ $X_t = X(\cdot, t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$

(X is adapted to $(\mathcal{F}_t)_{t \geq 0}$)

✓ $X(\omega, \cdot) : [0, \infty) \to \mathbb{R}^d$ is cadlag
$t \to X(\omega, t)$

Joint measurability in (t,ω)

One can view a stochastic process X either as

✓ a family of random variables with some measurability requirement on X_t for each $t \geq 0$

or as

✓ a map $X : \Omega \times [0,T] \to \mathbb{R}^d$ with some measurability requirement on $\Omega \times [0,\infty)$

The simplest measurability requirement on $\Omega \times [0,\infty)$ is with respect to the product σ-algebra generated by $\mathcal{F} \times \underbrace{\mathcal{B}([0,\infty))}_{\text{Borel } \sigma\text{-field}}$

X is then said to be **jointly measurable** in (t,ω).

But one can defined other σ-algebras on $\Omega \times [0,\infty)$ ...

Predictable and optional σ-algebras

Def: optional σ-algebra

$\mathcal{O}$ the σ-algebra on $\Omega \times [0,\infty)$ generated by continuous, $\mathcal{F}_t$-adapted processes.

If $X : \Omega \times [0,\infty) \longrightarrow \mathbb{R}^d$ is $\mathcal{O}$-measurable

it is called an optional process.

X $(\mathcal{F}_t)$-adapted $+$ X continuous $\Rightarrow$ X optional

Def: the predictable σ-algebra $\mathcal{P}$ is the σ-algebra on $\Omega \times [0,\infty)$ generated by, equivalently

✓ $A \times (s,t]$ where $A \in \mathcal{F}_s$, $s \leq t$

or

✓ all $(\mathcal{F}_t)_{t \geq 0}$-adapted left-continuous processes

X $(\mathcal{F}_t)_{t\geq 0}$-adapted + X left-continuous $\Rightarrow$ X predictable

In particular: if X is a (cadlag adapted) process then $Y(t) = X(t-) = \lim_{s \uparrow t} X(t)$ defines a predictable process.

Note: none of these notions refers to a probability measure !

Stieltjes integration

Partition Π of $[0,T]$ = sequence $\Pi = (\Pi_n)_{n\geq 1}$

$$\Pi_n = \{ t_0^n = 0 < t_1^n < \cdots < t_{k(n)}^n = T \}$$

such that $$|\Pi_n| = \sup_{i=0..k(n)-1} |t_{i+1}^n - t_i^n| \xrightarrow[n\to\infty]{} 0$$

- Π is said to be a refining partition if $\Pi_n \subset \Pi_m$ for $n \leq m$.

Example: $t_i^n = \dfrac{iT}{2^n}$ $\quad i = 0 .. 2^n \quad$ $|\Pi_n| = \dfrac{1}{2^n} \xrightarrow[n\to\infty]{} 0$

Riemann sums along a partition: $\pi_n = (t_i^n,\ i=0..k(n))$

Let A, H be a pair of (cadlag adapted) processes.

The **non-anticipative** Riemann sum of H w.r.t. to A along π_n

as

$$\sum_{i=0}^{k(n)-1} H(t_i^n)\cdot\left[A(t_{i+1}^n) - A(t_i^n)\right]$$

One can also consider other Riemann sums e.g. using mid-point rule

$$\sum_{i=0}^{k(n)-1} H\left(\frac{t_i^n + t_{i+1}^n}{2}\right)\cdot\left[A(t_{i+1}^n) - A(t_i^n)\right]$$

Question:
- when do these sums converge?
- in what sense do they converge?
- what properties does the (eventual) limit have?

Finite variation functions and processes

$f \in D([0,\infty), \mathbb{R})$

Def: $V_f([0,T]) = \sup_{\Pi_n \in \mathcal{P}([0,T])} \sum_{t_i^n \in \Pi_n} |f(t_i^n) - f(t_{i+1}^n)|$

- f is said to be 'of finite variation' (FV) if

$$\forall T > 0, \quad V_f([0,T]) < \infty$$

- Jordan decomposition: f is of finite variation $\iff$

$f = f_+ - f_-$ where f_+, f_- are increasing functions.

✓. A process X is said to be of finite variation if $t \to X(t,\cdot)$ is FV with probability 1.

Then $X = X_+ - X_-$ where X_+, X_- are a.s. increasing

Shieltjes integration

Let $g : [0,\infty) \to \mathbb{R}$ be increasing.

Then g is a.e. differentiable and

g' defines a positive measure $dg = \mu_g(dx)$

and for any continuous function $h : [0,\infty) \to \mathbb{R}$

✓ one can define the integral $\int_0^t h\, dg = \mu_g(h)$

✓ For any partition $\pi = (\pi_n)_{n \geq 1}$,

$$\sum_{\substack{t_i^n \in \pi_n \\ t_{i+1}^n < t}} h(t_i^n)\left[g(t_{i+1}^n) - g(t_i^n)\right] \to \int_0^t h\, dg$$

Integration wrt a FV process

$A_t = A^+_t - A^-_t$ where A^+, A^- are a.s. increasing.

