Lesson 6 Applications of the Ito formula

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The Ito Formula

Let S be a semimartingale, $f \in C^2([0,\infty), \mathbb{R})$.

Then $f(S)$ is also a semimartingale and $\forall t \geq 0$,

$$f(S_t) - f(S_0) = \int_0^t f'(S_{u-})\, dS_u + \int_0^t \frac{1}{2} f''(S_{u-})\, d[S]_u^c$$

$$+ \sum_{0 \leq s \leq t} \left\{ f(S_u) - f(S_{u-}) - \Delta S_u f'(S_u) \right\}$$

Ito formula : time dependent version

✓ If $f \in C^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$

(continuously differentiable in t,
twice continuously differentiable in x)

then for any semimartingale S, $\forall T \geq 0$,

$$f(T, S_T) - f(0, S_0) = \int_0^T \frac{\partial f}{\partial x}(t, S_{t-})\, dS_t + \int_0^T \frac{\partial f}{\partial t}(t, S_t)\, dt$$

$$+ \int_0^T \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_{t-})\, d[S]^c_t$$

$$+ \sum_{0 \leq t \leq T} f(t, S_t) - f(t, S_{t-}) - \Delta S_t \cdot \frac{\partial f}{\partial x}(t, S_{t-})$$

Proof: apply Ito formula to $X = (t, S_t)$ $\quad [t] = 0,\ [t, S] = 0$

Exponential of a semimartingale

X semi-martingale $\qquad S_t = S_0 e^{X_t} \qquad f(x) = S_0 e^{x}$

$$S_t = S_0 + \int_0^t f'(S_{u-})\, dX_u + \frac{1}{2}\int_0^t f''(S_{u-})\, d[X]^c_u$$

$$+ S_0 \sum_{0 \le u \le t} \left(e^{X_{u-} + \Delta X_u} - e^{X_{u-}} - e^{X_{u-}} . \Delta X_u \right)$$

$$= S_0 + \int_0^t S_{u-}\, dX_u + \frac{1}{2}\int_0^t S_{u-}\, d[X]^c_u + \sum_{0 \le u \le t} S_{u-} \left(e^{\Delta X_u} - 1 - \Delta X_u \right)$$

Ex. X Ito process $\quad X_t = X_0 + \int_0^t \sigma_u\, dW_u + \int_0^t \mu_u\, du$

$$[X]^c_t = [X]_t = \int_0^t \sigma_u^2\, du$$

$$S_T = S_0 + \int_0^T S_t \sigma_t\, dW_t + \int_0^T S_t \mu_t\, dt + \frac{1}{2}\int_0^T S_t \sigma_t^2\, dt$$

$$\text{“} \frac{dS_t}{S_t} = \sigma_t\, dW_t + \left(\mu_t + \frac{\sigma_t^2}{2}\right) dt \text{”} \qquad dS_t = S_t\, dX_t + \frac{1}{2}\sigma_t^2\, dt$$

Stochastic exponential of a semimartingale "Doléans-Dade" exponential

Let X be a semimartingale with $X_0 = 0$. Then

$$Z_t = \exp\left(X_t - \frac{1}{2}[X]_t\right) \prod_{0\leq s\leq t} (1+\Delta X_s)\exp\left(-\Delta X_s - \frac{(\Delta X_s)^2}{2}\right)$$

$$= \exp\left(X_t - \frac{1}{2}[X]_t^c\right) \prod_{0\leq s\leq t} (1+\Delta X_s)\, e^{-\Delta X_s}$$

is a semimartingale which verifies

$$Z_t = 1 + \int_0^t Z_{u-}\, dX_u \qquad \text{i.e.} \qquad dZ_t = Z_{t-}\, dX_t$$

Z is called the stochastic exponential of X : $Z = \mathcal{E}(X)$

Rem: if X local martingale then $\mathcal{E}(X)$ is also a local martingale.

