Lesson 5

The Ito formula

Quadratic variation of a path with respect to a sequence of partitions

Change of variable formula for functions with finite quadratic variation

Ito formula for functions of semimartingales

Multivariate extension

Kiyoshi Ito

(1915 – 2008)

Paul-André MEYER

(1934 – 2003)

Quadratic variation with respect to a partition

$(\Pi_n)_{n\geq 1}$ partition of $[0,T]$ $\begin{cases} \Pi_n = (t_i^n,\ i=0\ldots k(n)-1) \\ |\Pi_n| \to 0 \text{ as } n\to\infty \end{cases}$

Def: a cadlag function $f \in D([0,\infty),\mathbb{R})$ is said to have finite quadratic variation on $[0,T]$ w.r.t $\Pi = (\Pi_n)_{n\geq 1}$

if: $\forall t\in[0,T]$, $\displaystyle\sum_{\substack{t_i^n\in\Pi_n \\ t_{i+1}^n < t}} |f(t_{i+1}^n) - f(t_i^n)|^2$ has a limit as $n\to\infty$.

The limit $\displaystyle [f](t) := \lim_{n\to\infty} \sum_{\substack{t_i^n\in\Pi_n \\ t_{i+1}^n < t}} |f(t_{i+1}^n) - f(t_i^n)|^2$

is then an increasing function on $[0,T]$ called the quadratic variation of f along $\Pi = (\Pi_n)_{n\geq 1}$.

Denote $Q([0,T];\pi)$ the set of cadlag paths with finite quadratic variation with respect to the partitions $\pi = (\pi^n)_{n\geq 1}$

- $Q([0,T],\pi)$ is a subspace of $D([0,T],\mathbb{R})$

For $f \in Q([0,T],\pi)$: $\quad Q(\pi) = Q([0,\infty),\pi)$

- $[f]$ is an increasing function so defines a positive measure μ_f on $[0,T]$.
- The increasing function $t \to [f](t)$ has Lebesgue decomposition

$$[f](t) = \underbrace{[f]^c(t)}_{\text{continuous part}} + \sum_{0\leq s\leq t} \underbrace{|\Delta f(s)|^2}_{\text{jumps}}$$

$$\mu_f = \mu_f^c + \sum_{\Delta f(t)\neq 0} |\Delta f(t)|^2 \delta_t$$

"Examples":

✓ If X is a function with finite variation on $[0,T]$, then for ___any___ sequence of partitions with $|\Pi^n| \xrightarrow[n\to\infty]{} 0$, $X \in Q([0,T], \Pi)$

✓ If W is a Wiener process then for ___any___ sequence of partitions with $|\Pi^n| \xrightarrow[n\to\infty]{} 0$,

$$\mathbb{P}\left(W_\cdot \in Q([0,T],\Pi) \right) = 1$$

and in fact $[W](t) = t$

Theorem: Let S be a semimartingale. Then there exists a sequence $\pi = (\pi^n)_{n \geq 1}$ of partitions such that S has finite quadratic variation along π with probability 1: $\mathbb{P}(S_\cdot \in Q(\pi)) = 1$.

Proof: Let $\sigma^n = \left(t_k^n = \frac{kT}{2^n},\ k = 0 \dots 2^n \right)$ be the dyadic subdivision of $[0,T]$. Then

$$\sum_{k=0}^{2^n-1} \left(S_{t_{k+1}^n} - S_{t_k^n} \right)^2 \xrightarrow{n \to \infty} [S]_T \quad \text{in probability.}$$