$\exists \Omega_0 \subset \Omega$, $\mathbb{P}(\Omega_0) = 1$ such that $\forall \omega \in \Omega_0$, $\begin{matrix} A^+(.,\omega) \\ A^-(.,\omega) \end{matrix}$ are increasing.

$\begin{cases} dA^+ \\ dA^- \end{cases}$ define positive (random) measures

and for a continuous adapted process H

✓ the Stieltjes integrals $\int_0^t H_s \, dA^+_s$, $\int_0^t H_s \, dA^-_s$

can be defined pathwise (for each $\omega \in \Omega_0$).

Def: For any continuous adapted process H

$$\int_0^t H \, dA := \int_0^t H \, dA^+ - \int_0^t H \, dA^-$$

Variation of a FV process

Def: $$|A|_t = \sup_{n\geq 1} \sum_{k=1}^{2^n} \left| A_{\frac{tk}{2^n}} - A_{\frac{t(k-1)}{2^n}} \right| = V_A([0,t]) < \infty$$

$|A|$ is called the total variation process associated to A.

Properties: $|A|$ is an increasing process

- $$A_t^+ = \frac{1}{2}(|A|_t + A_t), \quad A_t^- = \frac{1}{2}(|A|_t - A_t)$$

  are increasing, $(\mathcal{F}_t)$ - adapted processes

  and $$A_t = A_t^+ - A_t^-$$

Let A be a FV process

H be a process with continuous path's

$\Pi = (\Pi_n)_{n \geq 1}$ a partition of $[0,T]$

Then : $\sum_{\Pi_n} H_{t_k^n} \cdot (A_{t_{k+1}^n} - A_{t_k^n})$ converges

almost surely and

$$\lim_{n \to \infty} \sum_{\Pi_n} H_{t_k^n} \cdot (A_{t_{k+1}^n} - A_{t_k^n}) = \int_0^t H_s \, dA_s$$

Note: Adaptedness of integrands does not play any role in convergence.

Change of variable formula for a Stieltjes integral

Let A be a FV process with continuous paths $f \in C^1(\mathbb{R})$. Then $Y = f(A)$ is a FV process and:
$$f(A_t) - f(A_0) = \int_0^t f'(A_s)\, dA_s \quad \text{a.s.}$$

SO: for FV processes construction of the integral and rules of calculus apply as in the case of deterministic functions: probabilistic properties do not play a role and ω appears as a "parameter".

Unfortunately...
the class of FV processes excludes a lot of fundamental examples:
Brownian motion, Levy processes-diffusions...

Meyer's 'impossibility' theorem:

If
$$\sum_{\Pi_n} h(t_k^n)\left(f(t_{k+1}^n)-f(t_k)\right)$$

converges as $n \longrightarrow \infty$ for every $h \in C(\mathbb{R}_+)$ then

f is FV.

In other " there is no theory of pathwise integration with respect to continuous integrands for function of infinite variation."

Banach-Steinhaus theorem:

Let E be a Banach space, F a normed linear space. $(T_n)_{n \geq 1}$ a family of bounded linear operators $T_n : E \to F$

If $\forall x \in E$ $(T_n(x))_{n \geq 1}$ is bounded

then $(\|T_n\|)_{n \geq 1}$ is bounded.

where $\|T_n\| = \sup_{\|x\|_E = 1} \|T_n(x)\|_F$

is the operator norm on $\mathcal{L}(E, F)$

Proof : $E = (C_0(\mathbb{R}), \| \ \|_\infty)$, $F = \mathbb{R}$

Take Π_n = dyadic partition $\Pi_n = (t_k^n)$ $t_k^n = \frac{kT}{2^n}$

$$T_n(h) = \sum_{\Pi_n} h(t_k^n)\left(f(t_{k+1}^n) - f(t_k^n)\right)$$

If $T_n(h)$ converges as $n \to \infty$ for each $h \in E$

then $(T_n(h))_{n \geq 1}$ is bounded for each $h \in E$

so by the Banach-Steinhaus theorem $(\|T_n\|)_{n \geq 1}$ is bounded:

$\forall n \geq 1$, $\|T_n\| \leq K$.

Now pick a sequence $h^i \in E$ such that

$$h^i(t_k^n) \xrightarrow{i \to \infty} \operatorname{sgn}\left(f(t_{k+1}^n) - f(t_k^n)\right)$$

Then $\|T_n(h^i)\|_\infty = \sup_{[0,T]} |T_n(h^i(t))| \leq K$

But if $V_f([0,T]) = \infty$ then we can extract a subsequence $T_n(h^{i(n)})$ such that

$$T_n(h^{i(n)}) \xrightarrow{n\to\infty} \infty$$

so f has to be of finite variation.

Quadratic variation with respect to a partition

$(\Pi_n)_{n\geq 1}$ partition of $[0,T]$ $\begin{cases} \Pi_n = (t_i^n,\ i=0 \dots k(n)-1) \\ |\Pi_n| \to 0 \text{ as } n\to\infty \end{cases}$

Def: a cadlag function $f \in D([0,\infty), \mathbb{R})$ is said to have finite quadratic variation on $[0,T]$ w.r.t $(\Pi_n)_{n\geq 1}$

if : $\forall t \in [0,T]$, $\displaystyle\sum_{\substack{t_i^n \in \Pi_n \\ t_{i+1}^n < t}} |f(t_{i+1}^n) - f(t_i^n)|^2$ has a limit as $n\to\infty$.