Proof: Let $V_t^\varepsilon = \prod_{\substack{0\le s\le t\\ |\Delta X_s|>\varepsilon}} (1+\Delta X_s)e^{-\Delta X_s} = \exp\Big(\sum_{\substack{0\le s\le t\\ |\Delta X_s|\ge\varepsilon}} \ln(1+\Delta X_s) - \Delta X_s\Big)$

$|\ln(1+u)-u| \le K_\varepsilon u^2$ for $|u|\le\varepsilon$ so

$\sum_{0\le s\le t} |\ln(1+\Delta X_s) - \Delta X_s| \le K \sum_{0\le s\le t} |\Delta X_s|^2 < [X]_t < \infty$ $\therefore$ $V_t^\varepsilon \xrightarrow{\varepsilon\to 0} V_t = \prod_{0\le s\le t} e^{-\Delta X_s}(1+\Delta X_s)$

Let $U_t = X_t - \frac{1}{2}[X]_t^c$. $Z_t = f(U_t, V_t)$ where $f(u,v) = ve^u$

$\frac{\partial f}{\partial u} = \frac{\partial^2 f}{\partial u^2} = f$; $\frac{\partial f}{\partial v}(u,v) = e^u$. Now: $[U]_t^c = [X]_t^c$

$$V_t = \prod_{0\leq s\leq t} e^{-\Delta X_s}(1+\Delta X_s) = \exp\Big(\underbrace{\sum_{0\leq s\leq t} \ln(1+\Delta X_s) - \Delta X_s}_{S_t}\Big)$$

S is a pure jump process

$$|\Delta S_t| = |\ln(1+\Delta X_t) - \Delta X_t| \leq K_\varepsilon |\Delta X_t|^2 \quad \text{for } |\Delta X_t| \leq \varepsilon$$

$$\sum_{\substack{|\Delta X_s|\leq\varepsilon\\ 0\leq s\leq t}} |\Delta S_u| \leq \sum_{\substack{|\Delta X_s|\leq\varepsilon\\ 0\leq s\leq t}} K_\varepsilon |\Delta X_s|^2 < [X]_t K_\varepsilon \quad \text{so} \quad \sum_{0\leq u\leq t} |\Delta S_u| < \infty \text{ a.s.}$$

So $V = e^S$ is a pure jump process of finite variation therefore

$$[V]_t = \sum_{0\leq s\leq t} (\Delta V_s)^2, \quad [V]^c_t = 0, \quad [V,U]^c_t = 0$$

Applying the Ito formula to $f(U_t, V_t)$ yields :

$$Z_t = 1 + \int_0^t Z_{s-}\, dU_s + \int_0^t e^{X_{s-}-[X]^c_{s-}} dV_s + \frac{1}{2}\int_0^t Z_{s-}\, d[U]^c_s$$

$$+ \sum_{0\leq s\leq t} Z_s - Z_{s-} - \Delta U_s \cdot Z_{s-} - \Delta V_s\, e^{U_{s-}}$$

$$U = X - \frac{1}{2}[X]^c$$

so

$$= 1 + \int_0^t Z_{s-}\, dX_s - \frac{1}{2}\int_0^t Z_{s-}\, d[X]^c_s + \frac{1}{2}\int_0^t Z_{s-}\, d[U]^c_s$$

$$+ \sum_{0\leq s\leq t} e^{U_{s-}} \Delta V_s + \sum_{0\leq s\leq t} Z_s - Z_{s-} - \Delta V_s e^{U_{s-}} - \Delta U_s \cdot Z_{s-}$$

$$\Delta U_t = \Delta X_t$$

$$= 1 + \int_0^t Z_{s-}\, dX_s + \sum_{0\leq s\leq t} Z_{s-}\left(1 + \Delta U_s - 1 - \Delta U_s\right)$$

since

$$Z_t = Z_{t-} e^{\Delta X_t}(1+\Delta X_t)\, e^{-\Delta X_t} = Z_{t-}(1+\Delta X_t) = Z_{t-}(1+\Delta U_t) \quad \text{so} \quad Z_t = 1 + \int_0^t Z_{s-}\, dX_s$$

Stochastic exponential : properties

(i) If X is continuous $\mathcal{E}(X) = e^{X - \frac{1}{2}[X]}$

(ii) If X is continuous with finite variation $\mathcal{E}(X) = e^{X}$

(iii) $\Delta \mathcal{E}(X)_t = \mathcal{E}(X)_{t-} \Delta X_t$ a.s.

