Stochastic Calculus

Rama CONT

Note Title

2/6/2011

Lecture 3: The Ito Stochastic Integral. Semimartingales.

- ✓ Stochastic integral for simple predictable integrands
- ✓ Martingale-preserving property
- ✓ Isometry formula
- ✓ Semi-martingales : definition
- ✓ Examples of semimartingales

Simple predictable processes

$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ filtered probability space

- $H = (H_t)_{t\geq 0}$ is a simple predictable process on $[0,T]$ if

$\exists n \geq 1 \quad H = \phi_0 \, 1_{[0]}(t) + \sum_{i=1}^{n-1} \phi_i \, 1_{(T_i, T_{i+1}]}(t)$ where

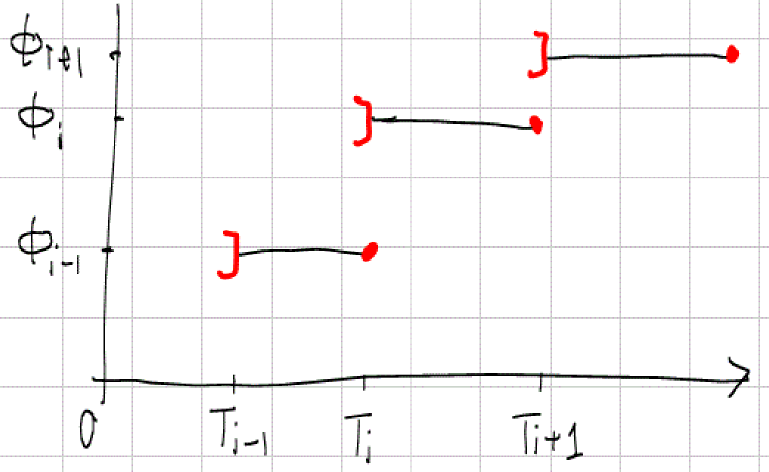
(i) $0 \leq T_1 \leq T_2 \leq \dots \leq T_n = T$ are finite stopping times

(ii) For $i = 1 \dots n$, ϕ_i is $\begin{cases} \mathcal{F}_{T_i}\text{-measurable} \\ \text{finite} : P(\phi_i < \infty) = 1 \end{cases}$ (= revealed at T_i)

✓ This definition depends on

- the filtration $\mathbb{F}$ through the measurability conditions
- the probability measure $\mathbb{P}$ through the finiteness conditions

Simple predictable processes



✓ Property: $t \mapsto H_t(\omega)$ is
left-continuous (caglad).

$S([0,T],\mathbb{F})$ $\left\{\begin{array}{l}\text{set of all simple} \\ \text{predictable processes} \\ \text{on } [0,T]\end{array}\right.$

. If $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is a smaller filtration $\mathcal{G}_t \subset \mathcal{F}_t \quad \forall t \geq 0$

then $S([0,T],\mathbb{G}) \subset S([0,T],\mathbb{F})$

. $S([0,T],\mathbb{F})$ only depends on $\mathbb{P}$ through its null sets:

if $\mathbb{P}, \mathbb{Q}$ are equivalent in the sense $\forall A \in \mathcal{F}, \ \mathbb{P}(A)=0 \iff \mathbb{Q}(A)=0$

then $\mathbb{P}, \mathbb{Q}$ define the same set of simple predictable processes.

Stochastic integral for a simple predictable process

$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$ filtered probability space. For a simple predictable process $H = H_0 \, 1_{[0]}(t) + \sum_{i=1}^{n-1} \phi_i \, 1_{(T_i, T_{i+1}]}(t) \in S([0,T], \mathbb{F})$ and a non-anticipative process X, define

$$\int_0^T H \, dX = \sum_{i=0}^{n-1} \phi_i \, (X_{T_{i+1}} - X_{T_i}) = \sum_{i=0}^{n-1} H_{T_i} \, (X_{T_{i+1}} - X_{T_i})$$ Random variable

More generally one can define the process $I_X(H) = \left(\int_0^t H \, dX \right)_{t \geq 0}$ as

$$\int_0^t H \, dX = \sum_{i=0}^{n-1} H_{T_i} \cdot (X_{T_{i+1} \wedge t} - X_{T_i \wedge t})$$ Stochastic integral of H with respect to X

