

## Lesson 5

### The Ito formula

Quadratic variation of a path with respect to a sequence of partitions

Change of variable formula for functions with finite quadratic variation

Ito formula for functions of semimartingales

Multivariate extension



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## Quadratic variation with respect to a partition

$$(\pi_n)_{n \geq 1} \text{ partition of } [0, T] \quad \left\{ \begin{array}{l} \pi_n = (t_i^n, i=0 \dots k(n)-1) \\ |\pi_n| \rightarrow 0 \text{ as } n \rightarrow \infty \end{array} \right.$$

Def: a cadlag function  $f \in D([0, \infty), \mathbb{R})$  is said

to have finite quadratic variation on  $[0, T]$  w.r.t  $\pi = (\pi_n)_{n \geq 1}$

if:  $\forall t \in [0, T], \sum_{\substack{t_i^n \in \pi_n \\ t_i^n < t}} |f(t_{i+1}^n) - f(t_i^n)|^2$  has a limit as  $n \rightarrow \infty$ .

The limit  $[f](t) := \lim_{n \rightarrow \infty} \sum_{\substack{t_i^n \in \pi_n \\ t_i^n < t}} |f(t_{i+1}^n) - f(t_i^n)|^2$

is then an increasing function on  $[0, T]$  called

the quadratic variation of  $f$  along  $\pi = (\pi_n)_{n \geq 1}$ .

Denote  $Q([0, T]; \pi)$  the set of  
 cadlag paths with finite quadratic variation  
 with respect to the partitions  $\pi = (\pi^n)_{n \geq 1}$

•  $Q([0, T], \pi)$  is a subspace of  $D([0, T], \mathbb{R})$

For  $f \in Q([0, T], \pi)$ :  $Q(\pi) = Q([0, \infty), \pi)$

•  $[f]$  is an increasing function so defines a positive  
 measure  $\mu_f$  on  $[0, T]$ .

• The increasing function  $t \rightarrow [f](t)$  has Lebesgue

decomposition  $[f](t) = \underbrace{[f]^c(t)}_{\text{continuous part}} + \sum_{0 \leq s \leq t} \underbrace{|\Delta f(s)|^2}_{\text{jumps}}$

$$\mu_f = \mu_f^c + \sum_{\Delta f(t) \neq 0} |\Delta f(t)|^2 \delta_t$$

'Examples':

✓ If  $X$  is a function with finite variation on  $[0, T]$ , then  
for any sequence of partitions with  $|\pi^n| \xrightarrow{n \rightarrow \infty} 0$ ,  $X \in Q([0, T], \pi)$

✓ If  $W$  is a Wiener process then  
for any sequence of partitions with  $|\pi^n| \xrightarrow{n \rightarrow \infty} 0$ ,

$$\mathbb{P}(W_\cdot \in Q([0, T], \pi)) = 1$$

and in fact  $[W](t) = t$

**Theorem:** Let  $S$  be a semimartingale. Then there exists a sequence  $\pi = (\pi^n)_{n \geq 1}$  of partitions such that  $S$  has finite quadratic variation along  $\pi$  with probability 1:  $\mathbb{P}(S \in Q(\pi)) = 1$ .

Proof: Let  $\sigma^n = (t_k^n = \frac{kT}{2^n}, k=0 \dots 2^n)$  be the dyadic subdivision of  $[0, T]$ . Then  $\sum_{k=0}^{2^n-1} (S_{t_{k+1}^n} - S_{t_k^n})^2 \xrightarrow{n \rightarrow \infty} [S]_T$  in probability.

Since every sequence converging in probability has an almost-surely convergent subsequence, there exists a subsequence  $\pi^n = (T_i^n, i=0 \dots k(n))$  of  $(\sigma^n)_{n \geq 1}$  such that:

$$\sum_{T_i^n \in \pi^n} (S_{T_{i+1}^n} - S_{T_i^n})^2 \xrightarrow{n \rightarrow \infty} [S]_T \quad \text{a.s.}$$

$$\Omega_T = \left\{ \omega \in \Omega, \sum_{T_i^n \in \pi^n} (S_{T_{i+1}^n}(\omega) - S_{T_i^n}(\omega))^2 \xrightarrow{n \rightarrow \infty} [S]_T(\omega) \right\}$$

then  $P(\Omega_T) = 1$ .

