• **Stochastic Calculus** **R Cont**

Note Title 2/8/2011

Equivalent changes of measure

Equivalence of probability measures on filtered spaces.
Local vs global equivalence of probability measures
The Girsanov-Meyer theorem.
Change of measure for Brownian motion
Change of measure for Poisson processes
Change of measure for counting processes
Change of measure for Poisson random measures

Equivalence of probability measures

Consider probability measures $\mathbb{P}, \mathbb{Q}$ on $(\Omega, \mathcal{F})$.

✓ $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ if $\quad \mathbb{Q} \ll \mathbb{P}$

$\forall A \in \mathcal{F}, \quad \mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$ $\quad$ Events impossible under $\mathbb{P}$ $\subset$ Events impossible under $\mathbb{Q}$

Radon-Nikodym theorem: there exists a random variable $Z \geq 0$,

$$\mathbb{Q}(A) = E^{\mathbb{P}}[1_A Z] \qquad E^{\mathbb{Q}}(X) = E^{\mathbb{P}}[ZX] \qquad Z = \frac{d\mathbb{Q}}{d\mathbb{P}} \geq 0$$

$$E^{\mathbb{P}}(Z) = 1$$

Equivalence: $\mathbb{Q} \sim \mathbb{P} \iff (\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}) \iff Z = \frac{d\mathbb{Q}}{d\mathbb{P}} > 0$

$Z^{-1} = d\mathbb{P}/d\mathbb{Q}$. Then $\mathbb{P}(A) = 1 \iff \mathbb{Q}(A) = 1$

✓ Invariance of convergence in probability: If $\mathbb{Q} \sim \mathbb{P}$, $\left(X_n \xrightarrow[\mathbb{P}]{n \to \infty} X \right) \iff \left(X_n \xrightarrow[n \to \infty]{\mathbb{Q}} X \right)$

Change of measure and expectations

$\mathbb{P} \sim \mathbb{Q}$ on $(\Omega, \mathcal{F})$ $\quad Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$

- For any $\mathcal{F}$-measurable random variable X

$$E^{\mathbb{P}}[X] = E^{\mathbb{Q}}[ZX]$$

Bayes formula

If $\mathcal{D} \subset \mathcal{F}$ is a sub σ-algebra

$$E^{\mathbb{P}}[X \mid \mathcal{D}] = \frac{E^{\mathbb{Q}}[XZ \mid \mathcal{D}]}{E^{\mathbb{Q}}[Z \mid \mathcal{D}]}$$

Change of measure on a filtered probability space

$\mathbb{Q} \sim \mathbb{P}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]})$, $\mathcal{F} = \mathcal{F}_T$ $\quad Z = \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^1(\mathbb{P})$

Then $Z_t = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]$ is a positive $\mathbb{P}$-martingale with

$E^{\mathbb{P}}(Z_t) = 1 \quad \forall t \in [0,T]$ (Density process)

✓ For any $\mathcal{F}_T$-measurable variable X: $\boxed{E^{\mathbb{Q}}(X \mid \mathcal{F}_t) = E^{\mathbb{P}}\left(\frac{Z_T}{Z_t} X \,\middle|\, \mathcal{F}_t\right)}$

Proof: $Y = E^{\mathbb{Q}}[X \mid \mathcal{F}_t]$ is the unique $\mathcal{F}_t$-measurable variable such that for any $\mathcal{F}_t$-measurable variable U, $E^{\mathbb{Q}}[XU] = E^{\mathbb{Q}}[YU] \iff$

$$E^{\mathbb{P}}(XZ_TU) = E^{\mathbb{P}}(YZ_TU) = E^{\mathbb{P}}\left(E^{\mathbb{P}}(YUZ_T \mid \mathcal{F}_t)\right) = E^{\mathbb{P}}(YUZ_t)$$

So: $YZ_t = E^{\mathbb{P}}(XZ_T \mid \mathcal{F}_t]$ so $Y = E^{\mathbb{P}}\left[\frac{Z_T}{Z_t} X \,\middle|\, \mathcal{F}_t\right]$

Change of measure on a filtered space

$Q \sim \mathbb{P}$, $Z_t = E^{\mathbb{P}}\left(\frac{dQ}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right)$

(i) $Z_t = E^{\mathbb{P}}\left(\frac{dQ}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right)$ is a positive $\mathbb{P}$-martingale

with $E(Z_t) = 1$, $\forall t \in [0,T]$.