Stochastic integral for simple predictable processes

Properties: For any non-anticipative cadlag process X, and any $H, H_1, H_2 \in S([0,T], \mathbb{F})$,

i) $I_X(H) = \left(\int_0^t H\,dX\right)_{t \geq 0}$ is a cadlag, non-anticipative process.

ii) $\int_0^t (\lambda H_1 + H_2)\,dX = \lambda \int_0^t H_1\,dX + \lambda \int_0^t H_2\,dX$

iii) Behavior under stopping: if τ is a stopping time then

$$\int_0^{t\wedge\tau} H.\,dX = \int_0^t H.\,dX^\tau = \int_0^t \left(H\, 1_{[0,\tau]}\right).\,dX$$

where $X_t^\tau = X_{t\wedge\tau}$ is the process X stopped at τ.

Stochastic integral for simple predictable processes

Martingale-preserving property: If X is a martingale then for any $H \in S([0,T], \mathbb{F})$, $G_t(H) = \int_0^t H\,dX$ is a martingale

Proof: $H = \sum_{i=0}^{n-1} H_i \mathbb{1}_{(T_i, T_{i+1}]}$ with H_i $\mathcal{F}_{T_i}$-measurable, $T_1 \leq T_2 \leq \dots \leq T_n = T$

$$G_t(H) = \sum_{i=0}^{n-1} H_i \left(X_{T_{i+1} \wedge t} - X_{T_i \wedge t} \right).$$

For $t \geq T$, $\quad G_t(H) = G_T(H) + H_T(X_t - X_T)$

so $\quad E[G_t(H) \mid \mathcal{F}_T] = G_T(H) + H_T E[X_t - X_T \mid \mathcal{F}_T] = G_T(H)$

so it is enough to show that for $t \leq T$, $\quad E[G_T \mid \mathcal{F}_t] = G_t$

$\forall t \leq T,$ $\{t \leq T_i\}$, $\{t > T_{i+1}\}$, $\{t \in (T_i, T_{i+1}]\}$ are $\mathcal{F}_t$-measurable

$E[G_T \mid \mathcal{F}_t] = E\left[\sum_{i=0}^{n-1} H_{T_i} (X_{T_{i+1}} - X_{T_i}) \mid \mathcal{F}_t\right]$. Each term is decomposable as

$$\underbrace{E[H_i (X_{T_{i+1}} - X_{T_i}) 1_{t > T_{i+1}} \mid \mathcal{F}_t]}_{= H_i (X_{T_{i+1}} - X_{T_i}) 1_{t > T_{i+1}}} + E[H_i (X_{T_{i+1}} - X_{T_i}) 1_{(T_i, T_{i+1}]}(t) \mid \mathcal{F}_t] + E[1_{[0,T_i]}(t) H_i (X_{T_{i+1}} - X_{T_i}) \mid \mathcal{F}_t]$$

$$E[1_{[0,T_i]}(t) H_{T_i} (X_{T_{i+1}} - X_{T_i}) \mid \mathcal{F}_t] = E[1_{[0,T_i]}(t) H_{T_i} \underbrace{E[X_{T_{i+1}} - X_{T_i} \mid \mathcal{F}_{T_i}]}_{= 0 \text{ by Doob's Optional Sampling Thm}} \mid \mathcal{F}_t]$$

$$E[H_{T_i} (X_{T_{i+1}} - X_{T_i}) 1_{(T_i, T_{i+1}]}(t) \mid \mathcal{F}_t] = H_{T_i} 1_{(T_i, T_{i+1}]}(t) E[X_{T_{i+1}} - X_{T_i} \mid \mathcal{F}_t]$$

$$= H_{T_i} 1_{(T_i, T_{i+1}]}(t) (X_t - X_{T_i}) = 1_{(T_i, T_{i+1}]} H_{T_i} (X_{T_{i+1} \wedge t} - X_{T_i \wedge t})$$

so summing the 3 terms: $E[H_{T_i} (X_{T_{i+1}} - X_{T_i}) \mid \mathcal{F}_t] = H_{T_i} (X_{T_{i+1} \wedge t} - X_{T_i \wedge t})$

Furthermore: if $M \in \mathcal{M}^2$ is a square-integrable martingale

$\forall\, H \in S([0,T])$,

- $\left(\int_0^t H\, dM \right)_{t \geq 0}$ is a square-integrable martingale

$$\int_0^{\cdot} H\, dM \in \mathcal{M}_2 \quad , \quad \text{and}$$

$$\left\| \int_0^{\infty} H\, dM \right\|_{\mathcal{M}_2}^2 = E\left| \int_0^{\infty} H\, dM \right|^2 = \sum_{k \geq 0} E\left[H_k^2 \left(M_{T_{k+1}}^2 - M_{T_k}^2 \right) \right] \leq \|H\|_{\infty}^2 \|M\|_{\mathcal{M}_2}^2$$