So  $P(\bigcap_{T \in \mathbb{Q}} \Omega_T) = 1$ . Since  $S, [S]$  are cadlag processes we conclude that on  $\Omega^\pi = \bigcap_{T \in \mathbb{Q}} \Omega_T$ ,

we have  $\sum_{T_i^n \in \pi^n \cap [0, t]} (S_{T_{i+1}^n} - S_{T_i^n})^2 \xrightarrow{n \rightarrow \infty} [S]_t$  for any  $t \geq 0$

for any  $t \geq 0$ : all paths in  $\Omega^\pi$  have finite quadratic variation along  $(\pi^n)_{n \geq 1}$

# Quadratic Riemann sums

**Proposition:** Let  $f \in \mathcal{Q}([0, T], \pi)$ .

$$\forall g \in C^0([0, T], \mathbb{R}), \quad \sum_{t_i^n \in \pi^n} g(t_i^n) |f(t_{i+1}^n) - f(t_i^n)|^2 \xrightarrow{n \rightarrow \infty} \int_0^T g d[f] \quad (\text{QR})$$

**Proof:** For  $g = 1_{[0, t]}$  using  $f \in \mathcal{Q}([0, T], \pi)$ ,

$$\sum_{t_i^n \in \pi^n} g(t_i^n) |f(t_{i+1}^n) - f(t_i^n)|^2 = \sum_{\substack{i=1 \\ t_{i+1}^n < t}}^d |f(t_{i+1}^n) - f(t_i^n)|^2 \xrightarrow{n \rightarrow \infty} [f](t) = \int_0^T g d[f]$$

More generally if  $g = \sum_{i=1}^d a_i 1_{[t_i, t_{i+1})} = \sum_{i=1}^d b_i 1_{[0, t_i]}$

we also have (QR) by linearity

For  $g \in C^0([0, T], \mathbb{R})$ , there exists  $g^n = \sum_{i=1}^d a_i^n 1_{[t_i^n, t_{i+1}^{n+1})}$

such that  $\|g - g^n\|_\infty \xrightarrow{n \rightarrow \infty} 0$  which allows to conclude.



## Weak convergence

A sequence  $(\mu_n)_{n \geq 1}$  of measures on  $[0, T]$  is said to converge weakly to a measure  $\mu$  on  $[0, T]$

$$\forall g \in C_b^0([0, T], \mathbb{R}), \int_0^T g d\mu^n \xrightarrow{n \rightarrow \infty} \int_0^T g d\mu$$

We denote  $\mu^n \xrightarrow{n \rightarrow \infty} \mu$

**Property:** If  $f \in Q([0, T]; \pi)$  then

$$\mu^n \approx \sum_{\pi^n} (f(t_{i+1}^n) - f(t_i^n))^2 \delta_{t_i^n} \xrightarrow{n \rightarrow \infty} \mu_f \approx d[f]$$

## Change of variable formula for functions of finite quadratic variation

**Theorem (Föllmer 1979):** If  $X \in Q(\Pi)$  has finite quadratic variation along a sequence  $\Pi = (\Pi^n)_{n \geq 1}$  of partitions, then for any  $f \in C^2(\mathbb{R}, \mathbb{R})$ ,

✓  $\sum_{t_i^n \in \Pi^n, 0 \leq t_i^n \leq t} f'(X_{t_i^n}) (X_{t_{i+1}^n} - X_{t_i^n})$  has a limit as  $n \rightarrow \infty$

✓ Denoting  $\int_0^t f'(X_{s-}) dX_s = \lim_{n \rightarrow \infty} \sum_{t_i^n \in \Pi^n, 0 \leq t_i^n \leq t} f'(X_{t_i^n}) (X_{t_{i+1}^n} - X_{t_i^n})$

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-}) dX_s + \frac{1}{2} \int_0^t f''(X_{s-}) d[X]_s + \sum_{0 \leq s \leq t} f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) - \frac{(\Delta X_s)^2}{2} f''(X_{s-})$$

Corollary :

✓ if  $x \in Q(\mathbb{T})$  is continuous then

$$f(x_t) - f(x_0) = \int_0^t f'(x_{s-}) dX_s + \frac{1}{2} \int_0^t f''(x_{s-}) d[X]_s$$

