# Stochastic Calculus R Cont

Note Title

## Lecture 4: Quadratic variation

Ito processes.
Quadratic variation of a semimartingale: properties, examples.
Isometry property.
Quadratic covariation
Integration by parts for stochastic integrals: Ito product formula
Integrals with respect to quadratic variation.
Quadratic Riemann sums
L2 stochastic integral with respect to a square integrable martingale.
Kunita - Watanabe inequality.

Let $W$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.
As observed in Lecture 3, Brownian motion is a prime example of semimartingale.

✓ Thus, for any caglad (left-continuous) adapted process $\sigma \in \mathbf{L}$, we can define $\int_0^t \sigma \cdot dW$

✓ For any $\sigma \in \mathbf{L}$, $\int \sigma \cdot dW$ is a local martingale.

✓ If $(b_t)_{t \geq 0}$ is a cadlag adapted process then $\left(\int_0^t b_u \, du\right)_{t \geq 0}$ is adapted cadlag with bounded variation.

$X_t = X_0 + \int_0^t \sigma \, dW + \int_0^t b_u \, du$ is thus a semimartingale.

# Ito processes

An Ito process is a semimartingale $X$ which may be represented as $X_t = X_0 + \int_0^t \sigma_u \, dW_u + \int_0^t b_u \, du$

where $W$ is a Brownian motion

$\sigma \in \mathbb{L}$ left-continuous adapted process

$b \in \mathbb{D}$ cadlag adapted process

$X_0$ $\mathcal{F}_0$-measurable random variable.

$X_t = X_0 + M_t + A_t$ $M$ continuous local martingale
$A$ adapted, bounded variation process

# $L^2$ extension of Brownian stochastic integral

$\forall H \in S([0,T],\mathbb{F})$, $\quad E\left[\left(\int_0^T H\,dW\right)^2\right] = E\left[\int_0^T H_t^2\,dt\right]$

Define, for $H : \Omega \times [0,T] \to \mathbb{R}$

$$\|H\|_{L^2(\Omega\times[0,T])} = E\left[\int_0^T H_t^2\,dt\right] = \overbrace{\int dt\,d\mathbb{P}(\omega)\,H_t^2(\omega)}^{L^2 \text{ norm w.r.t. } dt\times d\mathbb{P}}$$

Let $\mathcal{L}^2_T(\Omega, \mathcal{F}, \mathbb{F}=(\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be the closure of $S([0,T], \mathbb{F}, \mathbb{P})$ in $L^2(\Omega\times[0,T], dt\times d\mathbb{P})$

$\mathcal{L}^2_T(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is the set of square-integrable, $\mathbb{F}$-adapted processes which may be approximated in $L^2$ by a sequence of simple predictable processes.

Given $H \in \mathcal{L}^2_T(\Omega, \mathbb{F}, \mathbb{P})$, there exists therefore $(H^n)_{n\geq 1} \in S([0,T], \mathbb{F}, \mathbb{P})$ such that $E\left[\int_0^T \|H_t^n - H_t\|^2 dt\right] \xrightarrow{n\to\infty} 0$

$$\int_0^T H^n dW \in L^2(\Omega, \mathcal{F}, \mathbb{P})$$

We **define**

$$\int_0^T H\,dW = \lim_{n\to\infty} \int_0^T H^n dW \quad \text{in } L^2$$

This limit does not depend on the choice of $(H^n)_{n\geq 1}$ and verifies

$$E\left|\int_0^T H\,dW\right|^2 = E\left[\int_0^T H_t^2\,dt\right]$$

- For $H \in \mathcal{L}^2_T(\Omega, \bar{\mathbb{F}}, \mathbb{P})$, $\int_0^T H \, dW$ is called the $L^2$-extension of the Itô integral. It is due to Kunita–Watanabe.
- For $H \in \mathcal{L}^2_T(\Omega, \mathbb{F}, \mathbb{P}) \cap \mathbb{L}$ then the two constructions coincide.

  $\uparrow$ càglàd
- The $L^2$ construction does not involve pathwise regularity of integrands.