Since every sequence converging in probability has an almost-surely convergent subsequence, there exists a subsequence $\pi^n = (\tau_i^n,\ i = 0 .. k(n))$ of $(\sigma^n)_{n \geq 1}$ such that:

$$\sum_{\tau_i^n \in \pi^n} \left(S_{\tau_{i+1}^n} - S_{\tau_i^n} \right)^2 \xrightarrow{n \to \infty} [S]_T \quad \text{a.s.}$$

$$\Omega_\pi = \left\{ \omega \in \Omega,\ \sum_{\tau_i^n \in \pi^n} \left(S_{\tau_{i+1}^n}(\omega) - S_{\tau_i^n}(\omega) \right)^2 \xrightarrow[n \to \infty]{} [S]_T(\omega) \right\}$$

then $\mathbb{P}(\Omega_T) = 1.$

So $\mathbb{P}\left(\bigcap_{T \in \mathbb{Q}} \Omega_T\right) = 1$. Since $S, [S]$ are càdlàg processes we conclude that on $\Omega^{\pi} = \bigcap_{T \in \mathbb{Q}} \Omega_T$,

we have $\displaystyle\sum_{T_i^n \in \pi^n \cap [0,t]} \left(S_{T_{i+1}^n} - S_{T_i^n}\right)^2 \xrightarrow[n \to \infty]{} [S]_t$ for any $t \geq 0$

for any $t \geq 0$: all paths in Ω^{π} have finite quadratic variation along $(\pi^n)_{n \geq 1}$

Quadratic Riemann sums

Proposition: Let $f \in Q([0,T],\pi)$.

$$\forall g \in C^0([0,T],\mathbb{R}), \quad \sum_{t_i^n \in \pi^n} g(t_i^n)\,|f(t_{i+1}^n) - f(t_i^n)|^2 \xrightarrow{n\to\infty} \int_0^T g\, d[f] \quad \text{(QR)}$$

Proof: For $g = 1_{[0,t]}$ using $f \in Q([0,T],\pi)$,

$$\sum_{t_i^n \in \pi^n} g(t_i^n)\,|f(t_{i+1}^n) - f(t_i^n)|^2 = \sum_{\substack{\pi^n \\ t_{i+1}^n < t}} |f(t_{i+1}^n) - f(t_i^n)|^2 \xrightarrow[n\to\infty]{} [f](t) = \int_0^T g\, d[f]$$

More generally if $g = \sum_{i=1}^{d} a_i 1_{[t_i, t_{i+1})} = \sum_{i=1}^{d} b_i 1_{[0,t_i]}$

we also have (QR) by linearity

For $g \in C^0([0,T],\mathbb{R})$, there exists $g^n = \sum_{i=1}^{d(n)} a_i^n 1_{[t_i^n, t_i^{n+1})}$

such that $\|g - g^n\|_\infty \xrightarrow{n\to\infty} 0$ which allows to conclude.

Weak convergence

A sequence $(\mu_n)_{n\geq 1}$ of measures on $[0,T]$ is said to converge weakly to a measure μ on $[0,T]$

$$\forall g \in C_0^b([0,T],\mathbb{R}), \int_0^T g\, d\mu^n \xrightarrow[n\to\infty]{} \int_0^T g\, d\mu$$

We denote $\mu^n \overset{n\to\infty}{\Longrightarrow} \mu$

Property : If $f \in Q([0,T];\pi)$ then

$$\mu^n = \sum_{\pi^n} \left(f(t_{i+1}^n) - f(t_i^n)\right)^2 \delta_{t_i^n} \overset{n\to\infty}{\Longrightarrow} \mu_f = d[f]$$

Change of variable formula for functions of finite quadratic variation

Theorem (Föllmer 1979): If $X \in Q(\Pi)$ has finite quadratic variation along a sequence $\Pi = (\Pi^n)_{n \geq 1}$ of partitions, then for any $f \in C^2(\mathbb{R}, \mathbb{R})$,

✓ $\displaystyle\sum_{t_i^n \in \Pi^n,\ 0 \leq t_i^n \leq t} f'(X_{t_i^n})\,(X_{t_{i+1}^n} - X_{t_i})$ has a limit as $n \to \infty$

✓ Denoting $\displaystyle\int_0^t f'(X_{s-})\,d^{\Pi}X_s = \lim_{n\to\infty} \sum_{t_i^n \in \Pi^n,\ 0 \leq t_i^n \leq t} f'(X_{t_i^n})\,(X_{t_{i+1}^n} - X_{t_i})$