✓ if  $x \in Q(\mathbb{T})$  has bounded variation then

$$f(x_t) - f(x_0) = \int_0^t f'(x_{s-}) dX_s + \sum_{0 \leq s \leq t} f(x_s) - f(x_{s-}) - \Delta X_s f'(x_{s-}) - \frac{(\Delta X_s)^2}{2} f''(x_{s-})$$

Proof: ✓ by right continuity of  $x$ ,

$$f(x_t) - f(x_0) = \lim_{n \rightarrow \infty} \sum_{t_i^n \in [0, t]} f(x_{t_i^n}) - f(x_{t_{i-1}^n})$$

✓ Taylor's Theorem:  $\forall x, y \in [0, a]$ ,  $f \in C^2$

$$f(y) - f(x) = (y-x) f'(x) + \frac{(y-x)^2}{2} f''(x) + (y-x)^2 r(x, y)$$

Since  $x$  is cadlag therefore bounded on  $[0, T]$ ,  $I = x([0, T])$   
 $r$  is continuous on  $I^2$  therefore uniformly continuous.

where, by uniform continuity,  $r(x, y) \leq \varphi(|y-x|)$   
for some increasing function  $\varphi: [0, \infty[ \rightarrow [0, \infty[$   
with  $\varphi(\varepsilon) \xrightarrow{\varepsilon \downarrow 0} 0$

Proof: continuous case  $x \in C_0([0, T]) \cap Q(\pi)$

$x \in C_0([0, T])$  is uniformly continuous on  $[0, T]$  so

$$\forall \varepsilon > 0, \exists \delta > 0, |s_1 - s_2| \leq \delta, |x_{s_1} - x_{s_2}| \leq \varepsilon$$

Since  $|\pi^n| \xrightarrow{n \rightarrow \infty} 0$ , for  $n$  large enough,  $|\pi^n| \leq \delta$

so  $\exists n_0 \geq 1, \forall n \geq n_0, |\pi^n| \leq \delta$  and  $|x_{b_{i+1}^n} - x_{b_i^n}| \leq \varepsilon$   
for  $b_i \in \pi^n$ .

$$f(x_t) - f(x_0) = \sum_{t_i^n \in [0, t]} f(x_{t_{i+1}^n}) - f(x_{t_i^n}) =$$

$$\sum_{t_i^n \in [0, t]} f'(x_{t_i^n}) (x_{t_{i+1}^n} - x_{t_i^n})$$

converges since all other terms have a finite limit

$$+ \sum_{t_i^n \in [0, t]} \frac{1}{2} f''(x_{t_i^n}) (x_{t_{i+1}^n} - x_{t_i^n})^2 \xrightarrow{n \rightarrow \infty} \frac{1}{2} \int_0^t f''(x_s) d[X]_s$$

since  $x \in Q(\pi)$   
 $f'' \in C_0$

$$+ \sum_{t_i^n \in [0, t]} (x_{t_{i+1}^n} - x_{t_i^n})^2 \underbrace{r(x_{t_i^n}, x_{t_{i+1}^n})}_{\leq \varepsilon \text{ for } |\pi^n| \leq \delta} \leq C \quad \varepsilon \xrightarrow{n \rightarrow \infty} 0$$

$\xrightarrow{n \rightarrow \infty} [X]_t$  so is bounded wrt  $n$

Proof : discontinuous case

Let  $\varepsilon > 0$ . Since  $\sum_{0 \leq s \leq t} (\Delta X_s)^2 \leq [X]_t < \infty$ ,

the series  $\sum_{0 \leq s \leq t} (\Delta X_s)^2$  only contains a finite number of terms in any interval bounded away from zero.