The limit $\displaystyle [f](t) := \lim_{n\to\infty} \sum_{\substack{t_i^n \in \Pi_n \\ t_{i+1}^n < t}} |f(t_{i+1}^n) - f(t_i^n)|^2$

is then an increasing function on $[0,T]$ called the quadratic variation of f along $(\Pi_n)_{n\geq 1}$.

- $[f]$ is an increasing function so defines a positive measure μ_f on $[0,T]$.
- The Lebesgue decomposition of μ_f has the form

$$\mu_f = \mu_f^c + \sum_{\Delta f(t) \neq 0} |\Delta f(t)|^2 \delta_t$$

$$[f](t) = [f]^c(t) + \sum_{0 \leq s \leq t} |\Delta f(s)|^2$$

where $[f]^c_t = \mu_f^c([0,t])$

✓ Let $W=(W_t)_{t\geq 0}$ be a Wiener process and $\Pi=(\Pi_n)_{n\geq 1}$ be any partition of $[0,T]$ with $|\Pi_n|\xrightarrow{n\to\infty}0$ then the paths of W belong to $QV(\Pi,[0,T])$ with probability 1 and $[W](t)=t$.

✓ In particular the paths of W have infinite variation almost surely.

Denote $QV(\pi,[0,T])$ the set of all cadlag functions with finite quadratic variation along $\pi = (\pi_n)_{n\geq 1}$.

. Properties :

✓ If f is continuous and has FV then for any partition π

$$f \in QV(\pi,[0,T]) \quad \text{and} \quad [f] = 0$$

Proof : since f is (uniformly) continuous on $[0,T]$; so for n large ($n \geq N_0$) enough $|f(t_{i+1}^n) - f(t_i^n)| \leq \varepsilon$. Then for $n \geq N_0$

$$\sum_{\pi_n} |f(t_{i+1}^n) - f(t_i^n)|^2 \leq \varepsilon \sum_{\pi_n} |f(t_{i+1}^n) - f(t_i^n)|$$

$$= \varepsilon \, V_f([0,T]) \quad \text{so} \quad [f] = 0$$

Failure of the change of variable formula

- For a FV function g and $f \in C^1(\mathbb{R})$, $f(g(T)) - f(g(0)) = \int_0^T f'(g(t))\, dg$

where $$\int_0^T f'(g(t))\, dg = \lim_{\pi \to \infty} \sum_{t_k^n \in \pi_n} f'(g(t_k^n)) \cdot [g(t_{k+1}^n) - g(t_k^n)]$$

In particular: $$g(T)^2 - g(0)^2 = 2\int_0^T g\, dg = \lim_{n\to\infty} 2 \sum_{\pi_n} g(t_k^n)\left(g(t_{k+1}^n) - g(t_k^n)\right)$$

Proposition: Let $g \in QV(\pi, [0,T])$. Then

$$g(T)^2 - g(0)^2 = \lim_{n\to\infty} 2\sum_{\pi_n} g(t_k^n)\left(g(t_{k+1}^n) - g(t_k^n)\right) + [g](T)$$

So: the change of variable formula fails as soon as $[g] \neq 0$.

Proof: $\quad 2a(b-a) = -(b-a)^2 + b^2 - a^2$

$$2\sum_{t_k^n \in \Pi_n} g(t_k^n)\left(g(t_{k+1}^n) - g(t_k^n)\right) = -\underbrace{\sum_{\Pi_n} \left(g(t_{k+1}^n) - g(t_k^n)\right)^2}_{\xrightarrow[n\to\infty]{} [g](T)} + \underbrace{\sum_{\Pi_n} g(t_{k+1}^n)^2 - g(t_k^n)^2}_{g(T)^2 - g(0)^2}$$

so: $\quad g(T)^2 - g(0)^2 = \lim\limits_{n\to\infty} 2\sum_{t_k^n \in \Pi_n} g(t_k^n)\left(g(t_{k+1}^n) - g(t_k^n)\right) + [g](T)$

Summary :

* For a process A with finite variation one can define a pathwise Riemann-Stieltjes integral (ω by ω)

$$\int_0^T H_s(\omega)\,dA_s(\omega)$$

for H jointly measurable, continuous, as a pathwise (almost-sure) limit of Riemann sums. Adaptedness of integrands does not play any role in convergence.

* This integral verifies the usual charge of variable formula.
* However, this construction of the integral may not be carried out for all continuous integrands if A has infinite variation (Meyer's impossibility theorem).
* However Riemann sums may still converge for various subclasses of integrands!
* Moreover, if the paths of A have non-zero quadratic variation along some partition then the usual change of variable formula fails BUT one can obtain new change of variable formulas with extra terms involving quadratic variation
* All these results hold pathwise and do not involve probabilistic properties of A, H.