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})\,d^{\Pi}X_s + \frac{1}{2}\int_0^t f''(X_{s-})\,d[X]_s$$
$$+ \sum_{0 \leq s \leq t} f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) - \frac{(\Delta X_s)^2}{2} f''(X_{s-})$$

Corollary :

✓ if $x \in Q(\Pi)$ is continuous then

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})\,dX_s + \frac{1}{2}\int_0^t f''(X_{s-})\,d[X]_s$$

✓ if $x \in Q(\Pi)$ has bounded variation then

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})\,dX_s + \sum_{0\leq s\leq t} f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) - \frac{(\Delta X_s)^2}{2} f''(X_{s-})$$

Proof :✓ by right continuity of X,

$$f(X_t) - f(X_0) = \lim_{n\to\infty} \sum_{t_i^n \in [0,t]} f(X_{t_{i+1}^n}) - f(X_{t_i^n})$$

✓ Taylor's theorem : $\forall x, y \in [0,a]$, $f \in C^2$

$$f(y) - f(x) = (y-x) f'(x) + \frac{(y-x)^2}{2} f''(x) + (y-x)^2 r(x,y)$$

Since x is cadlag therefore bounded on $[0,T]$, $I = x([0,T])$

r is continuous on I^2 therefore uniformly continuous.

where, by uniform continuity, $r(x,y) \leq \varphi(|y-x|)$

for some increasing function $\varphi : [0,\infty[\longrightarrow [0,\infty[$

with $\varphi(\varepsilon) \xrightarrow{\varepsilon \downarrow 0} 0$

Proof: continuous case $\quad x \in C_0([0,T]) \cap Q(\Pi)$

$x \in C_0([0,T])$ so uniformly continuous on $[0,T]$ so

$\forall \varepsilon > 0, \ \exists \delta > 0, \ |s_1 - s_2| \leq \delta \ , \ |x_{s_1} - x_{s_2}| \leq \varepsilon$

Since $|\Pi^n| \xrightarrow{n\to\infty} 0$, for n large enough, $|\Pi^n| \leq \delta$

so, $\exists n_0 \geq 1, \ \forall n \geq n_0, \ |\Pi^n| \leq \delta$ and $|x_{t_{i+1}^n} - x_{t_i^n}| \leq \varepsilon$

for $t_i \in \Pi^n$.

$$f(X_t) - f(X_0) = \sum_{t_i^n \in [0,t]} f(X_{t_{i+1}^n}) - f(X_{t_i^n}) =$$

$$\sum_{t_i^n \in [0,t]} f'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n})$$

Converges since all other terms have a finite limit

$$+ \sum_{t_i^n \in [0,t]} \frac{1}{2} f''(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n})^2 \xrightarrow{n\to\infty} \frac{1}{2}\int_0^t f''(X_s)\, d[X]_s$$

since $x \in Q(\pi)$, $f'' \in C_0$

$$+ \underbrace{\sum_{t_i^n \in [0,t]} (X_{t_{i+1}} - X_{t_i})^2}_{\xrightarrow{n\to\infty} [X]_t} \underbrace{r(X_{t_i}, X_{t_{i+1}})}_{\leq \varepsilon \text{ for } |\pi^n| \leq \delta} \leq C \varepsilon \xrightarrow[n\to\infty]{} 0$$

so is bounded wrt n

Proof : discontinuous case

Let $\varepsilon > 0$. Since $\sum_{0 \leq s \leq t} (\Delta X_s)^2 \leq [X]_t < \infty$,

the series $\sum_{0 \leq s \leq t} (\Delta X_s)^2$ only contains a finite number of terms in any interval bounded away from zero.