(ii) $\frac{1}{Z} = E^{Q}\left[\frac{d\mathbb{P}}{dQ} \,\middle|\, \mathcal{F}_t\right]$ is a Q-martingale, $E^{Q}\left(\frac{1}{Z_t}\right) = 1$

(iii) $(M_t)_{t \geq 0}$ is a $\mathbb{P}$-(local) martingale $\iff$ $(M_t Z_t)_{t \geq 0}$ is a Q-(local) martingale.

(iv) If X is a semi-martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ then X is a semi-martingale on $(\Omega, \mathcal{F}, \mathbb{F}, Q)$.

(v) $[X]^{\mathbb{P}} = [X]^{Q}$ $\mathbb{P}$-a.s. : quadratic variation is invariant under equivalent change of measure.

Proof:

(iii) If M is a $\mathbb{P}$-local martingale

$\exists\, (T_n)_{n\geq 1}$ stopping times such that M^{T_n} is a $\mathbb{P}$-martingale ; $T_n \underset{n\to\infty}{\uparrow} \infty$ a.s.

$$\forall t \geq s \quad E^{\mathbb{P}}[\, M_t^{T_n} \mid \mathcal{F}_s\,] = M_s^{T_n} \qquad E^{\mathbb{Q}}\left[\, M_{t\wedge T_n} \frac{Z_{t\wedge T_n}}{Z_{s\wedge T_n}} \,\Big|\, \mathcal{F}_s\right] = M_s^{T_n}$$

$$E^{\mathbb{Q}}[\, M_t^{T_n} Z_{t\wedge T_n} \mid \mathcal{F}_s\,] = E^{\mathbb{Q}}[(MZ)_{t\wedge T_n} \mid \mathcal{F}_s] = M_{s\wedge T_n} Z_{s\wedge T_n}$$

which, by the optional sampling theorem, is equal to

$$= M_{s\wedge T_n} Z_{s\wedge T_n}$$

so $(MZ)^{T_n}$ is a $\mathbb{Q}$-martingale

Since $T_n \uparrow \infty$ $\mathbb{P}$-a.s., $T_n \uparrow \infty$ $\mathbb{Q}$-a.s.

Therefore MZ is a $\mathbb{Q}$-local martingale

Local martingales under change of measure $Q \sim \mathbb{P}$ $L_T = \frac{dQ}{d\mathbb{P}}$ $L(t) = E[L \mid \mathcal{F}_t]$

Theorem 12.4 *If M is a Q-local martingale, then*

$$Z(t) = M(t) - \int_0^t \frac{1}{L(s)} d[L, M]_s \tag{12.3}$$

is a P-local martingale. (Note that the integrand is $\frac{1}{L(s)}$, not $\frac{1}{L(s-)}$.)

Proof. Note that $LM - [L, M]$ is a Q-local martingale. We need to show that LZ is a Q-local martingale. But letting V denote the second term on the right of (12.3), we have

$$L(t)Z(t) = L(t)M(t) - [L, M]_t - \int_0^t V(s-)dL(s),$$

and both terms on the right are Q-local martingales. $\square$

Recall the Itô product formula: for semimartingales X, Y

$$X_t Y_t = \int_0^t X_- \, dY + \int_0^t Y_- \, dX + [X,Y]_t$$

In particular if M^1, M^2 are (local) martingales

$$M^1_t M^2_t - [M^1, M^2]_t = \int_0^t M^1_- \, dM^2 + \int_0^t M^2_- \, dM^1$$

which is a sum of two local martingales

✓ Equivalently: if M is a $\mathbb{Q}$-local martingale then under $\mathbb{P}$: $M_t = Z_t + \int_0^t \frac{1}{L(u)} \, d[L,M]_u$

$\uparrow$ $\mathbb{P}$-local martingale

M acquires a drift component $\int_0^t \frac{d[L,M]_u}{L(u)}$ under $\mathbb{P}$

✓ Note the difference between $\int \phi_{u-} dX_u$ and $\int \phi_u dX_u$:

if ϕ is a cadlag adapted process

$$\int_0^t \phi_u \, dX_u = \int_0^t \phi_{u-} \, dX_u + \sum_{0 \leq s \leq t} \Delta\phi(s) \, \Delta X(s)$$

Girsanov - Meyer Theorem :

Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$, $Z_t = E[Z \mid \mathcal{F}_t]$.