# $L^2$ stochastic integral : indefinite case

Let $\mathcal{L}^2(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ the closure in $L^2(\Omega \times [0,\infty), dt \times d\mathbb{P})$

of $S(\mathbb{F}, \mathbb{P})$ $\leftarrow$ simple predictable processes defined on $[0,\infty)$

If $\sigma \in \mathcal{L}^2(\Omega, \mathbb{F}, \mathbb{P})$, then

✓ $\int_0^T \sigma \, dW$ may be defined for each $T$ as above

✓ $E\left|\int_0^T \sigma \, dW\right|^2 = E\left[\int_0^T \sigma_t^2 \, dt\right] < \infty \quad \forall T > 0$

✓ $\sup\limits_{T>0} \left(\int_0^T \sigma \, dW\right)^2 = E\int_0^\infty \sigma_t^2 \, dt = \|\sigma\|_{L^2(\Omega \times [0,\infty), dt \times d\mathbb{P})} < \infty$

So $M_t = \int_0^t \sigma \, dW$ is a square-integrable martingale

# Ito Isometry Formula

Let $X_t = X_0 + \int_0^t \sigma . dW + \int_0^t b_u du$ an Ito process with

$$\sigma \in \mathcal{L}^2_T(\Omega, \mathbb{F}, \mathbb{P}) \qquad b \text{ adapted process}$$

$$\forall T>0, \qquad \mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] < \infty \qquad \mathbb{E}\left[\int_0^T b_t dt\right] < \infty$$

Then $X$ is square integrable and :

$$\mathbb{E}[X_t] = \mathbb{E}\left[\int_0^t b_u du\right] \qquad var(X_t) = \mathbb{E}\left[\int_0^t \sigma_u^2 du\right]$$

Furthermore : $M = \left(\int_0^t \sigma_u dW_u\right)_{t \in [0,T]}$ is a square-integrable martingale.

and $\quad \|M\|_{M^2_T} = \|\sigma\|_{L^2(\Omega \times [0,T])} = \mathbb{E}\left[\int_0^T \sigma_t^2 dt\right]$

# Quadratic variation of a semimartingale

Let $X$ be a cadlag semimartingale, $T>0$.

Consider a sequence $(\Pi^n)_{n\geq 1}$ of random partitions

$\Pi^n = (0 = T_0^n \leq T_1^n \leq \dots \leq T_{k_n-1}^n \leq T_{k_n}^n = T)$ of $[0,T]$

with $|\Pi^n| = \sup |T_{i+1}^n - T_i^n| \xrightarrow[n\to\infty]{} 0$ a.s. where $(T_i^n)$ are stopping times.

$$(X_{T_{i+1}} - X_{T_i})^2 = X_{T_{i+1}}^2 - X_{T_i}^2 - 2X_{T_i}(X_{T_{i+1}} - X_{T_i}) \quad \forall i\geq 1, \text{ so}$$

$$\sum_{i=1}^{k_n} (X_{T_{i+1}} - X_{T_i})^2 = X_T^2 - X_0^2 - 2\sum_{i=1}^{k_n} X_{T_i}(X_{T_{i+1}} - X_{T_i})$$

Since $X$ is a semi-martingale: $\sum_{T_i\in\Pi^n} X_{T_i^n}(X_{T_{i+1}^n} - X_{T_i^n}) \xrightarrow{ucp} \int_0^T X_{t-}\,dX$

$\sum_{\Pi^n} (X_{T_{i+1}^n} - X_{T_i^n})^2$ converges in probability as $n\to\infty$ to a limit which does not depend on $\Pi = (\Pi^n)_{n\geq 1}$ a.s.

$$[X]_T := \lim_{n\to\infty} \sum_{T_i^n \in \Pi^n} (X_{T_{i+1}} - X_{T_i})^2$$ exists and defines

a process $([X]_t)_{t\geq 0}$ called the quadratic variation of $X$.

which verifies : $$[X]_T = X_T^2 - X_0^2 - 2\int_0^T X_{t-} \cdot dX_t \quad \text{a.s.}$$

$$X_T^2 - X_0^2 = 2\int_0^T X_- \cdot dX + [X]_T$$

$$\neq 2\int_0^T X_- dX \qquad \text{unless} \quad [X]_T = 0$$

# Properties of quadratic variation   $X$ semimartingale

i) $[X]$ is a cadlag, adapted process with increasing paths.
(so has finite variation)

ii) $[X]_t - [X]_{t-} = (X_t - X_{t-})^2 = (\Delta X_t)^2$

In particular if $X$ has continuous paths then $[X]$ is continuous.

ii) For any stopping time $\tau$, $[X^\tau]_t = [X]_{t\wedge\tau}$

iii) If $X$ has continuous paths of finite variation, $[X] = 0$.