Separate the jump times of  $X$  in  $[0, t]$  into two classes

✓ a finite set  $C_1(\varepsilon, t)$

(large jumps)

✓ a set  $C_2(\varepsilon, t)$  such that  $\sum_{s \in C_2} (\Delta X_s)^2 \leq \varepsilon^2$  (small jumps)

with  $C_1(\varepsilon, t) \cup C_2(\varepsilon, t) = \{s \in [0, t], \Delta X_s \neq 0\}$

and  $\bigcup_{\varepsilon > 0} C_1(\varepsilon, t) = \{s \in [0, t], \Delta X_s \neq 0\}$

$$\sum_{\pi^n \cap [0, t]} f(x_{t_{i+1}^n}) - f(x_{t_i^n}) = \sum_{]t_i, t_{i+1}] \cap C, (\varepsilon, t) \neq \emptyset} f(x_{t_{i+1}^n}) - f(x_{t_i^n}) + \sum_{]t_i, t_{i+1}] \cap C, (\varepsilon, t) = \emptyset} f(x_{t_{i+1}^n}) - f(x_{t_i^n})$$

↑  
at least one large jump

↑  
no large jump

$$\checkmark \lim_{n \rightarrow \infty} \sum_{\substack{]t_i^n, t_{i+1}^n] \cap C, (\varepsilon, t) \neq \emptyset \\ t_i^n \in \pi^n}} f(x_{t_{i+1}^n}) - f(x_{t_i^n}) = \sum_{s \in C, (\varepsilon, t)} f(x_s) - f(x_{s-}) \quad \text{for each } \varepsilon$$

✓ Taylor's formula applied to the second sum gives

$$\sum_{]t_i, t_{i+1}] \cap C, (\varepsilon, t) = \emptyset} f(x_{t_{i+1}^n}) - f(x_{t_i^n}) = \sum_{\pi^n \cap [0, t]} f'(x_{t_i^n}) (x_{t_{i+1}^n} - x_{t_i^n}) + \frac{1}{2} \sum_{\pi^n \cap [0, t]} f''(x_{t_i^n}) (x_{t_{i+1}^n} - x_{t_i^n})^2$$

$$+ \sum_{]t_i, t_{i+1}] \cap C, (\varepsilon, t) = \emptyset} (x_{t_{i+1}^n} - x_{t_i^n})^2 r(x_{t_i^n}, x_{t_{i+1}^n}) - \sum_{]t_i, t_{i+1}] \cap C, (\varepsilon, t) \neq \emptyset} \left[ f'(x_{t_i^n}) (x_{t_{i+1}^n} - x_{t_i^n}) + \frac{1}{2} f''(x_{t_i^n}) (x_{t_{i+1}^n} - x_{t_i^n})^2 \right]$$



✓  $f''$  is uniformly continuous on the bounded set

$$I = \{x_s, s \in [0, t]\} \text{ so } \forall u, v \in I^2, r(u, v) \leq \varphi(\|u-v\|)$$

where  $\varphi(\varepsilon) \xrightarrow{\varepsilon \downarrow 0} 0$ . So,  $\forall \varepsilon > 0$

$$\limsup_{n \rightarrow \infty} \sum_{\substack{]t_i, t_{i+1}] \cap C_1(\varepsilon, t) = \emptyset \\ (x_{t_{i+1}} - x_{t_i})^2 r(x_{t_i}, x_{t_{i+1}}) \leq \varphi(\varepsilon) [X]_t \xrightarrow{\varepsilon \downarrow 0} 0}}$$

$$\checkmark \sum_{s \in C_1(\varepsilon, t)} |f(x_s) - f(x_{s-}) - \Delta x_s f'(x_{s-})| \leq \frac{\|f''\|_{\infty}}{2} \sum_{s \in C_1(\varepsilon, t)} (x_s - x_{s-})^2 \quad \text{by Taylor's formula}$$

so the series is **absolutely** convergent and

$$\sum_{s \in C_1(\varepsilon, t)} f(x_s) - f(x_{s-}) - \Delta x_s f'(x_{s-}) \xrightarrow{\varepsilon \rightarrow 0} \sum_{0 \leq s \leq t} f(x_s) - f(x_{s-}) - \Delta x_s f'(x_{s-})$$