If $X = M + A$ where M $\mathbb{P}$-local (square integrable) martingale

A continuous process with finite variation

then X is also a $\mathbb{Q}$-semimartingale with decomposition

$X = L + B$ where

$$L_t = M_t - \int_0^t \frac{1}{Z_s} \, d[Z,M]_s$$ is a $\mathbb{Q}$-local martingale

and $B_t = A_t + \int_0^t \frac{d[Z,M]_s}{Z_s}$ is a continuous process of finite variation on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$.

Girsanov - Meyer Theorem : interpretation

- when switching from $\mathbb{P}$ to $\mathbb{Q}$, $E\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t\right] = Z_t$

✓ A semimartingale X under $\mathbb{P}$ remains a semimartingale under $\mathbb{Q}$

✓ A local martingale M under $\mathbb{P}$ is not anymore a local martingale under $\mathbb{Q}$ but

- M_t/Z_t is a $\mathbb{Q}$-local martingale

- $M_t - \int_0^t \frac{1}{Z_s} d[Z,M]_s$ is a $\mathbb{Q}$-local martingale

In particular if $[Z,M]=0$, M is still a $\mathbb{Q}$-local martingale.

Proof : M, Z are $\mathbb{P}$-local martingales so $Z_t = E^{\mathbb{P}}[Z \mid \mathcal{F}_t]$

$$Z_t M_t - [Z, M]_t = \int_0^t M_- \, dZ + \int_0^t Z_- \, dM$$

is a $\mathbb{P}$-local martingale. Dividing by Z, we get a $\mathbb{Q}$-local martingale:

$$M_t - \frac{1}{Z_t}[Z,M]_t = \frac{1}{Z_t}\left(\int_0^t M_- \, dZ + \int_0^t Z_- \, dM\right)$$

Using the Ito product formula under $\mathbb{Q}$:

$$\frac{1}{Z_t}[Z,M]_t = \int_0^t \frac{1}{Z_{s-}} d[Z,M]_s + \underbrace{\int_0^t [Z,M]_{s-} \, d\left(\frac{1}{Z_s}\right)}_{N_t} + \left[[Z,M], \frac{1}{Z}\right]_t$$

Since $\frac{1}{Z}$ is a $\mathbb{Q}$-local martingale, N is also a $\mathbb{Q}$-local martingale since it is a stochastic integral with respect to the $\mathbb{Q}$-local mart. $\frac{1}{Z}$

✓ Since $[Z,M]$ has finite variation

$$\left[[Z,M], \frac{1}{Z} \right]_t = \sum_{0 \leq s \leq t} \Delta([Z,M])_s \, \Delta\left(\frac{1}{Z}\right)_s$$

so

$$\int_0^t \frac{1}{Z_{s-}} d[Z,M]_s + \left[[Z,M], \frac{1}{Z} \right]_t = \int_0^t \frac{1}{Z_s} d[Z,M]_s$$

using the above Lemma

So

$$\frac{1}{Z_t} [Z,M]_t = \int_0^t \frac{1}{Z_s} d[Z,M]_s + N_t$$

✓ So the sum

$$L_t = \underbrace{N_t}_{Q\text{-loc. mart.}} + \underbrace{M_t - \frac{1}{Z_t}[Z,M]_t}_{Q\text{-loc. mart}}$$

$$= \frac{1}{Z_t}[Z,M]_t - \int_0^t \frac{d[Z,M]}{Z} + M_t - \frac{1}{Z_t}[Z,M]_t = M_t - \int_0^t \frac{1}{Z_s} d[Z,M]$$

is a Q-local martingale.