Proof: ii) $\Delta\left(\int Y\,dX\right) = Y\,\Delta X$ so

$$[X]_T - [X]_{T-} = \Delta(X_T^2) - 2X_{T-}\,\Delta X_T = X_T^2 - X_{T-}^2 - 2X_{T-}(X_T - X_{T-}) = (X_T - X_{T-})^2$$

iii) $\displaystyle\int_0^{t\wedge\tau} Y\,dX = \int_0^t Y\,dX^\tau$

- $[a,b] \to [X]_b - [X]_a$ defines a (positive) measure on $[0,\infty)$
- Since $[X]$ is an increasing process, its paths can be decomposed into a continuous part and its jumps:

$$[X]_t = [X]^c_t + \sum_{0<s\leq t} (\Delta X_s)^2 \qquad \text{(Lebesgue decomposition)}$$

The increasing process $[X]^c$ is called the **continuous quadratic variation** of $X$.

Ex. Brownian motion $W$: $[W]_t = t = [W]^c_t$

Ex. Counting process $N$: $N_t = \sum_{n\geq 1} 1_{t \leq T_n}$ $\quad (T_n)_n \uparrow \infty$ $\; n\to\infty$

$$[N]_t = N_t \qquad [N]^c_t = 0$$

# Continuous quadratic variation: properties

Let $X$ be a semimartingale with $X_0 = 0$.

0) $[X]^c$ is increasing and continuous a.s.

1) If $X$ is cadlag, adapted, with paths of finite variation then $[X]^c = 0$ a.s.

2) If $X$ is a continuous local martingale.

Then (i) $X_t^2 - [X]_t$ is a continuous local martingale.

(ii) $(\forall t \geq 0,\ [X]_t = 0)$ a.s. $\iff$ $(\forall t \geq 0,\ X_t = 0)$ a.s.

Proof: 1) Since $X$ has finite variation, Riemann – Stieltjes change of variable formula applies: $[X]_t = X_t^2 - 2\int_0^t X_- \, dX = \sum (\Delta X_s)^2$ so $[X]^c = 0$

2) $X_t^2 - [X]_t = X_0^2 + 2\int_0^t X_{u-} \, dX_u \qquad (X_{u-})_{u \geq 0} \in L$

So if $X$ is a local martingale by the local martingale preserving property $X_t^2 - [X]_t$ is a local martingale.

Furthermore $\Delta(X^2 - [X]) = \Delta(X^2) - \Delta[X] = 0$

since $(\Delta[X])_t = (\Delta X_t)^2$ so $\Delta(X^2 - [X]) = 0$

$\Rightarrow$ $X^2 - [X]$ is a.s. continuous.

$\{$Continuous local martingales$\}$ $\cap$ $\{$Processes of finite variation$\}$ = Constants

Proposition: If $X$ is a continuous local martingale, then for any stopping times $T_1 \le T_2$, if $X$ has paths of finite variation on $[T_1, T_2]$ then $X$ is constant on $[T_1, T_2]$.

Proof: $M = X^{T_2} - X^{T_1}$ is a continuous local martingale

Since $M$ has finite variation $[M] = 0$

Since $[M] = [X]^{T_2} - [X]^{T_1}$ this implies $[X]^{T_2} = [X]^{T_1}$

So, $t^n_k = \frac{k}{2^n}$ $\quad \sum_k (X_{t^n_{k+1} \wedge T_2} - X_{t^n_k \wedge T_2})^2 - \sum_k (X_{t^n_{k+1} \wedge T_1} - X_{t^n_k \wedge T_1})^2 \xrightarrow[n \to \infty]{} 0$

So $X$ is constant on $[T_1, T_2]$.

# Pure-jump semimartingales

A semimartingale $X$ such that $[X]^c = 0$ is called a **pure-jump semimartingale**.

Ex. Poisson process $(N_t)_{t\geq 0}$ $\quad [N]_t = N_t$

$X_t = N_t - \lambda t \qquad [X] = [N] = N$

Rem: for a general semimartingale $X$ one cannot decompose $X$ into a "continuous" component and 'jumps': such a decomposition is not unique.

Ex. $N_t = (N_t - \lambda t) + \lambda t = X_t + \lambda t$

This decomposition is unique for $[X]$ since $t \to [X]_t$ is increasing.

