Stochastic Calculus

Note Title

Lectures 7 & 8:
Jump processes and random measures

Random measures.

Counting processes as random measures

Integer-valued random measures

Jump measure of a semimartingale

Poisson Random measures

Stochastic integration with respect to Poisson measures

Compensated Poisson integrals

Levy processes.

Levy-Ito representation. Levy Khinchin formula

Motivation

Let S be a semimartingale, $f \in C^2([0,\infty), \mathbb{R})$.

Then $f(S)$ is also a semimartingale and $\forall t \geq 0$,

stochastic integral with respect to S

$$f(S_t) - f(S_0) = \int_0^t f'(S_{u-})\, dS_u + \int_0^t \frac{1}{2} f''(S_{u-})\, d[S]^c_u$$

$$+ \sum_{0 \leq s \leq t} \underbrace{f(S_u) - f(S_{u-}) - \Delta S_u f'(S_u)}$$

can be expressed as a stochastic integral with respect to a measure J_S describing locations and sizes of jumps of S

Radon measures

Generalization of the notion of Lebesgue measure

✓ Def: Let $(E, \mathcal{B})$ be a measurable space with $E \subset \mathbb{R}^d$. A **Radon measure** on $(E, \mathcal{E})$ is a measure μ such that for every compact subset $B \in \mathcal{E}$, $\mu(B) < \infty$.

✓ Ex 1: the Lebesgue measure on $\mathbb{R}^d$ is a Radon measure.

✓ Ex 2: any (finite) linear combination of point masses

$$\mu = \sum_{i=1}^{n} a_i \delta_{x_i} \qquad x_i \in \mathbb{R}^d$$

$$a_i \geq 0$$

is a Radon measure: $\mu(B) = \sum_{i=1}^{n} a_i 1_{x_i \in B}$

✓ Ex 3: a probability measure on $\mathbb{R}^d$ is a Radon measure.

✓ Ex 4: $E = \mathbb{R} - \{0\}$, $\mu(A) = \int_A \frac{dx}{|x|^{1+\alpha}}$ $\qquad \alpha \in]0,2[$

In this case $\mu(E) = \infty$

Integration with respect to a Radon measure

μ Radon measure on $(E, \mathcal{B})$

✓ For $f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$ with $\begin{cases} A_i \subset \mathcal{B} \\ a_i \geq 0 \end{cases}$, define

$$\int_E f \, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i)$$

✓ For $f : (E, \mathcal{E}) \longrightarrow [0, \infty)$ there exists

$$f^n = \sum_{i=1}^{n} a_i^n \mathbb{1}_{A_i^n} \quad \text{with } a_i^n \geq 0 \qquad f^n \xrightarrow[n \to \infty]{\uparrow} f$$

Define: $\mu(f) = \int f \, d\mu := \lim_{n \to \infty} \sum_{1}^{n} a_i^n \mu(A_i^n)$

Def: $L^1(\mu) = \{ f : (E, \mathcal{B}) \to \mathbb{R}, \int |f| \, d\mu < \infty \}$

$f = f^+ - f^- \in L^1(\mu)$, $\int f \, d\mu := \int f^+ d\mu - \int f^- d\mu$

Signed measures

If μ_+, μ_- are Radon measures on $(E, \mathcal{B})$, $\mu = \mu_+ - \mu_-$ is called a signed measure on $(E, \mathcal{B})$

If $f \in L^1(\mu_+) \cap L^1(\mu_-)$ then we set

$$\int f \, d\mu = \int f \, d\mu_+ - \int f \, d\mu_-$$

Convergence of measures

$C_K((E,\mathcal{E}),\mathbb{R})$ continuous functions with compact support in E

$M_+(E)$ Radon measures on $(E,\mathcal{E})$

- $\forall f \in C_K(E,\mathbb{R})$, $\forall M_+(E)$, $\int f\,d\mu$ is well defined
- A sequence $(\mu_n)_{n\geq 1}$ of Radon measures is said to converge (vaguely) to $\mu \in M_+(E)$ if

$$\forall f \in C_K(E,\mathbb{R}),\quad \int f\,d\mu^n \xrightarrow[n\to\infty]{} \int f\,d\mu$$

- This defines a topology on $M_+(E)$
  $\to$ measurable space $(M_+(E), \mathcal{B})$

Random measures

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space.

Def: A measurable map $M : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathcal{M}_+(E), \mathcal{B})$

is called a **random measure** on E.

✓ For $\omega \in \Omega$, $M(\omega) \in \mathcal{M}_+(E)$ is a (Radon) measure on E.

✓ A random measure M may also be viewed as a map

subsets of E

$$\begin{aligned} M : \Omega \times \mathcal{E} &\longrightarrow [0, \infty) \\ (\omega, A) &\longrightarrow M(\omega, A) \end{aligned}$$

For $A \subset E$ measurable subset, $M(\cdot, A)$ is a random variable.

Counting measures

Ex 1. Let $(X_n)_{n\geq 1}$ a sequence of random variables with values in E such that for any compact $K \subset E$, $\#(\{X_n, n\geq 1\} \cap K) < \infty$ $\mathbb{P}$-a.s.