So: $\frac{dQ}{d\mathbb{P}} = Z_T$ $\qquad Z_t = E^{\mathbb{P}}[Z_T \mid \mathcal{F}_t]$

M $\mathbb{P}$-(local) martingale $\Rightarrow$ $M_t - \int_0^t \frac{d[Z,M]}{Z}$ Q-(local) martingale

If $M - \int_0^t \frac{d[Z,M]}{Z}$ is a martingale : $\forall t \geq s$,

$$E^{Q}\left[M_t - \int_0^t \frac{d[Z,M]}{Z} \,\middle|\, \mathcal{F}_s \right] = M_s - \int_0^s \frac{d[Z,M]}{Z}$$

$$E^{Q}[M_t \mid \mathcal{F}_s] = M_s + E^{Q}\left[\int_s^t \frac{d[Z,M]}{Z} \,\middle|\, \mathcal{F}_s \right]$$

Example : change of measure for Brownian motion

Let W be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and define $\mathcal{F}_t = \mathcal{F}_t^W$ natural filtration of W and $U_t = \int_0^t \theta_s \, dW_s$ where $\theta \in L$ bounded, caglad, $\mathcal{F}_t$-adapted

Then : $Z_t = \mathcal{E}(U)_t = \exp\left(\int_0^t \theta_s \, dW_s - \frac{1}{2} \int_0^t \theta_s^2 \, ds \right)$

is a $\mathbb{P}$-martingale and $E(Z_T) = 1$.

Now fix $T>0$ and define $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$:

$\forall t \leq T, \forall A \in \mathcal{F}_t, \quad \mathbb{Q}(A) = E^{\mathbb{P}}[1_A Z_T] = E^{\mathbb{P}}[1_A Z_t]$.

Cameron-Martin-Girsanov Theorem :

$\tilde{W}_t = W_t - \int_0^t \theta_s \, ds$ is a Brownian motion on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{Q})$.

Proof: $Z_t = \mathcal{E}(U)_t = \mathcal{E}\left(\int_0 \theta \, dW\right)_t = 1 + \int_0 Z_s \, dU_s$

so $dZ_t = Z_t \, dU_t = Z_t \theta_t \, dW_t$

So $[Z, W]_t = \left[\int Z\theta \, dW, \int dW\right]_t = \int_0^t Z_u \theta_u \, du$

Therefore $W_t - \int_0^t \frac{d[Z,W]}{Z} = W_t - \int_0^t \frac{Z_u \theta_u \, du}{Z_u} = W_t - \int_0^t \theta_u \, du$

is a $\mathbb{Q}$-local martingale

Interpretation W is Brownian motion under $\mathbb{P}$

$$\mathbb{P} \longrightarrow \mathbb{Q} \text{ defined as } \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds}$$

then W 'acquires a drift $\int_0^t \theta_s ds$ under $\mathbb{Q}$':

W has the law of $B_t^{\mathbb{Q}} + \int_0^t \theta_s ds$ where

$B^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion

Adding a drift $\int_0^t \theta_s ds$ to Brownian motion W $\iff$ Weighting the probabilities of paths on $[0,T]$ with $\exp\left(\int_0^T \theta \, dW - \frac{1}{2}\int_0^T \theta_s^2 ds\right)$

Proof: ✓ $[\tilde{W}]_t = [W]_t = t$ ✓ $Z = \mathcal{E}(\int \theta\, dW)$ is a continuous process.

✓ $\tilde{W}$ is continuous Q-a.s. since W is continuous P-a.s.

$$Z_t = 1 + \int_0^t Z_{s-}\, dU_s = 1 + \int_0^t Z_{s-}\,\theta_s\, dW_s$$

So $$[Z, W]_t = \int_0^t Z_{s-}\,\theta_s\, ds = \int_0^t Z_s \theta_s\, ds$$ since $\left[\int \varphi\, dW, \int \psi\, dW\right]_t = \int_0^t \varphi\psi$

Applying the Girsanov Meyer theorem,

$$W_t - \int_0^t \frac{d[Z,W]_s}{Z_s}$$ is a Q-local-martingale

$$= W_t - \int_0^t \frac{Z_s\,\theta_s\, ds}{Z_s} = W_t - \int_0^t \theta_s\, ds.$$ So by Lévy's theorem $\tilde{W}$ is a Q-BM.