(iv) $\mathcal{E}(X) > 0$ a.s. $\Leftrightarrow$ $\Delta X > -1$ a.s.

(v) $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X,Y])$

(vi) Preservation of local martingale property:

If X is a local martingale then $\mathcal{E}(X)$ is also a local martingale.

$$Z = \mathcal{E}(X) \qquad Z_t = 1 + \int_0^t Z_{u-}\, dX_u = e^{X_t - \frac{[X]_t}{2}} \prod_{0 \le s \le t} e^{-\Delta X_s}(1 + \Delta X_s)$$

Examples :

✓ $\mathcal{E}(\lambda W) = e^{\lambda W - \frac{\lambda^2 t}{2}}$

✓ N counting process : $N_t = \sum_{n \geq 1} 1_{t \leq T_n}$ $\quad (T_n)_{n \geq 1} \uparrow$ sequence of stopping times $\underset{n \to +\infty}{\longrightarrow} \infty$

$$\mathcal{E}(N)_t = e^{N_t} \prod_{0 \leq s \leq t} (1 + \Delta N_s) e^{-\Delta N_s} = e^{N_t} e^{-\overbrace{\sum_{s \leq t} \Delta N_s}^{N_t}} \prod_{0 \leq s \leq t} (1 + \Delta N_s)$$

$$= 2^{N_t} \qquad \mathcal{E}(N) = 2^N \qquad \text{since } [N]^c_t = 0$$

✓ Itô process : $X_t = \int_0^t \sigma_u \, dW_u + \int_0^t \mu_u \, du$

$$\mathcal{E}(X) = \exp\left(X_t - \frac{1}{2} \int_0^t \sigma_u^2 \, du\right)$$

✓ Finite variation process : $X_t = \int_0^t b_s \, ds + \sum_{0 \leq s \leq t} \Delta X_s$ $\quad \left| \begin{array}{l} \sum_{s \leq t} |\Delta X_s| < \infty \\ \Delta X_s > -1 \end{array} \right.$

$$\mathcal{E}(X) = e^{\int_0^t b_s \, ds} \prod_{0 \leq s \leq t} (1 + \Delta X_s) = \exp\left(X_t + \sum_{0 \leq s \leq t} \{\ln(1 + \Delta X_s) - \Delta X_s\}\right)$$

since $[X]^c = 0$

Exponential martingales

- If X is a martingale then $\mathcal{E}(X)$ is a **local** martingale

- If $\Delta X > -1$ then $\mathcal{E}(X)$ is a positive supermartingale, with $\mathcal{E}(X)_0 = 1$.

✓ If: $\forall t \geq 0$, $E(\mathcal{E}(X)_t) = 1$ then $\mathcal{E}(X)$ is a martingale, called the exponential martingale associated to X.

✓ It is not always easy to show that $E(\mathcal{E}(X)_t) = 1$.

✓ One method: write $\mathcal{E}(X)_t = \dfrac{e^{X_t}}{E(e^{X_t})}$

Proposition: Let M be a continuous local martingale.
If for each t there exists a constant $K_t > 0$, $[M]_t < K_t$ a.s.
then $\mathbb{P}(\max_{s \leq t} M_s > y) \leq \exp(-y^2/2K_t)$ $(*)$
and $\mathcal{E}(\theta M)$ is a martingale for every $\theta \in \mathbb{R}$.

Corollary: if $H \in \mathbf{L}$ is a bounded, left-continuous adapted process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and W a Brownian motion then

$$M_t = e^{\int_0^t H_s dB_s - \frac{1}{2}\int_0^t |H_s|^2 ds} = \mathcal{E}\left(\int H dB\right)$$

is a martingale.