**Proposition:** Let $M$ be a local martingale.

$M$ is a square-integrable martingale $\iff$ $\forall t \geq 0$, $E([M]_t) < \infty$

In this case $E(M_t^2) = E([M]_t)$

Proof: $\Rightarrow$ If $M$ martingale with $E(M_t^2) < \infty$, $Z_t = M_t^2 - [M]_t = 2\int_0^t M_{s-}\, dM_s$

is a locally square-integrable: there exist stopping times $T_n \uparrow \infty$ such that $Z^{T_n}$ is a square-integrable martingale. Then

$$E(Z_t^{T_n}) = E(Z_0) = 0 \quad \text{so} \quad E(M^2_{t\wedge T^n}) = E([M]_{t\wedge T^n})$$

By Doob's quadratic inequality for martingales:

$$E\,|\sup_{[0,t]} M|^2 \leq 4\,E(M_t^2) < \infty$$

So we can apply the dominated convergence theorem to interchange $E(\ )$ and $n \to \infty$:

$$E(M_t^2) = \lim_{n\to\infty} E(M^2_{t\wedge T^n}) = \lim_{n\to\infty} E([M]_{t\wedge T^n}) = E([M]_t)$$

by the Monotone Convergence Theorem.

$\Leftarrow$ : Assume $E([M]_t) < \infty$ and define $T^n = \inf\{t>0, |M_t|>n\} \wedge n$

$T^n \uparrow_{n\to\infty} \infty$ are stopping times and: $\sup_{t\in[0,T^n]} |M_t| \leq n + |\Delta M_{T^n}| \leq n + \sqrt{[M]_n} < \infty$

Since: $[M]_{T_n} = [M]_{T_n-} + (\Delta M_{T_n})^2$ so $|\Delta M_{T_n}| \leq \sqrt{[M]_{T_n}} \leq \sqrt{[M]_n}$

So $M^{T^n}$ is uniformly integrable $\left\{\sup_{t\geq 0} |M_t^{T_n}|\right\} \leq n + \sqrt{[M]_n}$

and $E(M_t^{T_n})^2 \leq E \sup_{t\geq 0} |M^{T_n}|^2 < \infty$

Applying the first part of the theorem, $E\left[(M^{T_n}_t)^2\right] = E\left([M^{T_n}]_t\right)$

Again, by Doob's maximal inequality:

$$E\left(\sup_{[0,t]} |M^{T_n}|^2\right) \leq 4\, E\left(M^2_{t\wedge T_n}\right) = 4\, E\left([M^{T_n}]_t\right)$$
$$\leq 4\, E\left([M]_t\right) < \infty$$

So by the monotone convergence theorem $(n \to \infty)$:

$$E \sup_{[0,t]} |M_t|^2 = \lim_{n\to\infty} E\left(\sup_{[0,t]} |M^{T_n}|^2\right) \leq 4\, E\,[M]_t < \infty$$

**Note:** a continuous adapted process is locally bounded.

Proof: Let $X$ be $\begin{cases}\text{continuous} \\ \text{adapted}\end{cases}$ and define

$T_n = \inf \{ t \geq 0, \; |X_t| \geq n \}$

$T_n$ is a hitting time of an open set, so a stopping time.

By definition of $T_n$, $\sup_{t \in [0, T_n)} |X_t| < n$

Since $X$ is continuous a.s.: $\mathbb{P}\left( \sup_{[0,T_n]} |X_t| \leq n \right) = 1$

**Theorem:** If $X$ is a continuous (local) martingale then $X$ and $[X]$ are constant on the same intervals a.s.

**Proof:** $u \in \mathbb{Q}$. Let $T_u = \inf\{t > u,\ X_t \neq X_u\}$.

$X^{T_u}$ is a bounded martingale.

$$E(X_{T_u} - X_u)^2 = E(X_{T_u}^2) - E(X_u^2)$$ using Doob's optional sampling theorem + martingale property

$$= E([X]_{T_u}) - E([X]_u) \geq 0$$ since $[X]$ is increasing

so $X_{T_u} = X_u$ a.s. $\Rightarrow$ $[X]_{T_u} = [X_u]$ a.s.

Conversely: by stopping $X$ at $T_n = \inf\{t > 0, |X_t| > n\}$ we obtain a bounded martingale, since $X$ is continuous, so assume $X$ bounded.

Define for $u \in \mathbb{Q} \cap [0, \infty)$, $\quad \tau_u = \inf\{t \geq u, [X]_t > [X]_u\} \geq u$ a.s.

$\tau_u$ is the hitting time of an open set by a cadlag process, therefore a stopping time.