Transformation of Itô processes under change of measure

$$X_t = X_0 + \int_0^t \mu_u \, du + \int_0^t \sigma_u \, dW_u^{\mathbb{P}}$$

where $(W_t^{\mathbb{P}})_{t \geq 0}$ is standard BM on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

$Z = \mathcal{E}\left(\int_0 \theta \, dW^{\mathbb{P}}\right)$. If $E[Z_T] = 1$ then under $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ on $\mathcal{F}_T$, X is an Itô process with decomposition $\quad W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} - \int_0^t \theta_s \, ds$

$$X = X_0 + \underbrace{\int_0^t \sigma_u \, dW_u^{\mathbb{Q}}}_{\mathbb{Q}\text{-local martingale}} + \int_0^t \underbrace{(\mu_u + \sigma_u \theta_u) \, du}_{\text{continuous process of finite variation}}$$

Example: constant drift $\theta_t = \theta$, W BM under $\mathbb{P}$

$Z_t = \mathcal{E}(U)_t = \exp\left(\theta W_t - \frac{\theta^2}{2} t\right)$ $\quad \frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ on $\mathcal{F}_T$

Then under $\mathbb{Q}$, $W^{\mathbb{Q}} = W - \theta t$ is a Brownian motion.

BUT Z is not a uniformly integrable martingale.

So the equivalence on $\mathcal{F}_T$ cannot be extended to $\mathcal{F}_\infty$.

An 'orthogonal' change of measure

W^1, W^2 independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$

$\mathcal{F}_t = \mathcal{F}_t^W$, $W = (W^1, W^2)$ $\quad \mathcal{F}_t^i = \mathcal{F}_t^{W_i}$

$$Z_t^1 = \exp\left(\int_0^t \theta_s \, dW_s^1 - \frac{1}{2} \int_0^t \theta_s^2 \, ds \right) \text{ where}$$

θ is $\mathcal{F}^1$-adapted, bounded; $\quad \frac{dQ}{d\mathbb{P}} = Z_T^1$

$Z_t^1 = 1 + \int_0^t Z_s^1 \theta_s \, dW_s^1$ so $[Z, W^2] = 0$

So W^2 is (still) a Brownian motion under Q.

W BM under $\mathbb{P}$, $\mathcal{F}_t = \mathcal{F}_t^W$

$B^1 = W$; $B^2_t = \int_0^t \sigma_u \, dW_u$ σ $\mathcal{F}_t$-adapted

where $\sigma \neq 1$ $dt \times d\mathbb{P}$ a.e.

Then there is no $\mathbb{Q} \sim \mathbb{P}$ such that

B^2 is $\mathbb{Q}$- Brownian motions

Proof: $[B^i]^{\mathbb{Q}}_t = [B^i]^{\mathbb{P}}_t \neq t$

Change of measure for counting processes

Let N be a standard Poisson process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

$M_t = N_t - t$ the compensated Poisson martingale.

$h = (h_t)_{t\geq 0}$ be a caglad adapted process with $\mathbb{P}(h_t > -1) = 1$.

$$\int_0^t h\, dM = \int_0^t h\, dN - \int_0^t h_s ds = \sum_{0\leq s\leq t} h_s \Delta N_s - \int_0^t h_s ds$$

$$Z_t = 1 + \int_0^t Z_{s-} h_s\, dM_s \quad \text{i.e.} \quad Z = \mathcal{E}\left(\int h\, dM\right)$$

$$U_t = \int_0^t h_u\, dM_u = \int_0^t h_u\, dN_u - \int_0^t h_u du = \sum_{0\leq T_n\leq t} h_{T_n} - \int_0^t h_u du$$ Pure jump martingale

so $[U]^c_t = 0$

$$Z_t = \mathcal{E}(U)_t = \exp\left(-\int_0^t h_u du\right) \prod_{0\leq T_n\leq t} (1 + h_{T_n})$$

$$= \exp\left(-\int_0^t h_u du - \sum_{0\leq T_n\leq t} \ln(1 + h_{T_n})\right)$$

$$Z_t = \mathcal{E}\left(\int h\, dM\right)_t = \exp\left(\int_0^t dN_u \ln(1+h_u) - \int_0^t h_u\, du \right)$$

Z_t is a strictly positive local martingale

Assume $E[Z_T]=1$: then Z is a martingale.

(True for ex if h bounded). and define

$$\frac{dQ}{dP} = Z_T \quad \text{on} \quad \mathcal{F}_T.$$

Intensity of a counting process

Consider a counting process $N_t = \sum_{n \geq 1} 1_{t \leq T_n}$

on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Def : An $\mathbb{F}$-adapted process $(\lambda_t)_{t \geq 0}$ is

an $\mathbb{F}$-intensity for N if

$$M_t = N_t - \int_0^t \lambda_u \, du$$

is a local martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

✓ For the Poisson process this coincides with the usual definition of intensity by

Watanabe's theorem: Any counting process with constant intensity is a Poisson process.