Separate the jump times of X in $[0, t]$ into two classes

✓ a finite set $C_1(\varepsilon, t)$ (large jumps)

✓ a set $C_2(\varepsilon, t)$ such that $\sum_{s \in C_2} (\Delta X_s)^2 \leq \varepsilon^2$ (small jumps)

with $C_1(\varepsilon, t) \cup C_2(\varepsilon, t) = \{ s \in [0, t], \Delta X_s \neq 0 \}$

and $\bigcup_{\varepsilon > 0} C_1(\varepsilon, t) = \{ s \in [0, t], \Delta X_s \neq 0 \}$

$$\sum_{\pi^n \cap [0,t]} f(X_{t^n_{i+1}}) - f(X_{t_i}) = \sum_{]t_i, t_{i+1}] \cap C_1(\varepsilon,t) \neq \emptyset} f(X_{t_{i+1}^n}) - f(X_{t_i}) + \sum_{]t_i, t_{i+1}] \cap C_1(\varepsilon,t) = \emptyset} f(X_{t_{i+1}^n}) - f(X_{t_i^n})$$

at least one large jump

no large jump

$$\checkmark \quad \lim_{n\to\infty} \sum_{\substack{]t_i^n, t_{i+1}^n] \cap C_1(\varepsilon,t) \neq \emptyset \\ t_i^n \in \pi^n}} f(X_{t_{i+1}^n}) - f(X_{t_i}) = \sum_{s \in C_1(\varepsilon,t)} f(X_s) - f(X_{s-})$$

for each ε

✓ Taylor's formula applied to the second sum gives

$$\sum_{]t_i, t_{i+1}] \cap C_1(\varepsilon,t) = \emptyset} f(x_{t_{i+1}}) - f(X_{t_i}) = \sum_{\pi^n \cap [0,t]} f'(x_{t_i})(x_{t_{i+1}} - x_{t_i}) + \frac{1}{2} \sum_{\pi^n \cap [0,t]} f''(x_{t_i})(x_{t_{i+1}} - x_{t_i})^2$$

$$+ \sum_{]t_i, t_{i+1}] \cap C_1(\varepsilon,t) = \emptyset} (X_{t_{i+1}} - X_{t_i})^2 \, r(X_{t_i}, X_{t_{i+1}}) - \sum_{]t_i, t_{i+1}] \cap C_1(\varepsilon,t) \neq \emptyset} \Big[f'(x_{t_i})(x_{t_{i+1}} - x_{t_i}) + \frac{1}{2} f''(x_{t_i})(x_{t_{i+1}} \ldots$$

✓ f'' is uniformly continuous on the bounded set

$I = \{ X_s, \ s \in [0,t] \}$ so $\forall \ u,v \in I^2, \ r(u,v) \leq \varphi(|u-v|)$

where $\varphi(\varepsilon) \xrightarrow{\varepsilon \downarrow 0} 0$. So, $\forall \varepsilon > 0$

$$\limsup_{n \to \infty} \sum_{]t_i, t_{i+1}] \cap C_1(\varepsilon,t) = \emptyset} (X_{t_{i+1}^n} - X_{t_i^n})^2 \, r(X_{t_i^n}, X_{t_{i+1}^n}) \leq \varphi(\varepsilon) \, [X]_t \xrightarrow{\varepsilon \downarrow 0} 0$$

✓ $$\sum_{s \in C_1(\varepsilon,t)} | f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) | \leq \frac{\|f''\|_\infty}{2} \sum_{s \in C_1(\varepsilon,t)} (X_s - X_{s-})^2$$ by Taylor's formula

So the series is absolutely convergent and

$$\sum_{s \in C_1(\varepsilon,t)} f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) \xrightarrow{\varepsilon \to 0} \sum_{0 \leq s \leq t} f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-})$$