Since $X$ bounded, we can apply the optional sampling theorem:

$$E(X_{\tau_u} X_u) = E(E(X_{\tau_u} X_u | \mathcal{F}_u)) = E(X_u E(X_{\tau_u} | \mathcal{F}_u))$$

$$= E(X_u^2) \quad \text{So}$$

$$E(X_{\tau_u} - X_u)^2 = E(X_{\tau_u}^2) - 2E(X_{\tau_u} X_u) + E(X_u^2) = E(X_{\tau_u}^2) - E(X_u^2)$$

Since $X$ is bounded it is square-integrable so

$$E(X_{\tau_u}^2) - E(X_u^2) = E([X]_{\tau_u}) - E([X]_u) = 0$$

by definition of $\tau_u$. So : $E(X_{\tau_u} - X_u)^2 = 0$ : $X_{\tau_u} = X_u$ a.s.

- We have shown that $X$ is constant a.s. on all intervals $[u, \tau_u]$, $u \in \mathbb{Q} \cap [0, \infty)$ where $[X]$ is constant

If $[a, \tau_a]$, $a > 0$ is another interval where $[X]$ is constant, $\exists (u_n)_{n \geq 1} \in \mathbb{Q}$, $u_n \geq a$ $u_n \xrightarrow[n \to \infty]{} a$ and $]a, \tau_a[ = \bigcup_{n \geq 1} [u_n, \tau_{u_n}]$

- So $X$ is a.s. constant on $]a, \tau_a[$

- Since $X$ is continuous, this implies that it is also constant on $[a, \tau_a]$

Rem: this result fails if $X$ is discontinuous.

Ex. $X = N_t - \lambda t$ Compensated Poisson process

$X$ is a martingale.

$[X] = [N] = N$ is piecewise constant ;

if $(T_i)_{i \geq 1}$ are the jumps of the Poisson process,

$N_t = i$ for $t \in [T_i, T_{i+1}[$ is constant on $[T_i, T_{i+1}[$

but $X_{T_{i+1}} = X_{T_i} - \lambda (T_{i+1} - T_i)$

# Quadratic covariation

For any sequence $(T_i)$ of stopping times:

$$(X_{T_{i+1}} - X_{T_i})(Y_{T_{i+1}} - Y_{T_i}) =$$

$$X_{T_{i+1}} Y_{T_{i+1}} - X_{T_i} Y_{T_i} - X_{T_i}(Y_{T_{i+1}} - Y_{T_i}) - Y_{T_i}(X_{T_{i+1}} - X_{T_i})$$

Summing over $i$,

$$\sum_i (X^{T^n_{i+1}} - X^{T^n_i})(Y^{T^n_{i+1}} - Y^{T^n_i}) = X_T Y_T - X_0 Y_0 - \underbrace{\sum_i X_{T_i}(Y_{T_{i+1}} - Y_{T_i})}_{\downarrow \text{ucp}} - \underbrace{\sum_i Y_{T_i}(X_{T_{i+1}} - X_{T_i})}_{\downarrow \text{ucp}}$$

If $X, Y$ are semimartingales;

as $\sup |T_{i+1} - T_i| \longrightarrow 0$ a.s., $\qquad \int X_- dY \qquad \int Y_- dX$

**Quadratic covariation** Let $X, Y$ be semimartingales.

The quadratic covariation (or bracket process) $[X,Y]$ of $X$ and $Y$ is defined as $[X,Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{u-} dY_u - \int_0^t Y_{u-} dX_u$

$[X,Y]$ is a cadlag adapted process of finite variation.

i) $(X,Y) \to [X,Y]$ is bilinear and $[X,Y] = [Y,X]$

ii) $[X,Y] = \frac{1}{2}([X+Y] - [X] - [Y])$ (Polarization)

iii) $[X,X] = [X]$ iv) $(\Delta [X,Y])_t = \Delta X_t \Delta Y_t$

v) If $(\Pi^n)$ is a sequence of random partitions,

$$\lim_{\substack{|\Pi^n| \to 0 \\ T_i^n \in \Pi^n}} \sum_i (X^{T_{i+1}^n} - X^{T_i^n})(Y^{T_{i+1}^n} - Y^{T_i^n}) \xrightarrow{ucp} [X,Y]$$

Proof: $(X_{T_{i+1}} - X_{T_i})(Y_{T_{i+1}} - Y_{T_i}) =$

$$\frac{1}{4}\left\{(X_{T_{i+1}} - X_{T_i} + Y_{T_{i+1}} - Y_{T_i})^2 - (X_{T_{i+1}} - X_{T_i})^2 - (Y_{T_{i+1}} - Y_{T_i})^2\right\}$$

so summing over $i$ and taking limits $|\Pi| \to 0$ yields

$$[X, Y] = \frac{1}{2}\left([X+Y] - [X] - [Y]\right)$$

# Ito's product differentiation rule

Let $X, Y$ be semimartingales.