So finally:  $\forall \varepsilon > 0, f(x_t) - f(x_0) = \sum_{\pi^n} f(x_{t_{i+1}}) - f(x_{t_i}) =$

$$\sum_{\substack{]t_i, t_{i+1}] \cap C, (\varepsilon, t) \neq \emptyset \\ \pi^n \wedge [0, t]}} f(x_{t_{i+1}}) - f(x_{t_i}) \xrightarrow{n \rightarrow \infty} \sum_{C, (\varepsilon, t)} f(x_s) - f(x_{s-})$$

$$+ \sum_{\pi^n \wedge [0, t]} f'(x_{t_i}) (x_{t_{i+1}} - x_{t_i}) \xrightarrow{n \rightarrow \infty} \text{converges as } n \rightarrow \infty \text{ since all other terms converge}$$

$$+ \frac{1}{2} \sum_{\pi^n \wedge [0, t]} f''(x_{t_i}) (x_{t_{i+1}} - x_{t_i})^2 \xrightarrow{n \rightarrow \infty} \frac{1}{2} \int_0^t f''(x_u) d[X]_u$$

$$+ \sum_{\substack{]t_i, t_{i+1}] \cap C, (\varepsilon, t) = \emptyset \\ \pi^n \wedge [0, t]}} (x_{t_{i+1}} - x_{t_i})^2 r(x_{t_i}, x_{t_{i+1}}) \leq k \varepsilon^2$$

$$- \sum_{\substack{]t_i, t_{i+1}] \cap C, (\varepsilon, t) \neq \emptyset \\ \pi^n \wedge [0, t]}} \left[ f'(x_{t_i}) (x_{t_{i+1}} - x_{t_i}) + \frac{1}{2} f''(x_{t_i}) (x_{t_{i+1}} - x_{t_i})^2 \right] \xrightarrow{n \rightarrow \infty} - \sum_{s \in C, (\varepsilon, t)} f'(x_{s-}) \Delta X_s + \frac{1}{2} (\Delta X_s)^2 f''(x_{s-})$$

Taking  $\varepsilon \rightarrow 0$  yields the Ito formula:

$$f(x_t) - f(x_0) = \int_0^t f'(x_{s-}) dX_s + \frac{1}{2} \int_0^t f''(x_{s-}) d[X]_s$$

$$+ \sum_{0 \leq s \leq t} f(x_s) - f(x_{s-}) - \Delta X_s f'(x_{s-}) - \frac{(\Delta X_s)^2}{2} f''(x_{s-})$$

## Different forms of the change of variable formula

$$f(x_t) - f(x_0) = \int_0^t f'(x_{s-}) d\bar{X}_s + \frac{1}{2} \int_0^t f''(x_{s-}) d[X]_s + \sum_{0 \leq s \leq t} f(x_s) - f(x_{s-}) - \Delta X_s f'(x_{s-}) - \frac{(\Delta X_s)^2}{2} f''(x_{s-})$$

$$= \int_0^t f'(x_{s-}) d\bar{X}_s + \frac{1}{2} \int_0^t f''(x_{s-}) d[X]_s^c + \sum_{0 \leq s \leq t} f(x_s) - f(x_{s-}) - \Delta X_s f'(x_{s-})$$

since

$$[X]_t = [X]_t^c + \sum (\Delta X_s)^2$$

$$\int_0^t f''(x_u) d[X] = \int_0^t f''(x_u) d[X]^c + \sum_{0 \leq s \leq t} f''(x_{s-}) (\Delta X_s)^2$$

however in general  $\sum_{\substack{0 \leq s \leq t \\ \Delta X_s \neq 0}} f(x_s) - f(x_{s-}) - \Delta X_s f'(x_{s-})$  may not be further decomposed:

in general  $\sum (\Delta X_s) f'(X_{s-})$  ; do not converge  
 $\sum f'(X_s) - f(X_{s-})$

unless  $\sum_{0 \leq s \leq t} |\Delta X_s| < \infty$  (the jumps have 'finite variation')

If  $\sum_{0 \leq s \leq t} |\Delta X_s| < \infty$  then we have

$$X_t = \overbrace{\left( X_t - \sum_{0 \leq s \leq t} \Delta X_s \right)}^{X_t^c} + \sum_{0 \leq s \leq t} \Delta X_s \text{ where } X_t^c \in C_0([0, T])$$

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-}) dX_s^c + \frac{1}{2} \int_0^t f''(X_{s-}) d[X]_s^c + \sum_{0 \leq s \leq t} f(X_s) - f(X_{s-})$$

## The Ito formula

Let  $S$  be a semimartingale,  $f \in C^2([0, \infty), \mathbb{R})$ .