Proof : $\mathbb{P}(\max_{s\le t} M_s > y) \le \mathbb{P}(\max_{s\le t} \mathcal{E}(\theta M)_s > e^{\theta y - \frac{K_t\theta^2}{2}}) \le e^{-\theta y + \frac{1}{2}\theta^2 K_t}$

Let $Z(t) = \max_{[0,t]} M$. $\quad E e^{\theta Z(t)} = 1 + \int dy\, \mathbb{P}(Z(t) \ge y)\, \theta e^{\theta y}$

$$\le 1 + \theta \int e^{\theta y} e^{-y^2/2K_t}\, dy < \infty$$

$Y = \mathcal{E}(\theta M) = \exp(\theta M - \frac{1}{2}\theta^2 [M])$ is a local martingale

so $\exists (T_n)_{n\ge 1}$ stopping times, $T_n \uparrow \infty$ a.s.,

$$E[\, Y_{T_n \wedge t} \mid \mathcal{F}_s \,] = Y_{T_n \wedge s} \quad \forall t \ge s$$

Since $Y_{T_n \wedge t}$ is dominated by $e^{\theta Z(t)}$, $E e^{\theta Z(t)} < \infty$

the dominated convergence theorem applies as $n \to \infty$:

$$E[\, Y_t \mid \mathcal{F}_s \,] = Y_s \quad \forall t \ge s \quad : \quad Y \text{ is a martingale.}$$

$$\mathcal{E}(\theta M) = \exp\left(\theta M - \frac{\theta^2}{2}[M]\right)$$

Examples of exponential martingales

✓ $\forall \lambda > 0$, $\mathcal{E}(\lambda W)_t = \exp\left(\lambda W_t - \frac{\lambda^2}{2} t\right)$ is a martingale.

✓ If N is a Poisson process and $M_t = N_t - \lambda t$

$$\mathcal{E}(M) = e^{N_t \ln 2 - \lambda t} \quad \text{is a martingale}$$

✓ If $\theta \in L$ is a bounded, $\mathcal{F}_t$-adapted caglad process

$$X_t = \int_0^t \theta_s \, dW_s \quad \text{is a square-integrable martingale}$$

$$\mathcal{E}(X)_t = \exp\left(\int_0^t \theta_s \, dW_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right) \quad \text{is a martingale}$$

Lévy's theorem or how to recognize a Brownian motion when you meet one

Theorem: A continuous local martingale X is a Brownian if and only if: $\forall t \geq 0 \quad [X]_t = t$

Proof: first note that $E(X_t^2) = E([X]_t) = t < \infty$
so X is a square-integrable martingale.
Apply Ito's formula to $f(t,x) = \exp\left(iux + \frac{u^2 t}{2}\right)$:

$$\underbrace{e^{iuX_t + \frac{u^2 t}{2}}}_{Y_t} = 1 + \int_0^t iu\, e^{iuX_s + \frac{u^2 s}{2}} dX_s + \int_0^t \frac{u^2}{2} e^{iuX_s + \frac{u^2 s}{2}} ds - \int_0^t \frac{u^2}{2} e^{iuX_s + \frac{u^2 s}{2}} d[X]_s$$

$$= 1 + \int_0^t iu\, e^{iuX_s + \frac{u^2 s}{2}} dX_s$$

By the martingale preserving property of the stochastic integral, Y is a local martingale

$[Y]_t = \int_0^t |u|^2 \, d[X]_s = |u|^2 t$ so $E(Y_t^2) = E[Y]_t < \infty$

so Y is a square-integrable martingale. So $\forall t \geq s$

$E[Y_t \mid \mathcal{F}_s] = Y_s$ which means $E[e^{iu(X_t - X_s)} \mid \mathcal{F}_s] = e^{-\frac{u^2}{2}(t-s)}$

so $X_t - X_s \sim N(0, t-s)$ is independent of $\mathcal{F}_s$

Thus X is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Lévy's theorem : original form

- Let M be a continuous local martingale with $M_0 = 0$.

If $M_t^2 - t$ is a local martingale then M is a Brownian motion.

Proof : By Itô's formula $M_t^2 = \int_0^t 2M_s \, dM_s + [M]_t$

so $M_t^2 - \int_0^t 2M_s \, dM_s$ has finite variation

$M_t^2 - [M]_t$ is a continuous local martingale

So $[M]_t - t = M_t^2 - t - (M_t^2 - [M]_t)$ is a continuous local martingale. But $[M]_t - t$ is a process of finite variation so $[M]_t - t = 0$. Thus by the previous theorem M is Brownian motion.