Theorem: Let $\frac{dQ}{dP} = Z_T$ on $\mathcal{F}_T$ where $Z_T = \mathcal{E}\left(\int h \, dM\right)_T$

Then $Q \sim P$ and N is a counting process with intensity $\lambda^Q_t = 1 + h_t$ under Q.

Proof: we need to show that $V_t = N_t - \int_0^t (1 + h_u) du$ is a Q-local martingale. But this is equivalent to requiring that $Z_t V_t$ is a P-local martingale.

$M_t = N_t - t$ is a P-martingale, so by Girsanov-Meyer

$$M_t - \int_0^t \frac{d[Z, M]}{Z}$$ is a Q-local martingale.

$$Z = \mathcal{E}\left(\int h \, dM\right)$$ so

$$Z_t = 1 + \int_0^t Z_{s-} h_s \, dM_s \quad \text{so} \quad \Delta Z_t = Z_{t-} h_t \underbrace{\Delta M_t}_{=1} \quad M_t = N_t - t$$

$$[Z,M]_t = \sum_{0 \leq u \leq t} \Delta Z_u \Delta M_u = \sum_{0 \leq T_n \leq t} Z_{T_n -} h_{T_n} = \int_0^t Z_{u-} h_u \, dN_u = \int_0^t Z_{u-} h_u \, dM_u + \int_0^t Z_{u-} h_u \, du$$

$$\int_0^t \frac{d[Z,M]_u}{Z_u} = \int_0^t \frac{Z_{u-} h_u \, dM_u}{Z_u} - \int \frac{Z_{u-} h_u \, du}{Z_u} = \underbrace{\int_0^t \frac{Z_{u-} h_u}{Z_u} dM_u}_{\text{local martingale}} - \int_0^t h_u \, du$$

So $M - \int_0^t \frac{Z_{u-} h_u}{Z_u} dM_u - \int_0^t h_u \, du$ Q-local martingale

So $N_t - t - \int_0^t h_u \, du$ is a Q-loc. mart. : so N has Q-intensity $1 + h_t$

Change of measure for point processes

Let N be a counting process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

with intensity $(\lambda^{\mathbb{P}}_t)_{t\geq 0}$, $M_t = N_t - \int_0^t \lambda^{\mathbb{P}}_u \, du$

Let $(h_t)_{t\geq 0}$ be predictable $\mathbb{P}(h_t > -1) = 1$

$Z_0 = 1$ $dZ_t = Z_{t-} h_t (dN_t - \lambda^{\mathbb{P}}_t dt)$ i.e. $Z = \mathcal{E}\left(\int_0^{\cdot} h \, dM\right)$ on $[0,T]$.

If $E^{\mathbb{P}}[Z_T] = 1$ then $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ is a prob on $\mathcal{F}_T$

and under $\mathbb{Q}$ N has intensity $\lambda^{\mathbb{Q}}_t = \lambda^{\mathbb{P}}_t (h_t + 1)$

Local vs global equivalence of measures

This result does not hold if $\mathcal{F}_T$ is replaced by $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$

Let $\mathbb{P}^\lambda$ be the law of a Poisson process with intensity λ.

Proposition: if $\lambda^1 \neq \lambda^2$ then $\mathbb{P}^{\lambda_1}$ and $\mathbb{P}^{\lambda_2}$ are not equivalent measures. In fact they are mutually singular: $\exists A \in \mathcal{F}$, $\mathbb{P}^{\lambda_1}(A) = 1$, $\mathbb{P}^{\lambda_2}(A) = 0$.

However for each $T>0$, $\mathbb{P}_{|\mathcal{F}_T} \sim \mathbb{Q}_{|\mathcal{F}_T}$:

$\mathbb{P}$ and $\mathbb{Q}$ are **locally equivalent** on $(\mathcal{F}_t)_{t\geq 0}$

Proof: $\lambda^1 \neq \lambda^2$. Then the strong law of large numbers implies

$$\mathbb{P}^{\lambda_1}\left(\frac{N_t}{t} \xrightarrow{t\to\infty} \lambda^1\right) = 1 = \mathbb{P}^{\lambda^2}\left(\frac{N_t}{t} \xrightarrow{t\to\infty} \lambda^2\right)$$

So if $A = \left\{\omega \in \Omega, \frac{N_t(\omega)}{t} \to \lambda^1\right\}$ $\quad \mathbb{P}^{\lambda_1}(A) = 1$, $\mathbb{P}^{\lambda^2}(A) = 0$.