Then $XY$ is a semimartingale and

$$X_t Y_t - X_0 Y_0 = \int_0^t X_{u-} \, dY_u + \int_0^t Y_{u-} \, dX_u + [X,Y]_t$$

# Quadratic covariation as 'compensator'

Let $X, Y$ be locally square-integrable local martingales.

Then $[X,Y]$ is the unique cadlag adapted process $A$ with paths of finite variation such that

(i) $M = XY - A$ is a local martingale with $M_0 = 0$

(ii) $\Delta A_t = \Delta X_t \Delta Y_t$

Proof: $XY - [X,Y] = \int_0^t X_- \, dY + \int_0^t Y_- \, dX$ by definition.

By the martingale preserving property of the stochastic integral, $XY - [X,Y]$ is a local martingale and from the polarization identity

$$\Delta\left( XY - [X,Y] \right) = \Delta X \cdot \Delta Y$$

To show uniqueness, take a cadlag adapted process $A$ with paths of finite variation such that (i)-(ii) hold.

$$Z_t = A_t - [X,Y]_t = \underbrace{X_t Y_t - [X,Y]_t}_{\text{local martingale}} - \underbrace{(X_t Y_t - A_t)}_{\text{local martingale}}$$

- So $Z$ is also a local martingale.
- Furthermore $\Delta Z_t = \Delta A_t - \Delta([X,Y])_t = 0$ by (ii)
- Also $A$, $[X,Y]$ are of finite variation so $Z$ is a continuous local martingale with paths of finite variation
  $\Rightarrow Z = Z_0 = 0$

# Quadratic covariation : examples

If $Y$ is an adapted cadlag process with bounded variation then for any semimartingale $X$, $[X,Y]=0$

In particular the usual product rule holds :

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{u-}\, dY + \int_0^t Y_{u-}\, dX$$

- Example : if $X$ is an Itô process $X_t = \int_0^t \sigma\, dW + \int_0^t b_u\, du$ and $N$ is a (cadlag adapted) counting process $N_t = \sum_{n \geq 1} 1_{t \geq T_n}$ then $[X,N]=0$

# Quadratic covariation of pure jump processes

If $X$ is a pure jump semimartingale then for any semimartingale $Y$,

$$[X]^c = 0 \quad \Rightarrow \quad [X,Y]_t = \sum_{0 \le s \le t} \Delta X_s \Delta Y_s$$

In particular if $X, Y$ have no common jumps with probability 1 then $[X,Y] = 0$

# Integrals with respect to quadratic variation.

$X, Y$ semimartingale, $[X]$, $[X,Y]$ are processes of finite variation, so we can define

$$\int_0^t \varphi \, d[X] = \lim_{n\to\infty} \sum_{T_i^n \in \Pi^n} \varphi_{T_i^n} \cdot \left([X]_{T_{i+1}^n} - [X]_{T_i^n}\right)$$

$$\int_0^t \varphi \, d[X,Y] = \lim_{n\to\infty} \sum_{T_i^n \in \Pi^n} \varphi_{T_i^n} \cdot \left([X,Y]_{T_{i+1}^n} - [X,Y]_{T_i^n}\right)$$

for $\varphi \in \mathbf{L}$ (left continuous, adapted, with right limits)

**Associativity property** $X, Y$ semimartingales
$\varphi, \psi \in L$

$$\left[ \int_0^{\cdot} \varphi \, dX , \int_0^{\cdot} \psi \, dY \right]_t = \int_0^t \varphi_s \psi_s \, d[X,Y]_s$$

$$\left[ \int \varphi \, dX \right]_t = \int_0^t \varphi_s^2 \, d[X]_s$$

Proof: let us first show that for $\varphi = U_i \mathbb{1}_{(T_i, T_{i+1}]}$ where $U_i$ is $\mathcal{F}_{T_i}$-measurable, $T_i, T_{i+1}$ are stopping times, we have

$$\left[ \int_0 \varphi \, dX , Y \right]_t = \int_0^t \varphi \, d[X,Y]$$

$$\int_0^t \varphi \, dX = U_i \left( X_{T_{i+1} \wedge t} - X_{T_i \wedge t} \right) \quad ; \quad \int_0^{\cdot} \varphi \, dX = U_i \left( X^{T_{i+1}} - X^{T_i} \right)$$

so $$\left[ \int_0^{\cdot} \varphi \, dX , Y \right] = \left[ U_i \left( X^{T_{i+1}} - X^{T_i} \right) , Y \right]$$

$$= U_i \left( [X^{T_{i+1}}, Y] - [X^{T_i}, Y] \right) \qquad \text{by bilinearity of } [\,,\,]$$

$$= U_i \left( [X,Y]^{T_{i+1}} - [X,Y]^{T_i} \right)$$

$$= \int_0^{t} \varphi \, d[X,Y]$$

By linearity if $\varphi \in \mathcal{S}$ : $\varphi = \sum_{i=0}^{n-1} U_i \, 1_{(T_i, T_{i+1}]}$, then $\left[ \int_0^{\cdot} \varphi \, dX, Y \right]_t = \int_0^t \varphi \, d