Then $M = \sum_{n\geq 1} \delta_{X_n}$ is an integer-valued random measure on E. M is called the 'counting measure' of $(X_n)_{n\geq 1}$.

$$M(\omega, A) = \#\{n\geq 1, \; X_n(\omega) \in A\} = \sum_{n\geq 1} 1_A(X_n(\omega))$$

Ex. 2. $X_n = \sum_{i\geq 1}^{n} T_i$ where $T_i \overset{iid}{\sim} \exp(\lambda)$, $E = [0,\infty)$

Then $M = \sum_{n\geq 1} \delta_{X_n}$ is the 'Poisson random measure';

$M([t_1, t_2]) = N^\lambda(t_2) - N^\lambda(t_1)$ where $N^\lambda_t = \sum_{n\geq 1} 1_{T_n \leq t}$ Poisson process

Poisson random measures

$$\begin{aligned} M : \Omega \times \mathcal{E} &\to \mathbb{N} \\ (\omega, A) &\mapsto M(\omega, A), \end{aligned}$$

$E \subset \mathbb{R}^d$

such that

1. *For (almost all)* $\omega \in \Omega$, $M(\omega, .)$ *is an integer-valued Radon measure on* E*: for any bounded measurable* $A \subset E$, $M(A) < \infty$ *is an integer valued random variable.*

2. *For each measurable set* $A \subset E$, $M(., A) = M(A)$ *is a Poisson random variable with parameter* $\mu(A)$*:*

$$\forall k \in \mathbb{N}, \quad \mathbb{P}(M(A) = k) = e^{-\mu(A)} \frac{(\mu(A))^k}{k!}. \tag{2.86}$$

3. *For disjoint sets* $A_1, \ldots, A_n \in \mathcal{E}$*, the variables* $M(A_1), \ldots, M(A_n)$ *are independent.*

μ is called the intensity (measure) of M

Ex. N^λ Poisson process with intensity λ, then $M^\lambda([t_1, t_2]) = N^\lambda(t_2) - N^\lambda(t_1)$ is a Poisson random measure on $[0, \infty)$ with intensity measure $\mu = \lambda \times \text{Leb}$

$\tilde{M} = M - \mu$ is called the "compensated Poisson measure" associated to M.

Poisson random measures : construction

PROPOSITION 2.14 Construction of Poisson random measures
For any Radon measure μ on $E \subset \mathbb{R}^d$, there exists a Poisson random measure M on E with intensity μ.

PROOF We give an explicit construction of M from a sequence of independent random variables. We begin by considering the case $\mu(E) < \infty$.

1. Take $X_1, X_2, \ldots$ to be i.i.d. random variables so that $\mathbb{P}(X_i \in A) = \frac{\mu(A)}{\mu(E)}$.

2. Take $M(E)$ to be a Poisson random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with mean $\mu(E)$, independent of the X_i.

3. Define $M(A) = \sum_{i=1}^{M(E)} 1_A(X_i)$, for all $A \in \mathcal{E}$.

It is then easily verified that this M is a Poisson random measure with intensity μ. If $\mu(E) = \infty$, since μ is a Radon measure we can represent $E \subset \mathbb{R}^d$ as a union of disjoint sets $E = \bigcup_{i=1}^{\infty} E_i$ with $\mu(E_i) < \infty$ and construct Poisson random measures $M_i(\cdot)$, where the intensity of M_i is the restriction of μ to E_i. Make the $M_i(\cdot)$ independent and define $M(A) = \sum_{i=1}^{\infty} M_i(A)$ for all $A \in \mathcal{E}$. The superposition and thinning properties of Poisson random variables (see Section 2.5) imply that M has the desired properties. $\square$

✓ This construction shows that a Poisson random measure can always be represented as a counting measure of some random sequence $(X_n)_{n \geq 1}$:

$$M(\omega, A) = \sum_{n \geq 1} 1_A(X_n(\omega)) \quad \text{or, equivalently}$$

as a sum of random point masses:

$$M = \sum_{n \geq 1} \delta_{X_n}$$

where the sequence $(X_n)_{n \geq 1}$ is an "independently scattered according to the measure $\mu(\cdot)$"

Lemma: $f_i \in L^1(\mu)$ $\quad supp(f_1) \cap supp(f_2) = \emptyset$

Then $\int_E f_1 \, d\mu$ and $\int_E f_2 \, d\mu$ are independent random variables.

Proof: $f_i = 1_{A_i}$ where $A_i \in \mathcal{E}$ measurable subset of E

$\mu(A_i) < \infty \quad supp(f_i) = A_i$

$$\int_E f_i \, d\mu = \mu(A_i)$$

$A_1 \cap A_2 = \emptyset \Rightarrow \mu(A_1), \mu(A_2)$ indep

Exponential formula for Poisson random measures

PROPOSITION 3.7 Exponential formula for Poisson random measures

Let M be a Poisson random measure with intensity measure μ. Then the following formula holds for every measurable set B such that $\mu(B)<\infty$ *and for all functions f such that $\int_B e^{f(x)}\mu(dx) < \infty$:*

$$E \exp\left\{\int_B f(x)M(dx)\right\} = \exp\left\{\int_B (e^{f(x)} - 1)\mu(dx)\right\}. \qquad (3.15)$$

- Allows to compute the 'Laplace transform' of the law of integrals with respect to M i.e. to characterize the distribution of M.

Proof :

First we note that, conditionally on $M(B)$, the restriction of M to B has the same distribution as a counting measure $\hat{M}_B$ on B defined by

$$\hat{M}_B(A) = \#\{i=1..M(B),\ X_i \in A\} \quad \text{for } A \subset B$$

where $(X_i)_{i\geq 1}$ are IID with law $\dfrac{\mu_{|B}}{\mu(B)}$ $\leftarrow$ restr. of μ to B

so:

$$E\exp\left\{\int_B f(x)M(dx)\right\} = E[E[\exp\left\{\int_B f(x)M(dx)\right\}|M(B)]]$$

$$= E[E[e^{\sum_{i=1}^{M(B)} f(X_i)}|M(B)]],$$

$$E[e^{\sum_{i=1}^{M(B)} f(X_i)}|M(B)] = \Bigg(\overbrace{\int_B \frac{\mu(dx)e^{f(x)}}{\mu(B)}}^{E[e^{f(X_i)}]}\Bigg)^{M(B)}.$$