So finally : $\forall \varepsilon > 0$, $f(X_t) - f(X_0) = \sum_{\pi^n} f(X_{t_{i+1}}) - f(X_{t_i}) =$

$$\sum_{]t_i, t_{i+1}] \cap C_1(\varepsilon,t) \neq \emptyset} f(X_{t_{i+1}}) - f(X_{t_i}) \xrightarrow[n\to\infty]{} \sum_{C_1(\varepsilon,t)} f(X_s) - f(X_{s-})$$

$$+ \sum_{\pi^n \cap [0,t]} f'(x_{t_i})(x_{t_{i+1}} - x_{t_i})$$ converges as $n \to \infty$ since all other terms converge

$$+ \frac{1}{2} \sum_{\pi^n \cap [0,t]} f''(x_{t_i})(x_{t_{i+1}} - x_{t_i})^2 \xrightarrow[n\to\infty]{} \frac{1}{2}\int_0^t f''(x_u)\, d[x]_u$$

$$+ \sum_{]t_i, t_{i+1}] \cap C_1(\varepsilon,t) = \emptyset} (X_{t_{i+1}} - X_{t_i})^2 \, r(X_{t_i}, X_{t_{i+1}}) \leqslant k \varepsilon^2$$

$$- \sum_{]t_i, t_{i+1}] \cap C_1(\varepsilon,t) \neq \emptyset} \left[f'(x_{t_i})(x_{t_{i+1}} - x_{t_i}) + \frac{1}{2} f''(x_{t_i})(x_{t_{i+1}} - x_{t_i})^2 \right] \xrightarrow{n\to\infty} -\sum_{s \in C(\varepsilon,t)} f'(X_{s-})\Delta X_s + \frac{1}{2}(\Delta X_s)^2 f''(X_{s-})$$

Taking $\varepsilon \to 0$ yields the Ito formula :

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})\, dX_s + \frac{1}{2}\int_0^t f''(X_{s-})\, d[X]_s$$

$$+ \sum_{0 \leqslant s \leqslant t} f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) - \frac{(\Delta X_s)^2}{2} f''(X_{s-})$$

Different forms of the change of variable formula

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})\, d\overset{\pi}{X}_s + \frac{1}{2}\int_0^t f''(X_{s-})\, d[X]_s$$
$$+ \sum_{0 \leq s \leq t} f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) - \frac{(\Delta X_s)^2}{2} f''(X_{s-})$$

$$= \int_0^t f'(x_{s-})\, d\overset{\pi}{x}_s + \frac{1}{2}\int_0^t f''(x_{s-})\, d[x]^c_s + \sum_{0 \leq s \leq t} f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-})$$

since $[x]_t = [x]^c_t + \sum (\Delta X_s)^2$

$$\int_0^t f''(x_u)\, d[x] = \int_0^t f''(x_u)\, d[x]^c + \sum_{0 \leq s \leq t} f''(X_{s-})\, (\Delta X_s)^2$$

However in general $\sum_{\substack{0 \leq s \leq t \\ \Delta X_s \neq 0}} f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-})$ may not be further decomposed :

in general $\sum (\Delta X_s) f'(X_{s-})$; do not converge
$\sum f(X_s) - f(X_{s-})$

<u>unless</u> $\sum_{0\le s\le t} |\Delta X_s| < \infty$ (the jumps have 'finite variation')

If $\sum_{0\le s\le t} |\Delta X_s| < \infty$ then we have

$$X_t = \overbrace{\left(X_t - \sum_{0\le s\le t} \Delta X_s \right)}^{X_t^c} + \sum_{0\le s\le t} \Delta X_s \quad \text{where } X^c \in C_0([0,T])$$

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-}) \, dX_s^c + \frac{1}{2}\int_0^t f''(X_{s-}) \, d[X]_s^c + \sum_{0\le s\le t} f(X_s) - f(X_{s-})$$

The Ito formula

Let S be a semimartingale, $f \in C^2([0,\infty), \mathbb{R})$.