$A \in \mathcal{F}_\infty$ but not to $\mathcal{F}_t$ for any $t \geq 0$.

- But at the same time we know that $\mathbb{Q}|_{\mathcal{F}_T} \sim \mathbb{P}|_{\mathcal{F}_T}$

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = Z_T \quad \text{where } Z = \mathcal{E}\left(\left(\frac{\lambda_2}{\lambda_1} - 1\right) M\right) \quad \text{where } M_t = N - \lambda_1 t.$$

In fact $Z_T = e^{-T\left(\frac{\lambda_2}{\lambda_1} - 1\right)} \left(\frac{\lambda_2}{\lambda_1}\right)^{N_T} > 0$ and $E(Z_T) = 1$

BUT Z is not a uniformly integrable martingale.

Change of measure for Poisson random measures

Let M be a counting measure on $[0,T] \times \mathbb{R}$.

M homogeneous Poisson random measure with intensity measure

$\mu(dt\, dz) = dt\; \nu^{\mathbb{P}}(dz)$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

$\Leftrightarrow$ $\forall A \in \mathcal{B}(\mathbb{R})$, $M_t^A = M([0,t] \times A)$ is a Poisson process with intensity $\nu^{\mathbb{P}}(A)$

$\Leftrightarrow$ $\forall A \in \mathcal{B}(\mathbb{R})$, $M([0,t] \times A) - \nu^{\mathbb{P}}(A)\, t$ is a local martingale (using Watanabe's characterization of the Poisson process)

Meyer theorem for Poisson random measures

Let $h : [0,T] \times \mathbb{R} \to]-1,\infty[$ with $E\left[\int_0^T \int |h(t,z)|^2 \, dt \, \nu(dz)\right] < \infty$.

Define $U_t = \int_0^t \int h(t,z) \, \tilde{M}(ds\,dz) = \int_0^t \int h(t,z) \{ M(ds\,dz) - ds\,\nu(dz) \}$

Let $Z = \mathcal{E}(U)$ and assume $E^{\mathbb{P}}[Z_T] = 1$.

$$Z_t = 1 + \int_0^t Z_{s-} \, dU_{s-} \quad , \quad \frac{dQ}{d\mathbb{P}} = Z_T \quad \text{on } \mathcal{F}_T$$

Then M is a Poisson random measure under Q

with intensity measure

$$\mu^Q(dt\,dz) = (1 + h(t,z)) \, \nu^{\mathbb{P}}(dz) \, dt$$

Proof: in the case $h(t,z) = h(z)$ (no time dependence)

M is an integer-valued random measure under $\mathbb{Q}$.

Using Watanabe's characterization of the Poisson process we need to show that: $\forall A \in \mathcal{B}(\mathbb{R})$

$M([0,t] \times A) - t\int_A (1+ (z))\nu^{\mathbb{P}}(dz)$ is a $\mathbb{Q}$-local martingale.

Since M is a PRM with intensity $\nu(dz)\,dt$ under $\mathbb{P}$

$M_t^A = M([0,t] \times A) - t\nu^{\mathbb{P}}(dz)$ is a compensated Poisson martingale under $\mathbb{P}$

By the Girsanov-Meyer theorem

$M_t^A - \int_0^t \frac{1}{Z_s} d[Z, M^A]_s$ is a $\mathbb{Q}$-local martingale

$$M_t^A = \int_0^t \int \mathbb{1}_A(z) \, \tilde{M}(ds\,dz) \quad \text{and}$$

$$Z_t = 1 + \int_0^t Z_{s-} \, dU_s = 1 + \int_0^t \int h(s,z) Z_{s-} \, \tilde{M}(ds\,dz), \text{ so}$$

$$[Z, M^A]_t = \int_0^t \int \mathbb{1}_A(z) h(s,z) Z_{s-} \, \mu(ds\,dz)$$

$$= \underbrace{\int_0^t \int \mathbb{1}_A(z) h(s,z) Z_{s-} \, \tilde{M}(ds\,dz)}_{\text{local martingale}} + \int_0^t \int \mathbb{1}_A(z) h(s,z) Z_{s-} \, \nu(dz)\, ds$$