Since $M(B) \sim \text{Poisson}(\mu(B))$ the outer expectation is computed as :

$$\sum_{n\geq 0} e^{-\mu(B)}\frac{\mu(B)^n}{n!}\left(\int_B \frac{\mu(dx)e^{f(x)}}{\mu(B)}\right)^n = \exp\left\{\int_B (e^{f(x)}-1)\mu(dx)\right\}.$$

Convergence of Poisson random measures

PROPOSITION 2.15 Convergence of Poisson random measures
Let $(M_n)_{n\geq 1}$ be a sequence of Poisson random measures on $E \subset \mathbb{R}^d$ with intensities $(\mu_n)_{n\geq 1}$. Then $(M_n)_{n\geq 1}$ converges in distribution if and only if the intensities (μ_n) converge to a Radon measure μ. Then $M_n \Rightarrow M$ where M is a Poisson random measure with intensity μ.

$$\left(M_n \overset{d}{\Rightarrow} M\right) \iff$$

$$\forall f \in C_K(E), \quad \int_E f \, d\mu^n \to \int_E f \, d\mu$$

Using the exponential formula:

$$E \exp\left(\lambda \int f \, dM_n\right) = E \exp\left[\int (e^{\lambda f} - 1) d\mu^n\right) \xrightarrow[n\to\infty]{} E \exp\left(\int (e^{\lambda f} - 1) d\mu\right)$$

So M verifies $E \exp\left(\lambda \int f \, dM\right) = E \exp\left(\int (e^{\lambda f} - 1) d\mu\right), \ \forall f \in C_K$

$\to$ M is a Poisson random measure with intensity $\mu(\cdot)$

Jump measure of a cadlag process

The construction of a (Poisson) random measure from the Poisson process can be carried out for any non-anticipative cadlag process X.

Since X is cadlag: X has countable jumps and for any compact A with $0 \notin A$, $\{t, \Delta X_t \in A\}$ is a.s. finite so defining

$J_X([0,t] \times A) :=$ number of jumps of X occurring between 0 and t whose amplitude belongs to A.

A compact subset of $\mathbb{R}^d \setminus \{0\}$
$\Rightarrow$ $J_X([0,t] \times A) < \infty$

J_X defines a random measure on $[0,T] \times \mathbb{R}^d \setminus \{0\}$, which is called the jump measure of the process X:

E

$$J_X = \sum_{t \in [0,T]}^{\Delta X_t \neq 0} \delta_{(t, \Delta X_t)}. \tag{2.96}$$

$$J_X = \sum_{n \geq 1} \delta_{(T_n, Y_n)} \quad \text{where} \begin{cases} T_n & \text{jump times of } X \\ Y_n = \Delta X_{T_n} = X_{T_n} - X_{T_n-} \end{cases}$$

• Integrating with respect to the jump measure:

$$\int_{[0,T]\times\mathbb{R}^d} f(t,x)\, J_X(dt\, dx) = \sum_{\substack{t\in[0,T],\\ \Delta X_t \neq 0}} f(t, \Delta X_t)$$

All quantities involving the jumps of X can be computed by integrating various functions against J_X. For example if $f(t,y) = y^2$ then one obtains the sum of the squares of the jumps of X:

$$\int_{[0,T]\times\mathbf{R}} y^2 J_X(dt\ dy) = \sum_{t\in[0,T]} (\Delta X_t)^2. \tag{2.97}$$

Jump measure of a compound Poisson process

PROPOSITION 3.6 Jump measure of a compound Poisson process

Let $(X_t)_{t\geq 0}$ be a compound Poisson process with intensity λ and jump size distribution f. Its jump measure J_X is a Poisson random measure on $[0,\infty)\times\mathbb{R}^d$ with intensity measure $\mu(dt\times dx) = dt\times\nu(dx) = dt\times\lambda f(dx)$.

Thus a compound Poisson process X with measure ν can be represented as:

$$X_t = \sum_{s\in[0,t]} \Delta X_s = \int_{[0,t]\times\mathbb{R}^d} x J_X(ds\times dx),$$

where J_X is a Poisson random measure with intensity measure $\mu(dt\ dx) = dt\cdot\nu(dx)$

- μ verifies $\mu(\{t\}\times(\mathbb{R}^d\setminus\{0\})) = 0 \quad \forall t\geq 0$: there are 'no jumps at deterministic times'

Proof : The jump measure of X is defined as

$$J_X([t_1,t_2]\times A) = \#\{s\in[t_1,t_2],\ \Delta X_s \in A\} = \sum_{i=N_{t_1}+1}^{N_{t_2}} 1_{Y_i\in A}$$

PROOF of Proposition 3.6 From the Definition (3.12) it is clear that J_X is an integer valued measure. Let us first check that $J_X(B)$ is Poisson distributed. It is sufficient to prove this property for a set of the form $B =$ $B=[t_1,t_2]\times A$ with $A\in\mathcal{B}(\mathbb{R}^d)$. Let $(N_t)_{t\geq 0}$ be the Poisson process, counting the jumps of X. Conditionally on the trajectory of N, the jump sizes Y_i are i.i.d. and $J_X([t_1,t_2]\times A)$ is a sum of $\underbrace{N(t_2)-N(t_1)}_{\sim \text{Poisson}(\lambda(t_2-t_1))}$ i.i.d. Bernoulli variables taking value 1 with probability $f(A)$. Therefore,

$$E[e^{iuJ_X([t_1,t_2]\times A)}] = E[E[e^{iuJ_X([t_1,t_2]\times A)}|N_t, t\geq 0]]$$

$$= E[\{e^{iu}f(A)+1-f(A)\}^{N(t_2)-N(t_1)}] = \exp\{\lambda(t_2-t_1)f(A)(e^{iu}-1)\}$$

because $N(t_2)-N(t_1)$ is Poisson distributed with parameter $\lambda(t_2-t_1)$. Thus, $J_X([t_1,t_2]\times A)$ is a Poisson random variable with parameter $f(A)\lambda(t_2-t_1)$ which was to be shown.