Then $f(S)$ is also a semimartingale and $\forall t \geq 0$,

$$f(S_t) - f(S_0) = \int_0^t f'(S_{u-})\, dS_u + \int_0^t \frac{1}{2} f''(S_{u-})\, d[S]_u$$

$$+ \sum_{0 \leq s \leq t} f(S_u) - f(S_{u-}) - \Delta S_u f'(S_u) - \frac{1}{2} f''(S_{u-}) \frac{(\Delta S_u)^2}{2}$$

Proof : Since S semimartingale there exists a sequence $(\pi^n)_{n\geq 1}$ of partitions such that $\mathbb{P}(\Omega^{\pi}) = 1$, where

$\Omega^{\pi} = \{\omega \in \Omega,\ S_\cdot(\omega) \in Q(\pi)\} \cdot S_0 :\ \forall \omega \in \Omega^{\pi}, \forall f \in C^2(\mathbb{R}),$

$$f(S_t(\omega)) - f(S_0) = \lim_{n\to\infty} \sum_{\pi^n} f'(S_{t_i^n}) \left(S_{t_{i+1}^n}(\omega) - S_{t_i^n}(\omega)\right)$$

$$+ \lim_{n\to\infty} \sum_{\pi^n} \frac{1}{2} f''(S_{t_i^n}) \left(S_{t_{i+1}^n}(\omega) - S_{t_i^n}(\omega)\right)^2$$

$$+ \sum_{0\leq s\leq t} f(S_u) - f(S_{u-}) - \Delta S_u f'(S_u) - \frac{1}{2} f''(S_{u-}) \left(\frac{\Delta S_u}{2}\right)^2$$

Since S is a semimartingale, since $|\pi^n| \to 0$,

$$\sum_{\pi^n} f'(S_{t_i^n})(S_{t_{i+1}^n} - S_{t_i^n}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^t f'(S_{u-})\,dS_u$$

and since $\omega \in \Omega^\pi$

$$\lim_{n\to\infty} \sum_{\pi^n} \frac{1}{2} f''(S_{t_i^n})\,(S_{t_{i+1}^n}(\omega) - S_{t_i^n}(\omega))^2 \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^t \frac{1}{2} f''(S_{u-})\,d[S]_u$$

non-anticipative quadratic Riemann sum

Ito formula: decomposition into 'martingale + finite variation' parts

If $S_t = M_t + A_t$ where M (locally square integrable) local martingale
A continuous process with finite variation

$$f(S_t) - f(S_0) = \underbrace{\int_0^t f'(S_{u-})\, dM_u}_{\text{local martingale}} + \underbrace{\int_0^t f'(S_{u-})\, dA_u}_{\text{finite variation process}}$$

$$+ \int_0^t \frac{1}{2} f''(S_{u-})\, d[M]_u$$

$$+ \sum_{0 \leq s \leq t} f(S_u) - f(S_{u-}) - \Delta S_u f'(S_u) - \frac{1}{2} f''(S_{u-})(\Delta S_u)^2$$

- If M is furthermore continuous then

$$f(S_t) = \underbrace{f(S_0) + \int_0^t f'(S)\, dM}_{\text{Local martingale}} + \underbrace{\int_0^t \frac{1}{2} f''(S_u)\, d[M] + f'(S_u)\, dA_u}_{\text{Finite variation}}$$

Ex. Change of variable formula for Ito processes

$$X_t = X_0 + \int_0^t \sigma_u \, dW_u + \int_0^t \mu_u \, du = M_t + A_t$$

$$E\left(\int_0^T \sigma_u^2 \, du\right) < \infty, \quad E\left(\int_0^T |\mu_u| du\right) < \infty, \quad [X]_t = \int_0^t \sigma_u^2 \, du$$

Then for $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(X_t)$ is also an Ito process with Ito decomposition

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) \sigma_u \, dW_u \quad \leftarrow \text{Local martingale term}$$

$$+ \int_0^t f'(X_u) \mu_u \, du + \frac{1}{2} \int_0^t f''(X_u) \sigma_u^2 \, du \quad \leftarrow \text{Finite variation part}$$

✓ Actually we have shown that for a semimartingale S there exists a set $\Omega^S = \Omega^\Pi$ with $\mathbb{P}(\Omega^S) = 1$ such that for all $f \in C^2(\mathbb{R}, \mathbb{R})$, the terms appearing in the Ito formula may be defined as pathwise limits of Riemann sums for any path in Ω^S and Ω^S is chosen independently of f.