So: the stochastic integral preserves $\mathcal{M}_2$

and $\quad M \longrightarrow \int H\, dM$ is continuous on $\mathcal{M}_2$

Remark: if $H \in S([0,T], \mathbb{F})$, X martingale then

$$Z_t = \sum_{i=0}^{n-1} H_{T_{i+1}} \left(X_{T_{i+1} \wedge t} - X_{T_i \wedge t} \right)$$

is <u>not</u> a martingale in general.

In the definition of simple predictable integrands it is in fact crucial to require **left-continuity** $H = \sum \phi_i 1_{(T_i, T_{i+1}]}$ and not right-continuity which would give $\sum \phi_i 1_{[T_i, T_{i+1})}$

Fundamental example: $X_t = N_t - \lambda t$ where N is a Poisson process with intensity λ

$T_1 = \inf\{t > 0, N_t \geq 1\}$ is a stopping time, $N_0 = 0$

$$\begin{cases} H = 1_{(T_1, T]} \in S([0,T]) \text{ and } \int_0^t H\,dX = (X_t - X_{T_1})\, 1_{t \geq T_1} \\ C = 1_{[T_1, T]} \notin S([0,T]) \quad \int_0^t C\,dX = (1 + X_t - X_{T_1})\, 1_{t \geq T_1} \end{cases}$$

$E\left(\int_0^t C\,dX\right) = E\left(1_{t \geq T_1}\right) = \mathbb{P}(T_1 \leq t) > 0$ so $\left(\int_0^t C\,dX\right)_{t \geq 0}$ is not a martingale

So for any $(\mathcal{F}_t)_{t\geq 0}$ - adapted process X we have defined

$$I_X : S([0,T], \mathbb{F}) \longrightarrow \text{Adapted cadlag processes}$$
$$H \longrightarrow \int_0 H\,dX$$

$X \in \mathcal{M}$ then $I_X(S([0,T],\mathbb{F})) \subset \mathcal{M}$

$X \in \mathcal{M}_2$ then $I_X(S([0,T],\mathbb{F})) \subset \mathcal{M}_2$

Semi-martingales

A $\begin{cases} \text{cadlag} \\ \text{non-anticipative} \end{cases}$ process X is called a **semimartingale**

if ✓ for each $T>0$

✓ for any sequence $(H^n)_{n\geq 1}$ in $S([0,T],\mathbb{F})$ of simple predictable processes

$$\left(\sup_{(t,\omega)\in[0,T]\times\Omega} |H^n_t(\omega) - H_t(\omega)| \xrightarrow{n\to\infty} 0 \right) \Rightarrow \left(\int_0^T H^n \cdot dX \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^T H \cdot dX \right)$$

Uniform convergence on $[0,T]\times\Omega$ of H^n

Convergence in probability of $\int_0^T H^n dX$

✓ By linearity: enough to check for $H=0$

✓ Enough to check for $H^n \in S$ bounded, $\sup|H^n| \to 0$

Semi-martingales as 'good' integrators

or in other words:

X is a semimartingale if : $\forall T>0$

the stochastic integral of simple processes with respect to X

topology of uniform convergence on $[0,T]\times\Omega$ | topology of convergence in probability

$$\left(S([0,T],\mathbb{F}),\ \|\ \|_\infty\right) \longrightarrow \left(L_0(\Omega,\mathcal{F},\mathbb{P}),\ d_{\mathbb{P}}\right)$$

$$H \longrightarrow \int_0^T H\,dX$$

is continuous

Semi-martingales : examples

✓ **Theorem** : Any non-anticipative cadlag process with paths of finite variation is a semimartingale.

Proof: for any $H \in S([0,T], \mathbb{F})$,

$$\left| \int_0^T H\,dX \right| = \left| \sum_{i=0}^{n-1} H_{T_i} (X_{T_{i+1}} - X_{T_i}) \right| \leq \sup_i |H_{T_i}| \sum_{i=0}^{n-1} |X_{T_{i+1}} - X_{T_i}|$$

$$\leq \underbrace{\sup_{[0,T]\times\Omega} |H_t(\omega)|}_{\|H\|_\infty} \cdot V_X([0,T]) \longrightarrow 0$$

$$\|H\|_\infty \to 0$$

So if $H^n \in S([0,T],\mathbb{F})$ with $\|H^n\|_\infty \to 0$ then $\left| \int_0^T H^n dX \right| \xrightarrow{n\to\infty} 0$ a.s.