Take now $\varphi \in L$. There exists a sequence $H^n \in S$ of simple predictable processes

$$H^n \xrightarrow{ucp} \varphi. \text{ Then: } Z^n = \int H^n dX \xrightarrow[n\to\infty]{ucp} Z = \int \varphi\, dX \quad \text{since}$$

$$\int H^n_s\, d[X,Y]_s \xrightarrow[ucp]{n\to\infty} \int \varphi_s\, d[X,Y] \quad \text{since } X, [X,Y] \text{ are semimartingales.}$$

$Z^n, Z$ are semimartingales and $\int_0^t H^n_s\, d[X,Y]_s = [Z^n, Y]_t$

so

$$[Z^n, Y] = YZ^n - \int Y_-\, dZ^n - \int Z^n_-\, dY = YZ^n - \int Y_- H^n dX - \int Z^n_-\, dY$$

so: $$\lim_{n\to\infty} [Z^n, Y]_t = Y_t Z_t - \int_0^t Y_{s-} \varphi_s\, dX_s - \int_0^t Z_{s-}\, dY_s$$

$$= Y_t Z_t - \int_0^t Y_{s-}\, dZ_s - \int_0^t Z_{s-}\, dY_s = [Z, Y]$$

# Quadratic covariation of Ito processes

Let
$$X_t^1 = X_0^1 + \int_0^t \sigma^1 \, dW^1 + \int_0^t b_u^1 \, du$$
$$X_t^2 = X_0^2 + \int_0^t \sigma^2 \cdot dW^2 + \int_0^t b_u^2 \cdot du$$

be Ito processes, where $W^1, W^2$ are Brownian motions, $\operatorname{cov}(W_t^1, W_t^2) = \int_0^t \rho_u^{12} du$

Then
$$[X^i]_t = \int_0^t |\sigma_u^i|^2 du$$
$$[X^1, X^2]_t = \int_0^t \sigma_u^1 \sigma_u^2 \rho_u^{12} du$$

since $\begin{cases} d[W^i]_t = dt \\ d[W^1, W^2]_t = \rho_t^{12} dt \end{cases}$

So: $d[X^1, X^2]_t = \sigma_t^1 \sigma_t^2 \rho_t^{12} dt =$ "$\operatorname{cov}(dX_t^1, dX_t^2)$"

can be interpreted as a "local covariance"

Combining this with the 'isometry' theorem we get: $\left( \begin{array}{l} M \text{ sq-int martingale} \\ E(M_t^2) = E[M]_t \end{array} \right.$

Let $X_t = X_0 + \int_0^t \sigma . dW + \int_0^t b_u \, du$ an Ito process. Then

$$[X]_t = \int_0^t \sigma_u^2 \, du = [X]_t^c \quad \text{and} \quad E[\int_0^t \sigma dW] = E([X]_t) = E(\int_0^t \sigma_u^2 du)$$

So: $X$ is square integrable $\Longleftrightarrow$ $\forall t \geq 0$ $E\left(\int_0^t \sigma_u^2 \, du\right) < \infty$

$\forall t \geq 0 \quad E(X_t^2) < \infty$

More generally if $M$ is a square-integrable local martingale then, for any $\varphi \in L$,

$$\boxed{E\left\{\left(\int_0^t \varphi\, dM\right)^2\right\}} = E\left(\left[\int \varphi\, dM\right]_t\right) = \boxed{E\left(\int_0^t \varphi_u^2\, d[M]_u\right)}$$

Ex. $M_t = \int_0^t \sigma_u\, dW_u$ then: $E\left\{\left(\int_0^t \varphi\, dM\right)^2\right\} = E\left(\int_0^t \varphi_u^2 \sigma_u^2\, du\right)$

So this result identifies the 'natural norm' associated with stochastic integrals with respect to $M$, and that is:

$$\left\| \int_0 \varphi\, dM \right\|^2 = E\left(\int_0^T \varphi_u^2\, d[M]_u\right)$$