Lévy's theorem : multidimensional version

If $(X^1, \dots, X^d)$ are continuous local martingales with $X^i_0 = 0$ and

$$[X^i]_t = t \qquad [X^i, X^j] = 0 \quad \forall i \neq j$$

the $X = (X^1, \dots, X^d)$ is a d-dimensional Brownian motion.

Proof: By Lévy's theorem, X^i is a Brownian motion

/compute $E(e^{iu \cdot X_t})$ for $u \in \mathbb{R}^d$ by applying Ito formula to $e^{iu \cdot X_t}$ for $u \in \mathbb{R}^d$

Dubins-Schwarz theorem:

Let M be a continuous local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $M_0 = 0$, $[M]_t \xrightarrow{t\to\infty} \infty$. Define

$$T_t = \inf\{u > 0,\ [M]_u > t\}$$

$\mathcal{G}_t = \mathcal{F}_{T_t}$. Then $B_t = M_{T_t}$ is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$ and $M_t = B_{[M]_t}$ $\mathbb{P}$-a.s.

Any continuous local martingale is a time-changed Brownian motion!

Proof : $T_t = \inf\{u \geq 0, [M]_u > t\}$

$[M]_\cdot(\omega) : [0,\infty) \to [0,\infty)$

$t \to [M]_t(\omega)$

is strictly $\uparrow$ continuous one-to-one

T_t is ✓ finite a.s. since $[M]_t \xrightarrow{t\to\infty} \infty$

✓ a hitting time of a cadlag adapted process so a stopping time.

$\{T_t \leq s\} = \{[M]_s > t\}$

Define $B_t = M_{T_t}$ and $\mathcal{G}_t = \mathcal{F}_{T_t}$.

Then $[B]_t = [M]_{T_t} = t$ by continuity.

The map $T : (0,\infty) \times \Omega \longmapsto (0,\infty)$

$(t,\omega) \longrightarrow \inf\{u > 0, [M]_u(\omega) > t\}$

is continuous, increasing in t almost surely

and $T \circ [M] = [M] \circ T = \mathrm{Id}$ $\mathbb{P}$- a.s.

so $B_t = M_{T_t}$ is a.s. continuous and $M_t = B_{[M]_t}$

$\exists (\tau_n)_{n\geq 1}$ $\mathbb{F}$ - stopping times $\tau_n \uparrow \infty$ a.s., M^{τ_n} is a martingale

$U_n = T \circ \tau_n$ are $(\mathcal{G}_t)$ - stopping times, $U_n \underset{a.s.}{\uparrow} \infty$

✓ By the optional sampling theorem : $\forall t \geq s$,

$$E[B_t^{U_n} - B_s^{U_n} \mid \mathcal{G}_{s \wedge U_n}] = E[M_{T_t}^{\tau_n} - M_{T_s}^{\tau_n} \mid \mathcal{F}_{T_s \wedge \tau_n}] = M_{T_s} - M_{T_s} = 0$$

So B is a continuous local martingale, $[B]_t = t$ a.s.

So by Lévy's theorem

B is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$.

And we have shown that $M_t = B_{[M]_t}$.

Example: X Itô process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

$$X_t = \int_0^t \sigma_u \, dW_u, \quad \sigma \in L$$

adapted left-continuous stochastic process

$$\int_0^\infty \sigma_t^2 \, dt = \infty$$

$$[X]_t = \int_0^t \sigma_u^2 \, du \qquad T_t(\omega) = \inf \left\{ u > 0, \int_0^u \sigma_s^2(\omega) ds > t \right\}$$

So: $B_t = X_{T_t}$ is a Brownian motion on $(\Omega, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$

where $\mathcal{G}_t = \mathcal{F}_{T_t}$

and $\boxed{X_t = B_{\int_0^t \sigma_u^2 du}}$ (apply to $u = T_t^{-1}(t)$)

In particular: $X \overset{d}{=} B_{\int_0^t \sigma_u^2 du}$

Watanabe's theorem or how to recognize a Poisson process next time you meet one

Theorem: Let N be a counting process

$$N_t = \sum_{n \geq 1} 1_{t \leq T_n}$$

If $M_t = N_t - \lambda t$ is a local martingale, then N is a Poisson process with intensity λ.