- When X has finite variation and

$$\sup_{(t,\omega)\in[0,T]\times\Omega} |H^n(t,\omega) - H(t,\omega)| \xrightarrow{n\to\infty} 0$$

✓ $\int_0^T H\,dX$ is well defined as pathwise Riemann-Stieltjes integral

✓ $\int_0^T H^n\,dX$ are Riemann sums approximating $\int_0^T H\,dX$

and $\int_0^T H^n dX \xrightarrow{n\to\infty} \int_0^T H\,dX$ is in fact pathwise (holds $\mathbb{P}$-a.s.)

- Example: counting processes

Let $(T_n)_{n\geq 0}$ be an increasing sequence of stopping times with $T_0 = 0$.

The integer-valued process $N_t = \sum_{n\geq 1} 1_{t \geq T_n}$ is called a 'counting process':

it counts how many events occured during $[0,t]$

If $\sup_{n\geq 0} T_n = \infty$ a.s. N is said to be non-explosive

A non-explosive counting process is a semimartingale.

it is cadlag, adapted and increasing (thus, of bounded variation).

Semi-martingales : examples

Theorem A cadlag square-integrable martingale is a semimartingale

Proof : For $H \in S([0,T],\mathbb{F})$ $E\left(\int_0^T H\,dX\right)^2 = E\left[\left(\sum_{i=0}^{n-1} H_{T_i}(X_{T_{i+1}} - X_{T_i})\right)^2\right]$

$= \sum_{i=0} E[H_{T_i}^2 (X_{T_{i+1}} - X_{T_i})^2] + 2 \sum_{i>j} E[H_{T_i} H_{T_j} (X_{T_{i+1}} - X_{T_i})(X_{T_{j+1}} - X_{T_j})]$

For $i > j$, $T_i \geq T_j$ a.s. so H_{T_j}, H_{T_i} and $X_{T_{j+1}} - X_{T_j}$ are $\mathcal{F}_{T_i}$ measurable, so

$E[H_{T_i} H_{T_j} (X_{T_{i+1}} - X_{T_i})(X_{T_{j+1}} - X_{T_j})] = E[E[H_{T_i} H_{T_j} (X_{T_{i+1}} - X_{T_i})(X_{T_{j+1}} - X_{T_j}) \mid \mathcal{F}_{T_i}]]$

$= E[H_{T_i} H_{T_j} (X_{T_{j+1}} - X_{T_j}) E[(X_{T_{i+1}} - X_{T_i}) \mid \mathcal{F}_{T_i}]] = 0$ since

$E[X_{T_{i+1}} \mid \mathcal{F}_{T_i}] = X_{T_i}$ by Doob's optional sampling theorem since T_i may be assumed bounded w.l.o.g.

$$E\left[\left(\int_0^T H\,dX\right)^2\right] = E\left[\sum_{i=0}^{n-1} H_{T_i}^2 (X_{T_{i+1}} - X_{T_i})^2\right]$$

$$\leq \sup_{[0,T]\times\Omega} |H_t(\omega)|^2 \cdot E\sum_{i=0}^{n-1} (X_{T_{i+1}} - X_{T_i})^2 = \|H\|_\infty^2 \, E(X_T - X_0)^2$$

since $$E\left[\sum_{i=0}^{n-1} (X_{T_{i+1}} - X_{T_i})^2\right] = E\left[\left(\sum_{i=0}^{n-1} (X_{T_{i+1}} - X_{T_i})\right)^2\right]$$ (using the above result for $H=1$)

$$= E(X_{T_n} - X_0)^2$$

so: $$E\left[\left(\int_0^T H\,dX\right)^2\right] \leq \|H\|_\infty^2 \, E(X_T - X_0)^2 \xrightarrow[\|H\|_\infty \to 0]{} 0$$

L^2-convergence $\Rightarrow$ conv in probability so $\int_0^T H\,dX \xrightarrow[\|H\|_\infty \to 0]{P} 0$