Proof works identically if S is not a semimartingale as long as it has finite quadratic variation alongvs some partition.

Ex. fractional Brownian motion with $H > \frac{1}{2}$

Multivariate extension:

$x = (x^1, \dots, x^d) \in D([0,T], \mathbb{R}^d)$ is said to have **finite quadratic variation with respect to $\pi = (\pi^n)_{n \geq 1}$** if

$\forall i,j = 1 \dots d, \quad x^i + x^j, \ x^i$ and $x^j \in Q(\pi)$

We then define $[x^i, x^j] = \frac{1}{2}\left([x^i + x^j] - [x^i] - [x^j]\right)$

$[x]_t = \left([x^i, x^j]\right)_{i,j=1 \dots d}$ may be seen as an increasing

function $[x] : [0,T] \longrightarrow Sym^+(d)$

$Sym^+(d)$ of $d \times d$ positive symmetric matrices.

Quadratic Riemann sums

$\forall X \in Q(\pi)$, $\forall f \in C_0([0,T], \mathbb{R}^{d\times d})$,

$$\sum_{[0,t)\cap \pi^n} tr\left\{ f(X_{t_i})^t (X_{t_{i+1}} - X_{t_i})(X_{t_{i+1}} - X_{t_i})\right\} =$$

$$\sum_{[0,t]\cap \pi^n} \left(\sum_{j,k} f_{jk}(X_{t_i})\,(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i}) \right)$$

$$\xrightarrow{n\to\infty} \int_0^t tr\left\{ f(X_{u-})\, d[X]_u \right\}$$

$$= \int_0^t \sum_{j,k=1}^{d} f_{ij}(X_{u-})\, d[X^i, X^j]_u$$

Multivariate Ito formula :

$S^1, \dots, S^d$ are semimartingales

$S = (S^1, \dots, S^d)$. Then for $f \in C^2(\mathbb{R}^d, \mathbb{R})$,

$$f(S_t) - f(S_0) = \sum_{j=1}^{d} \int_0^t \frac{\partial f}{\partial x_j}(S_{u-}) \cdot dS_u^j + \sum_{j=1}^{d} \int_0^t \frac{1}{2} \frac{\partial^2 f}{\partial x_j^2}(S_{u-}) d[S^j]_u^c$$

$$+ \sum_{j>k} \int_0^t \frac{\partial^2 f}{\partial x_j \partial x_k}(S_{u-}) \, d[S^k, S^j]_u^c$$

$$+ \sum_{0 < u \leq t} \left\{ f(S_u) - f(S_{u-}) - \Delta S_u \cdot \nabla f(S_{u-}) \right\}$$

$$= \int_0^t \nabla f(S_{u-}) \cdot dS_u + \int_0^t \frac{1}{2} \operatorname{tr}\left(\nabla^2 f(S_{u-}) \cdot d[S]^c_u \right)$$

$$+ \sum_{0 \le u \le t} f(S_u) - f(S_{u-}) - \Delta S_u \cdot \nabla f(S_{u-}) \Big\}$$

. If the jumps of S are summable : $\sum_{0 \le u \le t} \| \Delta S_u \| < \infty$ a.s.

then $$S_t = S^c_t + \sum_{0 \le u \le t} \Delta S_u$$ and

$$f(S_t) - f(S_0) = \int_0^t \nabla f(S_{u-}) \cdot dS^c_u + \int_0^t \frac{1}{2} \operatorname{tr}\left(\nabla^2 f(S_{u-}) \cdot d[S]^c_u \right)$$

$$+ \sum_{0 \le u \le t} \left[f(S_u) - f(S_{u-}) \right]$$