Then  $f(S)$  is also a semimartingale and  $\forall t \geq 0$ ,

$$f(S_t) - f(S_0) = \int_0^t f'(S_{u-}) dS_u + \int_0^t \frac{1}{2} f''(S_{u-}) d[S]_u \\ + \sum_{0 \leq s \leq t} f(S_u) - f(S_{u-}) - \Delta S_u f'(S_u) - \frac{1}{2} f''(S_{u-}) (\Delta S_u)^2$$

**Proof:** Since  $S$  semimartingale there exists a sequence  $(\bar{\pi}^n)_{n \geq 1}$  of partitions such that  $P(\Omega^{\bar{\pi}}) = 1$ , where

$$\Omega^{\bar{\pi}} = \{\omega \in \Omega, S_\cdot(\omega) \in Q(\bar{\pi})\}. \text{ So: } \forall \omega \in \Omega^{\bar{\pi}}, \forall f \in C^2(\mathbb{R}),$$

$$\begin{aligned} f(S_t(\omega)) - f(S_0) &= \lim_{n \rightarrow \infty} \sum_{\bar{\pi}^n} f'(S_{t_i^n}) (S_{t_{i+1}^n}(\omega) - S_{t_i^n}(\omega)) \\ &\quad + \lim_{n \rightarrow \infty} \sum_{\bar{\pi}^n} \frac{1}{2} f''(S_{t_i^n}) (S_{t_{i+1}^n}(\omega) - S_{t_i^n}(\omega))^2 \\ &\quad + \sum_{0 \leq s \leq t} f(S_u) - f(S_{u-}) - \Delta S_u f'(S_u) - \frac{1}{2} f''(S_{u-}) (\Delta S_u)^2 \end{aligned}$$

Since  $S$  is a semimartingale, since  $|\Pi^n| \rightarrow 0$ ,

$$\sum_{\Pi^n} f'(S_{t_i^n}) (S_{t_{i+1}^n} - S_{t_i^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t f'(S_{u-}) dS_u$$

and since  $\omega \in \Omega^{\Pi}$

$$\lim_{n \rightarrow \infty} \sum_{\Pi^n} \frac{1}{2} f''(S_{t_i^n}) (S_{t_{i+1}^n}(\omega) - S_{t_i^n}(\omega))^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t \frac{1}{2} f''(S_{u-}) d[S]_u$$

↑  
non-anticipative quadratic Riemann sum

# Ito formula : decomposition into 'martingale + finite variation' parts

If  $S_t = M_t + A_t$  where

$M$  (locally square integrable) local martingale  
 $A$  continuous process with finite variation

$$f(S_t) - f(S_0) = \int_0^t f'(S_{u-}) dM_u + \int_0^t f'(S_{u-}) dA_u + \int_0^t \frac{1}{2} f''(S_{u-}) d[M]_u$$

↑ local martingale
↑ finite variation process

$$+ \sum_{0 \leq s \leq t} f(S_u) - f(S_{u-}) - \Delta S_u f'(S_u) - \frac{1}{2} f''(S_{u-}) (\Delta S_u)^2$$

• If  $M$  is furthermore continuous then

$$f(S_t) = f(S_0) + \int_0^t f'(S_u) dM_u + \int_0^t \frac{1}{2} f''(S_u) d[M]_u + \int_0^t f'(S_u) dA_u$$

Local martingale

Finite variation



Ex. Change of variable formula for Ito processes

$$X_t = X_0 + \int_0^t \sigma_u dW_u + \int_0^t \mu_u du = M_t + A_t$$

$$E\left(\int_0^T \sigma_u^2 du\right) < \infty, \quad E\left(\int_0^T |\mu_u| du\right) < \infty, \quad [X]_t = \int_0^t \sigma_u^2 du$$

Then for  $f \in C^2(\mathbb{R}, \mathbb{R})$ ,  $f(X_t)$  is also an Ito process with Ito decomposition

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) \sigma_u dW_u \quad \leftarrow \text{Local martingale term}$$

$$+ \int_0^t f'(X_u) \mu_u du + \frac{1}{2} \int_0^t f''(X_u) \sigma_u^2 du \quad \leftarrow \text{Finite variation part}$$

✓ Actually we have shown that for a semimartingale  $S$

there exists a set  $\Omega^S = \Omega^{\bar{\pi}}$  with  $P(\Omega^S) = 1$

such that for all  $f \in C^2(\mathbb{R}, \mathbb{R})$ , the terms

appearing in the Ito formula may be defined as pathwise limits of Riemann sums for any path in  $\Omega^S$

and  $\Omega^S$  is chosen independently of  $f$ .