Proof : Apply Ito's formula to N with $f(x) = e^{i\theta x}$:

$$e^{i\theta N_t} = 1 + \sum_{0 \leqslant s \leqslant t} \left(e^{i\theta N_s} - e^{i\theta N_{s-}} \right)$$

$M_t = N_t - \lambda t$ local martingale

$$= 1 + \int_0^t \left(e^{i\theta(N_{s-} + 1)} - e^{i\theta N_{s-}} \right) dN_s$$

$$= 1 + \underbrace{\int_0^t \left(e^{i\theta(N_{s-}+1)} - e^{i\theta N_{s-}} \right) dM_s}_{Z_t} + \int_0^t \left(e^{i\theta(N_{s-}+1)} - e^{i\theta N_{s-}} \right) \lambda \, ds$$

By the martingale-preserving property of the stochastic integral, Z is a local martingale

$$[Z]_t = \sum_{0 \leqslant s \leqslant t} \left(e^{i\theta(N_{s-}+1)} - e^{i\theta N_{s-}} \right)^2 \leqslant 4 N_t$$

So: $E(Z_t^2) = E([Z]_t) \leqslant 2r\, E(N_t) < \infty$

so Z is a L^2-martingale.

$\forall t \geq s, \quad E[e^{i\theta N_t} \mid \mathcal{F}_s] = e^{i\theta N_s} + \lambda(e^{i\theta}-1)\int_s^t E[e^{i\theta N_u} \mid \mathcal{F}_s]\, du$

Let $\varphi_t(\theta, s) = E[\exp i\theta(N_t - N_s) \mid \mathcal{F}_s]$

$\varphi_t(\theta, s) = 1 + \lambda(e^{i\theta}-1)\int_s^t \varphi_u(\theta, s)\, du$ so $\varphi_\cdot(u, s)$

solves the ODE: $\frac{\partial}{\partial t}\varphi_t(\theta, s) = \lambda(e^{i\theta}-1)\varphi_t(\theta, s)$

whose solution is $\varphi_t(\theta, s) = \exp[(t-s)\lambda(e^{i\theta}-1)]$

Characteristic
function of Poisson distribution $(\lambda(t-s))$

By iterating, we obtain that for $0 = t_0 < t_1 < \cdots < t_k$

$$E\left[\prod_{k=1}^{n} e^{i\theta_k (N_{t_k} - N_{t_{k-1}})}\right] = \prod_{k=1}^{n} e^{\lambda(t_k - t_{k-1})(e^{i\theta_k}-1)}$$

so the increments of N are independent and $\sim$ Poisson$(\lambda(t_k - t_{k-1}))$

Intensity of a counting process

Consider a counting process $N_t = \sum_{n \geq 1} 1_{t \leq T_n}$ $(T_n)_{n \geq 1}$ increasing sequence of stopping times on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Def : An $\mathbb{F}$-adapted process $(\lambda_t)_{t \geq 0}$ is an $\mathbb{F}$-intensity for N if

$$M_t = N_t - \int_0^t \lambda_u \, du$$

is a local martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

- An increasing process $(H_t)_{t \geq 0}$ is called a $(\mathcal{F}_t)_{t \geq 0}$-hazard process for N if $N - H$ is a local martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

✓ For the Poisson process this coincides with the usual definition of intensity

✓ **Watanabe** : Any counting process with constant intensity is a Poisson process.