Therefore the intensity μ of J_X is such that $\mu([t_1,t_2]\times A) = \lambda f(A)\,(t_2-t_1) = \nu(A)\times \mathrm{Leb}([t_1,t_2])$

Now let us check the independence of measures of disjoint sets. First, let us show that if A and B are two disjoint Borel sets in $\mathbb{R}^d$ then $J_X([t_1,t_2]\times A)$ and $J_X([t_1,t_2]\times B)$ are independent. Conditionally on the trajectory of N, the expression $iuJ_X([t_1,t_2]\times A)+ivJ_X([t_1,t_2]\times B)$ is a sum of $N(t_2)-N(t_1)$ i.i.d. random variables taking values:

$$Z_k = \begin{cases} iu & \text{with probability } f(A); \\ iv & \text{with probability } f(B); \\ 0 & \text{with probability } 1-f(A)-f(B). \end{cases}$$

$E(e^{Z_k}) = (e^{iu}-1)f(A) + (e^{iv}-1)f(B) + 1$

Proceeding like in the first part of the proof, we factorize the characteristic function as follows

$$\begin{aligned} E[e^{iuJ_X([t_1,t_2]\times A)+ivJ_X([t_1,t_2]\times B)}] &= E[e^{\sum_{N(t_1)+1}^{N(t_2)} Z_k}] \\ &= E[\{(e^{iu}-1)f(A)+(e^{iv}-1)f(B)+1\}^{N(t_2)-N(t_1)}] \\ &= \exp\{\lambda(t_2-t_1)(f(A)(e^{iu}-1)+f(B)(e^{iv}-1))\} \\ &= E[e^{iuJ_X([t_1,t_2]\times A)}]E[e^{ivJ_X([t_1,t_2]\times B)}]. \end{aligned}$$

Second, let $[t_1,t_2]$ and $[s_1,s_2]$ be two disjoint intervals. The independence of $J_X([t_1,t_2]\times A)$ and $J_X([s_1,s_2]\times B)$ follows directly from the independence of increments of the process X.

Let $B = \bigcup_{n\geq 1} [t_n, s_n] \times A_n$ with A_n disjoint

$$J_x(B) = \sum_{n\geq 1} \underbrace{J_x([t_n, s_n] \times A_n)}$$

independent Poisson variables
with parameter
$\lambda f(A_n) \; (s_n - t_n) = \mu([t_n, s_n] \times A_n)$

so: $J_x(B) \sim \text{Poisson}\left(\sum_{n\geq 1} \lambda f(A_n)(s_n - t_n) \right)$

$$\sum_{n\geq 1} \lambda f(A_n)(s_n - t_n) = \sum_{n\geq 1} \mu([t_n, s_n] \times A_n) = \mu(B)$$

Homogeneous Poisson random measures

A Poisson random measure M on $[0,T]\times\mathbb{R}^d$ is called **homogeneous** if its intensity is given by

$$\mu(dt\,dx) = dt \times \nu(dx)$$

where ν is a Radon measure on $\mathbb{R}^d\setminus\{0\}$.

ν is sometimes called the 'Lévy measure' associated to M.

Let $M = \sum_{n\geq 1} \delta_{(T_n, Y_n)}$ for some random sequence $(T_n, Y_n)_{n\geq 1}$

. For any $A \subset \mathbb{R}^d\setminus\{0\}$ compact subset of $\mathbb{R}^d\setminus\{0\}$ then

$M([t, t+T]\times A)$ has a Poisson distribution with parameter $T\,\nu(A)$

So $\nu(A) = E\left[\begin{array}{l}\text{number of events per unit time}\\ \text{whose size belongs to } A\end{array}\right]$

M Poisson random measure $\qquad M = \sum_{n \geq 1} \delta_{(T_n, Y_n)}$

$A \subset \mathbb{R}^d \setminus \{0\}$ measurable set

$T_i \leq T_{i+1}$ stopping times

$$\int_0^t \int_{\mathbb{R}^d} \mathbb{1}_{]T_i, T_{i+1}] \times A} \, dM := M(]T_i, T_{i+1}] \times A)$$

$=$ number of events in $]T_i, T_{i+1}]$ whose size is in A

$$= \# \{ n \geq 1, \ (T_n, Y_n) \in]T_i, T_{i+1}] \times A \}$$

Let M be a Poisson random measure on $[0,\infty) \times \mathbb{R}^d \setminus \{0\}$

with intensity μ

$$M = \sum_{n \geq 1} \delta_{(T_n, Y_n)}$$

T_n : 'event times'

Y_n : 'event sizes'

$$\mathcal{F}_t^M := \sigma(\{T_n, Y_n, \ T_n \leq t\})$$ σ-algebra generated by events 'before t'

- We already know how to integrate a deterministic $\phi : [0,T] \times \mathbb{R}^d$ with respect to the Radon measure $M(\omega, \cdot)$ for a fixed $\omega \in \Omega$.
- We want to define $\displaystyle\int_0^T \int_{\mathbb{R}^d} \phi(t, y, \omega) \, M(dt \, dy)$

for integrands which are
- ✓ caglad in t
- ✓ adapted to $(\mathcal{F}_t^M)_{t \geq 0}$