## $L^2$ extension of the stochastic integral with respect to a square-integrable martingale $M$

Consider first

$$A_M = \left\{ \varphi \in L \ \left(\begin{matrix}\text{left continuous} \\ \text{adapted}\end{matrix}\right) \text{ with } E\left( \int_0^T \varphi_t^2 \, d[M]_t \right) < \infty \right\}$$

$\mathcal{L}^2(M)$ = closure of $A_M$

$$= \left\{ \begin{matrix} \varphi \text{ adapted process such that} \\ \varphi^n \in A_M, \quad E\left( \int_0^T (\varphi_t^n - \varphi_t)^2 \, d[M]_t \right) \xrightarrow[n \to \infty]{} 0 \end{matrix} \right.$$

For $\varphi \in \mathcal{L}^2(M)$ we define

$$\int_0^T \varphi \, dM \quad \text{as} \quad \lim_{n\to\infty} \int_0^T \varphi^n \, dM \quad \text{in } L^2(\Omega, \mathcal{F}_T, \mathbb{P})$$

✓ For $\varphi \in \mathcal{L}^2(M)$, $\int_0^T \varphi \, dM$ is thus defined as a limit in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ so as an element of $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

✓ For $\varphi \in \mathcal{L}^2(M)$, $\varphi \notin L$, $\int_0^T \varphi \, dM$ cannot be interpreted in general as a limit of Riemann sums.

## 'Quadratic Riemann sums'

Let $X, Y$ be semimartingales

$H$ be a cadlag adapted process, $H_0 = 0$.

$\Pi^n = (0 = T_0^n \leq T_1^n \leq \dots \leq T_{k_n}^n)$ be a sequence of random partitions with $|\Pi^n| \xrightarrow[n\to\infty]{} 0$ and $T_{k_n}^n \xrightarrow[n\to\infty]{} \infty$ a.s.

Then:

$$\sum_{\Pi^n} H_{T_i^n} \left(X^{T_{i+1}^n} - X^{T_i^n}\right)\left(Y^{T_{i+1}^n} - Y^{T_i^n}\right) \xrightarrow[n\to\infty]{\text{UCP}} \int H_- \, d[X,Y]$$

In particular:

$$\sum_{\Pi^n} H_{T_i^n} \left(X^{T_{i+1}^n} - X^{T_i^n}\right)^2 \xrightarrow[n\to\infty]{\text{UCP}} \int_0^{\cdot} H_{t-} \, d[X]_t$$

Proof: $\quad [X,Y]_t = X_t Y_t - \int_0^t X_{s-}\, dY_s - \int Y_{s-}\, dX_s \quad$ so

$$\int_0^t H_{s-}\, d[X,Y]_s = \int_0^t H_{s-}\, dZ_s - \int_0^t H_{s-} X_{s-}\, dY_s - \int_0^t H_{s-} Y_{s-}\, dX_s$$

where $Z_t = X_t Y_t$. But since $X, Y$ and $XY$ are semimartingales

$$\sum_{\Pi^n} H_{T_i^n} \left(X^{T_{i+1}^n} - X^{T_i^n}\right)\left(Y^{T_{i+1}^n} - Y^{T_i^n}\right) =$$

$$\sum_{\Pi^n} H_{T_i^n} \left(X^{T_{i+1}^n} Y^{T_{i+1}^n} - X^{T_i^n} Y^{T_i^n}\right) \xrightarrow{ucp} \int_0^{\cdot} H_{s-}\, dZ_s$$

$$- \sum_{\Pi^n} H_{T_i^n} X_{T_i^n} \left(Y^{T_{i+1}^n} - Y^{T_i^n}\right) \xrightarrow{ucp} \int_0^{\cdot} H_{s-} X_{s-}\, dY_s$$

$$- \sum_{\Pi^n} H_{T_i^n} Y_{T_i^n} \left(X^{T_{i+1}^n} - X^{T_i^n}\right) \xrightarrow{ucp} \int_0^{\cdot} H_{s-} Y_{s-}\, dX_s$$

# Kunita - Watanabe inequality

Let $X, Y$ be semi-martingales, $\varphi, \psi$ be (measurable) processes. Then

$$\int_0^\infty |\varphi_t||\psi_t| \, d[X,Y]_t \leq \sqrt{\int_0^\infty \varphi_t^2 \, d[X]_t} \cdot \sqrt{\int_0^\infty \psi_t^2 \, d[Y]_t}$$

In particular the measure defined by $[0,t] \to [X,Y]_t$ is absolutely continuous with respect to the measure defined by $[0,t] \to [X]_t$