Proof works identically if  $S$  is not a semimartingale as long as it has finite quadratic variation along vs some partition.

Ex. fractional Brownian motion with  $H > \frac{1}{2}$

## Multivariate extension:

$x = (x^1, \dots, x^d) \in D([0, T], \mathbb{R}^d)$  is said to have finite quadratic variation with respect to  $\bar{\pi} = (\bar{\pi}^n)_{n \geq 1}$  if

$$\forall i, j = 1 \dots d, \quad x^i + x^j, x^i \text{ and } x^j \in Q(\bar{\pi})$$

We then define  $[x^i, x^j] = \frac{1}{2} ([x^i + x^j] - [x^i] - [x^j])$

$[X]_{\bar{\pi}} = ([x^i, x^j])_{i, j = 1 \dots d}$  may be seen as an increasing

function  $[x] : [0, T] \rightarrow \text{Sym}^+(d)$

$\text{Sym}^+(d)$  of  $d \times d$  positive symmetric matrices.

Quadratic Riemann sums  $\forall X \in Q(\pi), \forall f \in C_0([0, T], \mathbb{R}^{d \times d})$ ,

$$\sum_{[0, t] \wedge \pi^n} \text{tr} \left\{ f(X_{t_i})^t (X_{t_{i+1}} - X_{t_i}) (X_{t_{i+1}} - X_{t_i}) \right\} \Rightarrow$$

$$\sum_{[0, t] \wedge \pi^n} \left( \sum_{j, k} f_{j, k}(X_{t_i}) (X_{t_{i+1}}^j - X_{t_i}^j) (X_{t_{i+1}}^k - X_{t_i}^k) \right)$$

$$\begin{matrix} n \rightarrow \infty \\ \longrightarrow \end{matrix} \int_0^t \text{tr} \left\{ f(X_{u-}) d[X]_u \right\}$$

$$= \int_0^t \sum_{j, k=1}^d f_{j, j}(X_{u-}) d[X^i, X^j]_u$$

Multivariate Ito Formula :

$S^1, \dots, S^d$  are semimartingales

$S = (S^1, \dots, S^d)$ . Then for  $f \in C^2(\mathbb{R}^d, \mathbb{R})$ ,

$$\begin{aligned} f(S_t) - f(S_0) &= \sum_{j=1}^d \int_0^t \frac{\partial f}{\partial x_j}(S_{u-}) \cdot dS_u^j + \sum_{j=1}^d \int_0^t \frac{1}{2} \frac{\partial^2 f}{\partial x_j^2}(S_{u-}) d[S^j]_u^c \\ &+ \sum_{j>k} \int_0^t \frac{\partial^2 f}{\partial x_j \partial x_k}(S_{u-}) d[S^k, S^j]_u^c \\ &+ \sum_{0 \leq u \leq t} \left\{ f(S_u) - f(S_{u-}) - \Delta S_u \cdot \nabla f(S_{u-}) \right\} \end{aligned}$$

$$= \int_0^t \nabla f(S_{u-}) \cdot dS_u + \int_0^t \frac{1}{2} \text{tr}(\nabla^2 f(S_{u-}) \cdot d[S]_u^c) \\ + \sum_{0 \leq u \leq t} \left\{ f(S_u) - f(S_{u-}) - \Delta S_u \cdot \nabla f(S_{u-}) \right\}$$

• If the jumps of  $S$  are summable :  $\sum_{0 \leq u \leq t} \|\Delta S_u\| < \infty$  a.s.  
 then  $S_t = S_t^c + \sum_{0 \leq u \leq t} \Delta S_u$  and

$$f(S_t) - f(S_0) = \int_0^t \nabla f(S_{u-}) \cdot dS_u^c + \int_0^t \frac{1}{2} \text{tr}(\nabla^2 f(S_{u-}) \cdot d[S]_u^c) \\ + \sum_{0 \leq u \leq t} \left\{ f(S_u) - f(S_{u-}) \right\}$$