Stochastic integration with respect to a Poisson random measure

$\phi : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is called a 'simple predictable random function' if

$$\phi(t,y) = \sum_{j=1}^{m} \phi_{0j} 1_{t=0} 1_{A_j}(y) + \sum_{i=0}^{n} \sum_{j=1}^{m} \phi_{ij} 1_{]T_i,T_{i+1}]}(t) 1_{A_j}(y), \qquad (8.20)$$

where $0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_n$ are nonanticipating random times, $(\phi_{ij})_{j=1\ldots m}$ are bounded $\mathcal{F}_{T_i}$-measurable random variables and $(A_j)_{j=1\ldots m}$ are disjoint subsets of $\mathbb{R}^d$ with $\mu([0,T] \times A_j) < \infty$. The stochastic integral $\int_{[0,T]\times\mathbb{R}^d} \phi(t,y) M(dt\ dy)$ is then defined as the random variable

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \phi(t,y) M(dt\ dy) &= \sum_{i=0}^{n} \sum_{j=1}^{m} \phi_{ij} M(\,]T_i, T_{i+1}] \times A_j) \\ &= \sum_{i=0}^{n} \sum_{j=1}^{m} \phi_{ij} [M_{T_{i+1}}(A_j) - M_{T_i}(A_j)]. \qquad (8.21)\end{aligned}$$

where $M_T(A) = M(]0,T] \times A)$ is a Poisson process with intensity $\nu(A)$

Similarly, one can define the process $t \mapsto \int_0^t \int_{\mathbb{R}^d} \phi(t,y) M(dt\, dy)$ by

$$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(s,y) M(ds\, dy) = \sum_{i=0}^{n} \sum_{j=1}^{m} \phi_{ij} [M_{T_{i+1}\wedge t}(A_j) - M_{T_i \wedge t}(A_j)].$$

Proposition: for every simple predictable function $\phi : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $X_t = \int_0^t \int_{\mathbb{R}^d} \phi(s,y) M(ds\, dy)$ is a cadlag adapted process.

Description: if $M = \sum_{n\geq 1} \delta_{(T_i, Y_i)}$ T_i jump times, Y_i jump sizes

then:
- $\Delta X_t \neq 0 \iff t \in \{T_n, n \geq 1\}$
- $\Delta X_{T_i} = \phi(T_i, Y_i)$

✓ In fact X is a semimartingale

Similarly, one can define the process $t \mapsto \int_0^t \int_{\mathbb{R}^d} \phi(t,y) M(dt\, dy)$ by

$$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(s,y) M(ds\, dy) = \sum_{i=0}^{n} \sum_{j=1}^{m} \phi_{ij} [M_{T_{i+1}\wedge t}(A_j) - M_{T_i \wedge t}(A_j)].$$

Property: $E(X_t) = E\left[\int_0^t \int_{\mathbb{R}^d} \phi(s,y)\, \mu(ds\, dy) \right]$

Proof: $E\left(\phi_{ij}\, M_{T_{i+1}\wedge t}(A_j) - M_{T_i \wedge t}(A_j) \right) =$

$$E\left(E\left(\phi_{ij}\, M_{T_{i+1}\wedge t}(A_j) - M_{T_i\wedge t}(A_j) \,\middle|\, \mathcal{F}_{T_i} \right)\right) =$$

since ϕ_{ij} $\mathcal{F}_{T_i}$ - measurable

$$= E\left(\phi_{ij}\, E\left(M_{T_{i+1}\wedge t}(A_j) - M_{T_i\wedge t}(A_j) \,\middle|\, \mathcal{F}_{T_i} \right)\right) =$$

$$= E\left(\phi_{ij}\, E\left(M_{T_{i+1}\wedge t}(A_j) - M_{T_i\wedge t}(A_j) \right)\right) =$$ by indep. of increments of $M(A_j)$

$$= E\left(\phi_{ij}\, E\left(\nu(A_j)\, (T_{i+1}\wedge t - T_i \wedge t)\right)\right) = E\left(\phi_{ij}\, (T_{i+1}\wedge t - T_i\wedge t)\, \nu(A_j) \right)$$

Stochastic integral with respect to a Poisson random measure

Denote $\mathcal{L}^1(\mu)$ the set of maps

$$\phi : \Omega \times [0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R} \quad \text{such that}$$

(i) For every $t \geq 0$, the mapping $(\omega, x) \mapsto \phi(\omega, t, x)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$-measurable.

(ii) For every (ω, x), the mapping $t \mapsto \phi(\omega, t, x)$ is caglad.

(iii) $\quad E\left[\int_0^T \int_{\mathbb{R}^d} |\phi(t,y)| \, \mu(ds\, dy)\right] < \infty$

Then for any $\phi \in \mathcal{L}^1(\mu)$, $\int_0^T \int_{\mathbb{R}^d} \phi(s,y) \, M(ds\, dy)$ may be defined pathwise: if $M = \sum_{n \geq 1} \delta_{(T_i, Y_i)}$

$$\int_0^t \int_{\mathbb{R}^d} \phi(s,y) \, M(ds\, dy) := \sum_{i \geq 1} \phi(T_i, Y_i) \, 1_{T_i \leq t}$$

which is absolutely convergent almost surely since

$$\sum_{i \geq 1} |\phi(T_i, Y_i)| \, 1_{T_i \leq t} = \int_0^t \int_{\mathbb{R}^d} |\phi| < \infty \quad \text{a.s.}$$

Poisson integral M Poisson random measure on $[0,T] \times \mathbb{R}^d$

μ intensity of M $\qquad M = \sum_{n \geq 1} \delta_{(T_i, Y_i)}$

For any predictable random function $\phi \in \mathcal{L}^1(\mu)$

$$E\left(\int_0^T \int_{\mathbb{R}^d} |\phi(t,y)| \mu(dt\,dy)\right) < \infty$$

✓ $$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(s,y) \; M(ds\,dy)$$

is a cadlag process, adapted to the filtration

$\mathcal{F}_t = \mathcal{F}_t^M$ generated by M.