Brownian motion : L^2 isometry formula

The above result implies that a Brownian motion is a semimartingale and

$$\forall H \in S([0,T],\mathbb{F}), \quad E\Big[\Big(\int_0^T H\,dW\Big)^2\Big] = E\Big[\int_0^T H_t^2\,dt\Big]$$

Proof : As shown in the previous proof, if $H = H_0 1_{\{0\}} + \sum_{i=0}^{n-1} H_{T_i} 1_{(T_i, T_{i+1}]}$,

$E\big(\int_0^T H\,dW\big)^2 = E\big[\sum_0^{n-1} H_{T_i}^2 (W_{T_{i+1}} - W_{T_i})^2\big]$. Conditioning on $\mathcal{F}_{T_i}$, we get

$$\sum_0^{n-1} E\big[E\big[H_{T_i}^2 (W_{T_{i+1}} - W_{T_i})^2 \mid \mathcal{F}_{T_i}\big]\big] = \sum_0^{n-1} E\big[H_{T_i}^2 \underbrace{E\big[(W_{T_{i+1}} - W_{T_i})^2 \mid \mathcal{F}_{T_i}\big]}_{= T_{i+1} - T_i}\big]$$

$$= E\Big[\sum_{i=0}^{n-1} H_{T_i}^2 (T_{i+1} - T_i)\Big] = E\Big[\int_0^T H_t^2\,dt\Big]$$

Localization

We say that a process X verifies 'locally' a property $(\mathcal{P})$ if there exists an increasing sequence $0 \leq T_1 \leq T_2 \leq \cdots \leq T_n \xrightarrow[n\to\infty]{} \infty$ of stopping times with $T_n \xrightarrow[n\to\infty]{} \infty$ a.s. such that

for each $n \geq 1$, $X^{T_n}_t = X_{t \wedge T_n}$ verifies $(\mathcal{P})$.

Ex. An adapted process X is called a **local martingale** if $\exists\, (T_n)_{n\geq 1}$ stopping times $T_n \xrightarrow{n\to\infty} \infty$ a.s. such that $\forall n \geq 1$, X^{T_n} is a martingale.

- **Ex:** X is said to be locally square-integrable if $\exists (T_n)_{n\geq 1}$ stopping times $T_n \xrightarrow{n\to\infty} \infty$ a.s. such that $\forall n \geq 1$, X^{T_n} is square-integrable.

Ex. X is said to be locally bounded if there exists a sequence $(T_n)_{n\geq 1}$ of stopping times $T_n \underset{n\to\infty}{\longrightarrow} \infty$ a.s. such that $\forall n \geq 1$, X^{T_n} is bounded.

'Locality' of the semimartingale property

- Let X be a cadlag adapted process. $(T_n)_{n\geq 1}$ an increasing sequence of positive variables with $T_n \xrightarrow[n\to\infty]{} \infty$ a.s.

  If there exists a sequence (X^n) of semimartingales such that $X^{T_n-} = (X^n)^{T_n-}$ (that is: $X_{t\wedge T_n-} = X^n_{T_n-\wedge t}$, $\forall t\geq 0$)

  then X is a semimartingale.

- In particular: a 'local semimartingale' is a semimartingale.

  If $\forall n\geq 1$, X^{T_n} is a semimartingale for some stopping times $T_n \to \infty$ a.s. then X is a semimartingale

Proof: Let $T>0$; $H^k \in S([0,T], \mathbb{F})$, with $\sup_{[0,T]\times\Omega} |H_t^k(\omega)| \xrightarrow{k\to\infty} 0$.

$X^{T_n-} = (X_n)^{T_n-}$

$= \int_0^{T_n} H^k dX$

$$\mathbb{P}\left(\left|\int_0^T H^k dX\right| \geq c\right) \leq \mathbb{P}\left(\left|\int_0^T H^k dX^n\right| \geq c\right) + \overbrace{\mathbb{P}(T_n \leq T)}^{\xrightarrow{n\to\infty} 0}$$ since $T_n \to \infty$ a.s.

$X^n = X^{T_n}$

$\forall \varepsilon > 0$, $\exists n_0 \geq 1$, $\forall n \geq n_0$, $\mathbb{P}(T_n \leq t) \leq \frac{\varepsilon}{2}$

Since X^n is a semimartingale $\mathbb{P}\left(\left|\int_0^T H^k dX^n\right| \geq c\right) \xrightarrow{n\to\infty} 0$ so

$\exists n_1 \geq 1$, $\forall k \geq n_1$, $\mathbb{P}\left(\left|\int_0^T H^k dX^{n_0}\right| \geq c\right) \leq \frac{\varepsilon}{2}$

Consequence: If X is a semimartingale T a stopping time the $X^T = (X_{t\wedge T})_{t\geq 0}$ is also a semimartingale.