Representation of continuous martingales as Brownian stochastic integrals

Let M be a continuous local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $t \longrightarrow [M]_t$ is absolutely continuous and non-vanishing:

There exists $a : [0,\infty) \times \Omega \to (0,\infty)$ such that

$$\frac{d}{dt}[M]_t(\omega) = a_t(\omega) > 0 \quad dt \times d\mathbb{P} \text{ a.e.} \qquad (*)$$

Then there exists a Brownian motion B on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that: $M_t = M_0 + \int_0^t \sqrt{a_u}\, dB_u$

Define : $B_t = \int_0^t \frac{dM_u}{\sqrt{a_u}}$: B is a continuous local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$

$[B]_t = \int_0^t \frac{d[M]_u}{(\sqrt{a_u})^2} = \int_0^t \frac{a_u}{a_u} \, du = t$ so by Lévy's theorem

B is a Brownia motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

And $M_t = M_0 + \int_0^t \sqrt{a_u} \, dB_u$

Representation of continuous martingales as Brownian stochastic integrals : multidimensional version

Let $(X^1, \dots X^d)$ be continuous local martingales on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

Such that $t \to [X](t) = \left([X^i, X^j]\right)_{i,j=1..d}(t)$ is absolutely continuous:

$\exists\, a : [0,\infty) \times \Omega \to Sym^+(d)$ adapted such that $\frac{d}{dt}[X](\omega) = a(t,\omega) \geq 0$

i.e. $\frac{d}{dt}[X^i, X^j](\omega) = a^{ij}(t,\omega)$

Then there exists a Brownian motion $(W^1, \dots, W^d)$ on $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \mathbb{P})$ and an adapted $\mathbb{R}^{d\times d}$-valued process $(\sigma_t)_{t\geq 0}$ such that

$$X_t = X_0 + \int_0^t \sigma_u \cdot dW_u$$

Proof: Let $T>0$. Let $\underline{\lambda}_t(\omega)$ = smallest eigenvalue of $a_t(\omega)$

$\underline{\lambda}_t(\omega) > 0$ and $t \to \underline{\lambda}_t(\omega)$ is caglad

So $\inf_{t\in[0,T]} \underline{\lambda}_t(\omega) = \min_{t\in[0,T]} \underline{\lambda}_t(\omega) > 0$ a.s.

Therefore there exists a measurable process

$$\sigma : [0,T]\times\Omega \longrightarrow M_{d\times d}(\mathbb{R})$$

such that $\sigma_t^* \sigma_t(\omega) = a_t(\omega)$ $dt\times d\mathbb{P}$ a.e.

In particular $\det(\sigma_t(\omega)) = \sqrt{\det|a_t(\omega)|} > 0$: $\sigma_t(\omega)$ is a.s. invertible.

Define now: $W_t = \int_0^t \sigma_u^{-1} . dX_u \doteq (W^1, \dots, W^d)$

$$[W]_t = \int_0^t (\sigma_u^{-1})(\sigma_u^{-1})^* d[X] = \int_0^t \sigma_u^{-1}(\sigma_u^{-1})^* a_u \, du$$

$$= \int_0^t \sigma_u^{-1}(\sigma_u^{-1})^* \sigma_u^* \sigma_u \, du = t I_d$$

Each W^i is a continuous local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

and $[W^i, W^j] = t\,\delta_{ij}$ so by

the multidimensional version of Lévy's theorem,

W is a standard d-dimensional Brownian motion

on the same space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

Furthermore $$X_t = X_0 + \int_0^t \sigma_u \cdot dW_u$$

W Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

$$B_t = \int_0^t \underbrace{\mathrm{sgn}(W_u)}_{\sigma_u} \, dW_u \qquad \mathrm{sgn}(x) = +1 \text{ if } x > 0, \quad = -1 \text{ if } x \leq 0$$

$$[B]_t = \int_0^t \sigma_u^2 \, du = t$$

B is a Brownian motion by Lévy's Theorem

B is $(\mathcal{F}_t)_{t\geq 0}$-adapted but W is not adapted to $(\mathcal{F}_t^B)_{t\geq 0}$.

$$[B, W] = \left[\int_0^t \mathrm{sgn}(W_u) \, dW_u , \int_0^t dW_u \right] = \int_0^t \mathrm{sgn}(W_u) \, du$$

$=$ Time spent > 0 during $[0,t]$ $-$ Time spent < 0 by W