✓ $$E(X_t) = E\left(\int_0^t \int_{\mathbb{R}^d} \phi(s,y)\, \mu(ds\,dy)\right)$$

✓ X has paths of bounded variation and

$$X_t = \sum_{n \geq 1} 1_{T_n \leq t} \, \phi(T_n, Y_n, \omega)$$

In particular X is a semimartingale.

Example: Let M be a homogeneous PRM on $[0,T] \times \mathbb{R}^d$ with intensity $\mu(dt\,dx) = dt \times \nu(dx)$

✓ If $\int_{\mathbb{R}^d} \|x\| \nu(dx) < \infty$ then $\int_0^t \int_{\mathbb{R}^d} x \, M(dt\,dx)$ is well-defined

Then: $X_t = \int_0^t \int_{\mathbb{R}^d} x \, M(dt\,dx)$ is a pure-jump process with independent increments with $E(e^{iu \cdot X_t}) = \exp\left(t \int (e^{iu \cdot x} - 1) \nu(dx)\right)$

If $M = \sum_{n \geq 1} \delta_{(T_n, Y_n)}$, $X_t = \sum_{T_n \leq t} Y_n$ and $\sum_{T_n \leq t} |Y_n| < \infty$ a.s.

✓ Note that X may have an infinite number of jumps in $[0,t]$..

$\cdots t_i \leq t_{i+1} \leq \cdots$: $X_{t_{i+1}} - X_{t_i} = \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} x \, M(dt\,dx)$ are independent variables

✓ $X_t^\varepsilon = \sum_{s \leq t, |\Delta X_s| > \varepsilon} \Delta X_s = \sum_{T_n \leq t} Y_n 1_{|Y_n| > \varepsilon} = \int_0^t \int_{\|x\| > \varepsilon} x \, M(dt\,dx)$

is a compound Poisson process with

intensity $\nu(\mathbb{R}^d \setminus B(0,\varepsilon))$, jump size distribution $\dfrac{\nu_{|\mathbb{R}^d - B(0,\varepsilon)}}{\nu(\mathbb{R}^d - B(0,\varepsilon))}$

✓ If $\int_{\mathbb{R}^d} \|x\| \nu(dx) < \infty$ and $\int_{\mathbb{R}^d} \nu(dx) < \infty$ then $X_t = \int_0^t \int_{\mathbb{R}^d} x \, M(dt\, dx)$

is a compound Poisson process : $X_t = \sum_{i=1}^{N_t} Y_i$ where

✓ $N_\cdot$ is a Poisson process with intensity $\nu(\mathbb{R}^d) < \infty$

✓ $Y_i \overset{IID}{\sim} F$ where $F = \frac{\nu(\cdot)}{\nu(\mathbb{R}^d)}$

Example where: $\int |x| \nu(dx) < \infty$ but $\nu(\mathbb{R}^d) = +\infty$

Ex 1. $\nu(dx) = \frac{e^{-K|x|}}{|x|^{1+\alpha}} dx \quad 0 \leq \alpha < 1 \qquad d = 1$

Then $\nu(\mathbb{R}) = \infty$, $\int |x| \nu(dx) < \infty$

$X_t = \int_0^t \int_{\mathbb{R}} x \, M(ds\, dx)$ is called a 'tempered α-stable process'

Compensated Poisson integral

M Poisson random measure on $[0,T]\times(\mathbb{R}^d\setminus\{0\})$
with intensity measure $\mu \in M_+([0,T]\times(\mathbb{R}^d\setminus\{0\}))$

The signed random measure $\tilde{M} = M - \mu$ is called
the **compensated Poisson measure** associated to M

- For $\phi = \sum_{i,j} \phi_{ij} 1_{]T_i,T_{i+1}]\times A_j}$ simple predictable function, define

$$\int_0^T \int_{\mathbb{R}^d} \phi(s,y)\tilde{M}(ds\,dy) = \sum_{i=0}^{n}\sum_{j=1}^{m} \phi_{ij}\tilde{M}(\,]T_i,T_{i+1}]\times A_j) \tag{8.23}$$

$$= \sum_{i=0}^{n}\sum_{j=1}^{m} \phi_{ij}\,[M(]T_i,T_{i+1}]\times A_j) - \mu(]T_i,T_{i+1}]\times A_j)].$$

By restricting to terms with $T_i \leq t$ (i.e., stopping at t), we obtain a stochastic process:

$$\int_0^t \int_{\mathbb{R}^d} \phi(s,y)\tilde{M}(ds\,dy) = \sum_{i=0}^{n}\sum_{j=1}^{m} \phi_{ij}[\tilde{M}_{T_{i+1}\wedge t}(A_j) - \tilde{M}_{T_i\wedge t}(A_j)]. \tag{8.24}$$

Compensated Poisson integrals: martingale property and isometry formula

For any simple predictable function $\phi : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ the process $(X_t)_{t\in[0,T]}$ defined by the compensated integral

$$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(s,y)\tilde{M}(ds\ dy)$$

is a square integrable martingale and verifies the isometry formula*:*

$$E[\ |X_t|^2] = E\left[\int_0^t \int_{\mathbb{R}^d} |\phi(s,y)|^2 \mu(ds\ dy)\right]. \tag{8.25}$$

Let $Y^j(t) = \underbrace{M(]0,t]\times A_j)}_{\substack{\text{Poisson process} \\ \text{with intensity} \\ \nu(A_j)}} - \underbrace{\mu(]0,t]\times A_j)}_{= t\nu(A_j)} = N_t^j - t\nu(A_j)$

$A_i \cap A_j = \phi \implies Y^i, Y^j$ independent compensated Poisson processes

PROOF For $j = 1 \ldots m$ define $Y_t^j = \tilde{M}(]0,t] \times A_j) = \tilde{M}_t(A_j)$. From Proposition 2.16, $(Y_t^j)_{t \in [0,T]}$ is a martingale with independent increments. Since the A_j are disjoint, the processes Y^j are mutually independent.