Semi-martingales : examples

- Any locally square-integrable local martingale is a semimartingale

- If $X = M + A$ where

  M is a (locally) square-integrable (local) martingale

  A is a cadlag adapted process with finite variation

  then X is a semimartingale

  Ex. Brownian motion + finite variation process

  $$X_t = W_t + A_t$$

Stochastic integrals : extension to caglad integrands

$D([0,\infty), \mathbb{R})$ cadlag (right-continuous functions with left limits)

$\mathbb{D}$: space of non-anticipative ($\mathbb{F}$-adapted) processes with cadlag paths

$\mathbf{L}$: space of non-anticipative processes with caglad paths (left-continuous with right limits)

Rem: if $X \in \mathbb{D}$, $Y_t = X_{t-} = \lim_{\substack{s \uparrow t \\ s < t}} X_s$ then $Y \in \mathbf{L}$

- If X is a semimartingale we can define $\int Y \, dX$ for $Y \in \mathbf{L}$

Uniform convergence in probability on compacts (UCP)

A sequence $(X^n)_{n\geq 1}$ of processes converges uniformly in probability on compacts to X if

$$\forall \varepsilon > 0, \quad \forall t \geq 0, \quad \mathbb{P}\left(\sup_{s\leq t} |X^n_s - X_s| > \varepsilon\right) \xrightarrow{n\to\infty} 0$$

We write $X^n \xrightarrow[\mathbb{P}]{ucp} X$

$$\Leftrightarrow \quad \sup_{s\in[0,t]} |X^n_s - X_s| \xrightarrow[n\to\infty]{\mathbb{P}} 0. \qquad \forall t \geq 0$$

ucp convergence is metrizable by: $d_{ucp}(X,Y) = \sum_{n\geq 1} \frac{1}{2^n} E\left[\min\left(1, \left(\overrightarrow{X-Y}\right)_n\right)\right]$

$(\mathbb{D}, d_{ucp})$ is a complete metric space.

Theorem : $S([0,\infty))$ is dense in $(\mathbf{L}, d_{ucp})$: for any $Y\in$

there exists a sequence $(Y^n)_{n\geq 1}$ of simple predictable process

such that $Y^n \xrightarrow[n\to\infty]{ucp} Y$.

Proof : Let $U^n_t = Y_{t\wedge R_n}$ where $R_n = \inf\{t, |Y_t| > n\}$. $U^n \in \mathbf{L}$ and U^n b

$U^n \xrightarrow[n\to\infty]{ucp} Y$, so it is enough to show the result for Y bound

Define $Z_t = \lim_{u\downarrow t} Y_u = Y_{t+}$. Then $Z\in\mathbb{D}$ is cadlag. Defin

$$T^\varepsilon_{n+1} = \inf\{t > T^\varepsilon_n, \ |Z_t - Z_{T^\varepsilon_n}| > \varepsilon\}, \quad T^\varepsilon_0 = 0$$

Since Z is cadlag, T^ε_n are stopping times.

$$Y^{n,\varepsilon} = Y_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} Z_{T_i^\varepsilon} \mathbb{1}_{(T_i^\varepsilon \wedge n,\, T_{i+1}^\varepsilon \wedge n]} \in S$$ is a simple predict process

then: $$\sup_{s\in[0,t]} \left| Y_s^{n,\varepsilon} - Y_s \right| \leq \varepsilon$$

So: $\forall Y \in \mathbf{L}$, $\exists Y^{n,\varepsilon} \in S$, $\sup_{s\in[0,t]} |Y_s^{n,\varepsilon} - Y_s| \leq \varepsilon$

Theorem: For any semimartingale X,

$$I_X : (S, d_{ucp}) \longrightarrow (\mathbb{D}, d_{ucp})$$ is continuous.