Writing $\tilde{M}(]T_i \wedge t, T_{i+1} \wedge t] \times A_j) = Y^j_{T_{i+1}\wedge t} - Y^j_{T_i \wedge t}$, the compensated integral X_t can be expressed as a sum of stochastic integrals:

$$X_t = \sum_{i=0}^{n} \sum_{j=1}^{m} \phi_{ij} (Y^j_{T_{i+1}\wedge t} - Y^j_{T_i \wedge t})$$

$$= \sum_{j=1}^{m} \underbrace{\int_0^t \phi^j dY^j}_{\text{martingales}} \quad \text{where} \quad \phi^j = \sum_{i=0}^{n} \phi_{ij} 1_{]T_i, T_{i+1}]}. \leftarrow \text{simple predictable process}$$

from the martingale-preserving property

Each ϕ^j is a simple predictable process so $\int \phi^j dY^j$ is a martingale $\Rightarrow$ X is a martingale. $E(X_t) = E(X_0) = 0$. To compute $E(X_T^2)$, using the independence of $(Y^j)_{j\geq 1}$,

$$
\begin{aligned}
E|X_T|^2 &= \sum_{j=1}^{m}\sum_{i=0}^{n}\sum_{k=1}^{m}\sum_{l=0}^{n} E\left[\phi_{ij}\phi_{lk}(Y^j_{T_{i+1}} - Y^j_{T_i})(Y^k_{T_{l+1}} - Y^k_{T_l})\right] \\
&= \sum_{i,j,k} E\left[\phi_{ij}\phi_{ik}(Y^j_{T_{i+1}} - Y^j_{T_i})(Y^k_{T_{i+1}} - Y^k_{T_i})\right] \\
&= \sum_{i,j,k} E\left[\phi_{ij}\phi_{ik}E[(Y^j_{T_{i+1}} - Y^j_{T_i})(Y^k_{T_{i+1}} - Y^k_{T_i})|\mathcal{F}_{T_i}]\right] \\
&= \sum_{i,j} E\left[|\phi_{ij}|^2 E[(Y^j_{T_{i+1}} - Y^j_{T_i})^2|\mathcal{F}_{T_i}]\right] \\
&= \sum_{i,j} E\left[|\phi_{ij}|^2 \mu(]T_i, T_{i+1}] \times A_j)\right], \ = E\left(\int_0^T \int_{\mathbb{R}^d} |\phi|^2 \mu(dt\, dy)\right)
\end{aligned}
$$

Compensated Poisson integral : L^2 extension

Denote $\mathcal{P}^2(\mu)$ the set of maps

$\phi : \Omega \times [0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

(i) For every $t \geq 0$, the mapping $(\omega, x) \mapsto \phi(\omega, t, x)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$-measurable.

(ii) For every (ω, x), the mapping $t \mapsto \phi(\omega, t, x)$ is caglad.

(iii) $$E \int_0^T \int_{\mathbb{R}^d} |\phi(t,y)|^2 \mu(dt\, dy) < \infty. \qquad \forall T > 0$$

Then for $\phi \in \mathcal{P}^2(\mu)$ there exists a sequence (ϕ^n) of simple predictable random functions such that

$$E\left[\int_0^T \int_{\mathbb{R}^d} |\phi^n(t,y) - \phi(t,y)|^2 \mu(dt\, dy)\right] \overset{n\to\infty}{\to} 0.$$

and the compensated Poisson integral may be defined

as : $$\int_0^T \int_{\mathbb{R}^d} \phi(s,y)\, d\tilde{M} = \lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} \phi^n(s,y)\, \tilde{M}(ds,dy) \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P})$$

Compensated Poisson integral : property

For every $\phi \in \mathcal{P}^2(\mu)$,

$$t \mapsto \int_0^t \int_{\mathbb{R}^d} \phi(s,y)\tilde{M}(ds\ dy) \quad \textit{is a square integrable martingale,}$$

$$E\left[\left|\int_0^t \int_{\mathbb{R}^d} \phi(s,y)\tilde{M}(ds\ dy)\right|^2\right] = E\left[\int_0^t \int_{\mathbb{R}^d} |\phi(s,y)|^2 \mu(ds\ dy)\right].$$

Lévy-Ito processes

M Poisson random measure on $[0,\infty)\times\mathbb{R}^d\setminus\{0\}$ with intensity measure μ

W Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Def: A Lévy-Ito process (also called 'Ito semimartingale') is a cadlag $\mathcal{F}_t$-adapted process

$$X_t = \int_0^t b_s\,ds + \int_0^t \phi_s\,dW_s + \int_0^t \int_{\mathbb{R}^d} \psi(s,y)\,M(ds\,dy)$$

where ϕ is a caglad adapted process

$\psi \in L_2(\mu)$ is a predictable random function

b is a locally integrable adapted process

- If $M = \sum_{n\geq 1} \delta_{(T_n, Y_n)}$ then

$$X_t = \int_0^t b_s\,ds + \int_0^t \phi_s\,dW_s + \sum_{T_n \leq t} \psi(T_n, Y_n)$$