Proof: First, we show that if $\sup_{[0,T]\times\Omega} |H^k_t(\omega)| \xrightarrow{k\to\infty} 0$ then

$$\int_0^{\cdot} H^k dX \xrightarrow{ucp} 0$$

For $\delta > 0$, define the stopping time: $T^k = \inf\{ t \geq 0, \ |\int_0^t H^k dX| \geq \delta \}$

$H^k 1_{[0,T^k]} \in S$ and $\| H^k 1_{[0,T^k]} \|_\infty \xrightarrow{k\to\infty} 0$

$$\forall t \geq 0, \quad \mathbb{P}\left(\overline{\left|\int_0^{\cdot} H^k dX\right|}_t > \delta \right) \leq \mathbb{P}\left(\left|\int_0^t H^k dX^{T^k}\right| \geq \delta \right)$$

where $\overline{U}_t = \sup_{0\leq s\leq t} U_s$

$$= \mathbb{P}\left(\left|\int_0^T H^k 1_{[0, T^k \wedge t]} dX\right| \geq \delta \right) \xrightarrow{k\to\infty} 0$$

So $I_X : (S, \| \ \|_\infty) \longrightarrow (\mathbb{D}, d_{ucp})$ is continuous.

• Now take $(H^k)_{k\geq 1}$ $H^k \xrightarrow[k\to\infty]{ucp} 0$. H^k bounded : $\|H^k\|_\infty \leq C$
Let $t>0$, $\delta>0$.

$$\mathbb{P}\left(\left|\int_0 H^k dX\right|_t \geq \delta\right) \leq \mathbb{P}\left[\left(\int H^k 1_{[0,R_k]}\, dX\right)_t \geq \delta\right] + \mathbb{P}(R_k \leq t)$$

where $R_k = \inf\{u,\ |H^k_u| > \eta\}$. is a stopping time.

$$\sup_{[0,T]\times\Omega} |H^k_t 1_{[0,R_k]}(t)| \leq \eta \xrightarrow[\eta\to 0]{} 0 \quad \text{so}$$

$$\exists \eta>0,\ \mathbb{P}\left[\left(\int_0 H^k 1_{[0,R_k]}\, dX\right)_t \geq \delta\right] \leq \frac{\varepsilon}{2}.$$

Since (H^k) is uniformly bounded, $\exists k_0 \geq 1$, $k \geq k_0$, $\mathbb{P}(R_k < t) \leq \frac{\varepsilon}{2}$.

So then: $\mathbb{P}\left(\left|\int H\, dX\right|_t \geq \delta\right) \leq \varepsilon$

Extension of the stochastic integral

Let X be a semimartingale, $Y \in \mathbf{L}$ be a left-continuous adapted process. Define the stochastic integral $I_X(Y) = \int Y\,dX$ by

$$\int_0^t Y\,dX = \lim_{n\to\infty} \int_0^t Y^n dX \quad \text{where } Y^n \in S \text{ is any}$$

sequence of simple predictable processes with $Y^n \xrightarrow[n\to\infty]{ucp} Y$.

i) $I_X : (\mathbf{L}, ucp) \to (\mathbb{D}, ucp)$ is continuous.

ii) $I_X(Y)$ does not depend on the approximating sequence $(Y^n)_{n\geq 1}$.

In particular we may choose $Y^n = Y_0 1_{\{0\}} + \sum_{T_i^n} Y_{T_i^n} 1_{(T_i^n, T_{i+1}^n]} \in S$

Then: $\int_0^t Y^n dX = \sum Y_{T_i^n} (X_{T_{i+1}^n} - X_{T_i^n})$

Stochastic integral as limit of non-anticipative Riemann sums

Let $(\Pi^n)_{n\geq 1}$ be a sequence of random partitions where

$\Pi^n = (T_0^n = 0 \leq T_1^n \leq \dots \leq T_{k_n}^n)$ is a sequence of stopping times

i) $\limsup T_k^n = \infty$ a.s.

ii) $\|\Pi^n\| = \sup_k |T_{k+1}^n - T_k^n| \longrightarrow 0$ a.s.

then $\forall Y \in \mathbb{D}$
or $Y \in \mathbb{L}$,

$$\sum_{T_i^n \in \Pi^n} Y_{T_i^n} \left(X_{T_{i+1}^n} - X_{T_i} \right) \xrightarrow[n\to\infty]{ucp} \int_0^T Y_{t-} \, dX_t$$

Proof: use $Y^n = \sum_{i=0}^{k_n - 1} Y_{T_i^n} 1_{(T_i^n, T_{i+1}^n]}$. $Y^n \in S$ and $Y^n \xrightarrow[n\to\infty]{ucp} Y$.

Case of finite variation processes

If X is a semimartingale with finite variation paths, $Y \in \mathbf{L}$ then the stochastic integral $\int_0^{\cdot} Y_- \, dX$ is a modification of the pathwise Riemann-Stieltjes integral of Y_- wrt X.

How does this construction depend on the probability measure $\mathbb{P}$?

Let $\mathbb{Q} \sim \mathbb{P}$: $\forall A \in \mathcal{F}$, $\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$

$$S([0,T], \mathbb{F}, \mathbb{P}) = S([0,T], \mathbb{F}, \mathbb{Q})$$

: $\mathbb{P}, \mathbb{Q}$ only enter in the requirement that H_{T_i} are a.s. bounded.

- The def of $\mathbb{P}$-semimartingales, involves convergence in probability of $\int_0^T H\,dX$ where $H \in S([0,T], \mathbb{F})$

  so X $\mathbb{P}$-semimartingale $\iff$ X $\mathbb{Q}$-semimartingale

- The notion of UCP convergence under $\mathbb{P}$ and $\mathbb{Q}$ are the same.

Stability with respect to changes of probability measure

Let X be a $\mathbb{P}$-semimartingale, $H \in \mathbf{L}$

If $\mathbb{Q} \sim \mathbb{P}$ are equivalent measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ then the stochastic integral $\int_{\mathbb{Q}} H \, dX$ constructed under $\mathbb{Q}$ is indistinguishable from the stochastic integral $\int_{\mathbb{P}} H \, dX$ constructed under $\mathbb{P}$.

Preservation of local martingale property

✓ If X is a (locally) square-integrable (local) martingale then for any $H \in \mathbf{L}$, $\left(\int_0^t H\,dX\right)_{t \geq 0}$ is also a locally square-integrable local martingale.

Remark : stochastic integration does not preserve in general the martingale property : if X is a square-integrable martingale $H \in \mathbf{L}$, $\int H\,dX$ is a *local martingale*, not a martingale.

(More on this later)

Proof: by localization it is sufficient to prove the result when X is a square-integrable martingale, $H \in \mathbf{L}$ bounded: $|H| \leq c$

$$\theta_n = \inf\{t>0, \|X_t\|_{L^2} + \|H_t\|_\infty \geq n\}$$

Let $H^n \in S$, $H^n \xrightarrow[ucp]{n\to\infty} H$, $|H^n| \leq c$ $\quad H^n = \sum H^n_{T_i^n} \mathbb{1}_{(T_i^n, T_{i+1}^n]}$

Then $M^n_t = \int_0^t H^n \cdot dX$ is a martingale for each $n \geq 1$ by the martingale preserving property of $I_X : S \to \mathbb{D}$, $\forall t \leq T$

$$E\left(\int_0^t H^n dX\right)^2 = E\left(\sum_{i=1}^{k_n} H^n_{T_i^n}\left(X_{t\wedge T_{i+1}^n} - X_{T_i^n \wedge t}\right)\right)^2 \leq c^2 \underbrace{E(X_t^2 - X_0^2)}_{\text{since } X^2 \text{ sq.-integrable martingale}} \leq c^2 E(X_T^2)$$

So $(M^n)_{n\geq 1}$ is a sequence of square-integrable martingales uniformly bounded in L^2, thus uniformly integrable.

so $M = \lim_{n\to\infty} M^n$ is also a square-integrable martingale.

Consider now the case of a general $H \in \mathbf{L}$, not necessarily bounded. Let $M_t = \int_0^t H\, dX$

Define $Z_t = H_{t+} = \lim_{u \downarrow t} H_u$. Then $Z \in \mathbb{D}$ is cadlag, adapted.

and $T_k = \inf\{ t > 0, |Z_t| > k \}$ is a stopping time.

Then the process $H_t^{T_k} = H_{t \wedge T_k}$ is bounded by k

and the above result implies that $\int H^{T_k} \cdot dX$ is a square-integrable martingale.

But $\int_0^t H^{T_k} \cdot dX = \left(\int_0^{t \wedge T_k} H \cdot dX \right) = M_{t \wedge T_k}$ so $\left(M_{t \wedge T_k} \right)_{t \geq 0}$ is a square-integrable martingale for all $k \geq 1$: M is a locally sq-integrable local martingale.

Actually we have shown in the proof that

- $H \in \mathbb{L}$ is bounded
- X square-integrable martingale

Then $\int_0^{\cdot} H \, dX$ is a square-integrable martingale.

References

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