Lévy-Ito processes : properties

$$X_t^i = \int_0^t b_s^i ds + \int_0^t \phi_s^i \, dW_s + \int_0^t \int_{\mathbb{R}^d} \psi^i(s,y) \, M(ds\, dy)$$

i) $$[X^i]_t = \int_0^t \|\phi_s^i\|^2 ds + \int_0^t \int_{\mathbb{R}^d} \|\psi^i(s,y)\|^2 \, M(ds\, dy)$$

ii) $$[X^1, X^2]_t = \int_0^t \langle \phi_s^1, \phi_s^2 \rangle ds + \int_0^t \int_{\mathbb{R}^d} \langle \psi^1(s,y), \psi^2(s,y) \rangle \, M(ds\, dy)$$

$$= \int_0^t \langle \phi_s^1, \phi_s^2 \rangle ds + \sum_{0 \le T_n \le t} \langle \psi^1(T_n, Y_n), \psi^2(T_n, Y_n) \rangle$$

iii) Semimartingale decomposition :

$$X_t^i = X_0^i + \int_0^t \phi_s^i \, dW_s$$ ← continuous local martingale

$$+ \int_0^t \int_{\mathbb{R}^d} \psi^i(s,y) \, \tilde{M}(ds\, dy)$$ ← 'compensated' pure-jump local martingale

$$+ \int_0^t b_s^i \, ds + \int_0^t \int_{\mathbb{R}^d} \psi^i(s,y) \, \mu(ds\, dy)$$ ← process of bounded variation

Lévy processes

Let ν be a Radon measure on $\mathbb{R}^d \setminus \{0\}$

with

$$(*) \int (\|x\|^2 \wedge 1)\, \nu(dx) < \infty \iff \begin{cases} \exists \varepsilon > 0 & \int_{\|x\| < \varepsilon} \|x\|^2 \nu(dx) < \infty \\ \int_{|x| > \varepsilon} \nu(dx) < \infty \end{cases}$$

and M a homogeneous Poisson random measure with intensity $\mu(dt\, dy) = dt\, \nu(dy)$ on $[0,\infty) \times \mathbb{R}^d$

Then : ✓ $\int_0^t \int_{\|x\| \le \varepsilon} x\, \tilde{M}(ds\, dx)$ is a well-defined square-integrable martingale

since $\psi(s,x,\omega) = x$ is a predictable random function with $\psi \in \mathcal{P}^2(\mu)$ by $(*)$

✓ $\int_0^t \int_{\|x\| > \varepsilon} x\, M(ds\, dx)$ is a well-defined compound Poisson process

$$M = \sum_{n \ge 1} \delta_{(T_n, Y_n)} \Rightarrow \int_0^t \int_{\|x\| > \varepsilon} x\, M(ds\, dx) = \sum_{T_n < t,\ \|Y_n\| > \varepsilon} Y_n$$

Lévy processes

W $\mathbb{R}^d$-valued Wiener process

M PRM with intensity $dt\,\nu(dx)$

W, M independent.

$b \in \mathbb{R}^d$, $\Sigma \in M_d(\mathbb{R})$, $A = {}^t\Sigma \cdot \Sigma$

where the 'Lévy measure' ν verifies:

$$\int_{|x|>1} \nu(dx) < \infty, \quad \int_{-1}^{1} \nu(dx)\, \|x\|^2 < \infty$$

Definition:

$$X_t = bt + \Sigma \cdot W + \int_0^t \int_{\|x\| \leqslant 1} x \, \tilde{M}(ds\, dx) + \int_0^t \int_{\|x\|>1} x \, M(ds\, dx) \qquad \text{(L-I}$$

is a cadlag process with independent increments, called a **Lévy process**.

(L-I) is called the Lévy-Ito decomposition of X

Lévy process

$$X_t = bt + \Sigma \cdot W + \int_0^t \int_{\|x\| \leq 1} x \ \widetilde{M}(ds\, dx) + \int_0^t \int_{\|x\| > 1} x \ M(ds\, dx)$$

$$= b_\varepsilon t + \Sigma \cdot W + \int_0^t \int_{\|x\| \leq \varepsilon} x \ \widetilde{M}(ds\, dx) + \int_0^t \int_{\|x\| > \varepsilon} x \ M(ds\, dx)$$

where $\quad b_\varepsilon = b - \int_{\varepsilon < \|x\| \leq 1} x \ \nu(dx)$

Lévy - Khinchin formula

$$X_t = bt + \Sigma \cdot W + \int_0^t \int_{\|x\| \leq 1} x \ \tilde{M}(ds\, dx) + \int_0^t \int_{\|x\| > 1} x \ M(ds\, dx)$$

The characteristic function of X_t is given by:

$$\phi_{X_t}(u) = E\left(e^{iuX_t}\right) = \exp(\ t\psi(u)) \quad \text{where}$$

$$\psi(u) = i\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u,x\rangle} - 1 - 1_{\|x\| \leq 1}\ i\langle u,x\rangle\right) \nu(dx)$$

is called the characteristic exponent of X.

- ϕ_{X_t} is entirely determined by the triplet (b, A, ν)

(b, A, ν) is called the **characteristic triplet** of X.

Lévy - Khinchin formula

$$X_t = bt + \Sigma \cdot W + \int_0^t \int_{\|x\| \leqslant 1} x \; \tilde{M}(ds\, dx) + \int_0^t \int_{\|x\| > 1} x \; M(ds\, dx)$$

The characteristic function of X_t is given by :

$$\phi_{X_t}(u) = E\left(e^{iuX_t}\right) = \exp\left(t\,\psi(u) \right) \quad \text{where}$$

$$\psi(u) = i\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u,x\rangle} - 1 - 1_{\|x\| \leqslant 1}\, i\langle u,x\rangle\right) \nu(dx)$$

is called the characteristic exponent of X.

- ϕ_{X_t} is entirely determined by the triplet (b, A, ν)

(b, A, ν) is called the **characteristic triplet** of X.