MACHINE LEARNING DETECTS TERMINAL SINGULARITIES

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BACKGROUND

Algebraic geometry is the study of shapes defined by solutions to systems of polynomial equations. **$\mathbb{Q}$-Fano varieties are the atomic pieces** of algebraic geometry. They are the positively curved shapes with $\mathbb{Q}$-factorial terminal singularities (they are not smooth, e.g. Figure 1a, but have *mild* singularities, e.g. Figures 1b and 1c). Their classification is an open problem: it can be thought as building a **Periodic Table** of geometrical shapes.

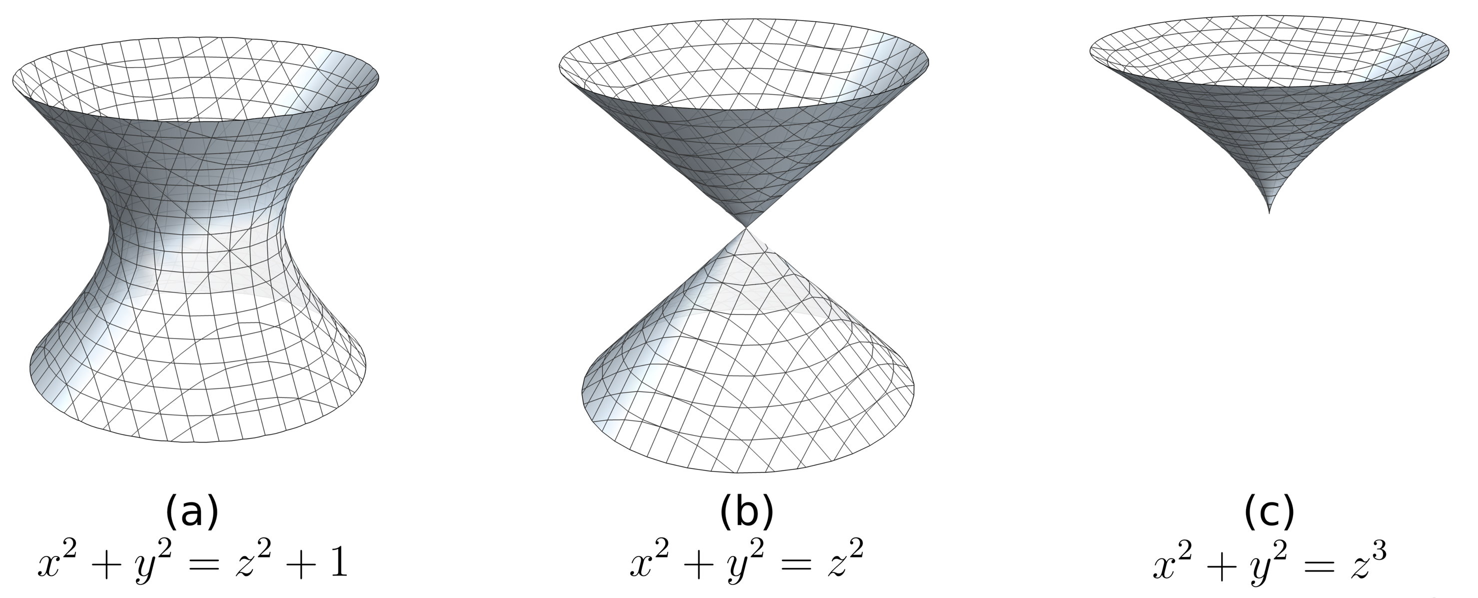


Figure 1: Algebraic varieties and their equations, visualised in $\mathbb{R}^3$.

QUESTION

We want to explore the landscape of $\mathbb{Q}$-Fano varieties. Checking $\mathbb{Q}$-factoriality is easy, checking terminality is challenging and time consuming. **Can machine learning detect terminal singularities?**

DATA

We look at $\mathbb{Q}$-factorial toric (i.e. highly symmetric) Fano varieties of dimension 8 and Picard rank 2 ($\rho = 2$). Their structure is encoded in an integer-valued (weight) matrix

$$\begin{bmatrix} a_1 & \cdots & a_{10} \\ b_1 & \cdots & b_{10} \end{bmatrix}$$

representing the action $(\mathbb{C}^\times)^2$ on $\mathbb{C}^{10}$,

$$(\lambda, \mu) \cdot (z_1, \ldots, z_{10}) = (\lambda^{a_1}\mu^{b_1}z_1, \ldots, \lambda^{a_{10}}\mu^{b_{10}}z_{10}) .$$

Its geometric quotient is a toric variety.

TERMINAL SINGULARITIES

An n-dimensional toric variety X determines a convex lattice polytope P in $\mathbb{R}^n$. Its vertices $e_1, \ldots, e_{n+2} \in \mathbb{Z}^n$ satisfy

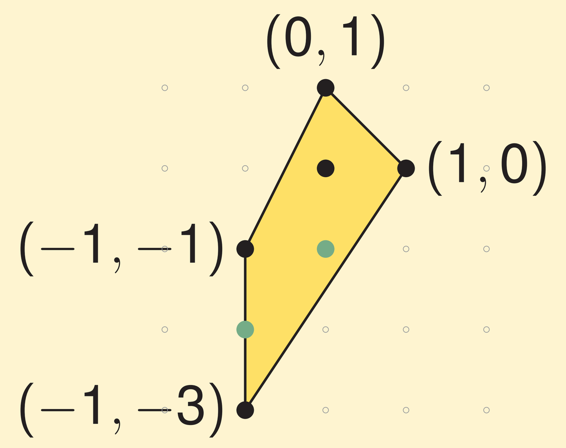
$$\begin{cases} a_1e_1 + \cdots + a_{n+2}e_{n+2} = 0 \\ b_1e_1 + \cdots + b_{n+2}e_{n+2} = 0 \end{cases}$$

X has terminal singularities if and only if the only lattice points in P are the origin and its vertices. **This is easy to state but computationally expensive**.

E.g. The toric variety with weight matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

corresponds to the yellow polytope P: its vertices and the origin are in black, the blue dots mean that it is not terminal.



APPROACH

Why do we consider $\mathbb{Q}$-factorial toric Fanos in dimension 8 and $\rho = 2$?

- In low dimension the problem is easier (terminal $\Longrightarrow$ smooth when dim $= 2$) and there is not enough data for ML (there are only 34 examples for dim $= 3$ and $\rho = 2$).
- There is a fast criterion on the weights that determines terminality for $\rho = 1$. This is used to generate Figure 2a, from [2].

THE NEURAL NETWORK

We train a fully-connected feed-forward neural network with 3 hidden layers $(512, 768, 512)$, taking flattened weight matrices as input, on a balanced dataset of **5 million samples**. The matrices are put in a *standard form*, specifying a representative of the orbit, so that the model is equivariant.

ANSWER

The neural network detects terminality for $\mathbb{Q}$-factorial toric Fano varieties (in dimension 8 and $\rho = 2$) with **95% accuracy**.

CONSEQUENCES

- The model **inspires** a new algorithm to test terminality for $\mathbb{Q}$-factorial toric Fanos, looking directly at the weights. It is **15 times faster** than the original method (since it does not need to compute the polytope).
- The model allows us to start **exploring** the toric $\mathbb{Q}$-Fano landscape, by quickly generating huge amount of data (**100 million** in 120 CPU hours, **30 000 times** faster than the original method). A visualisation is in Figure 2b, from [3].

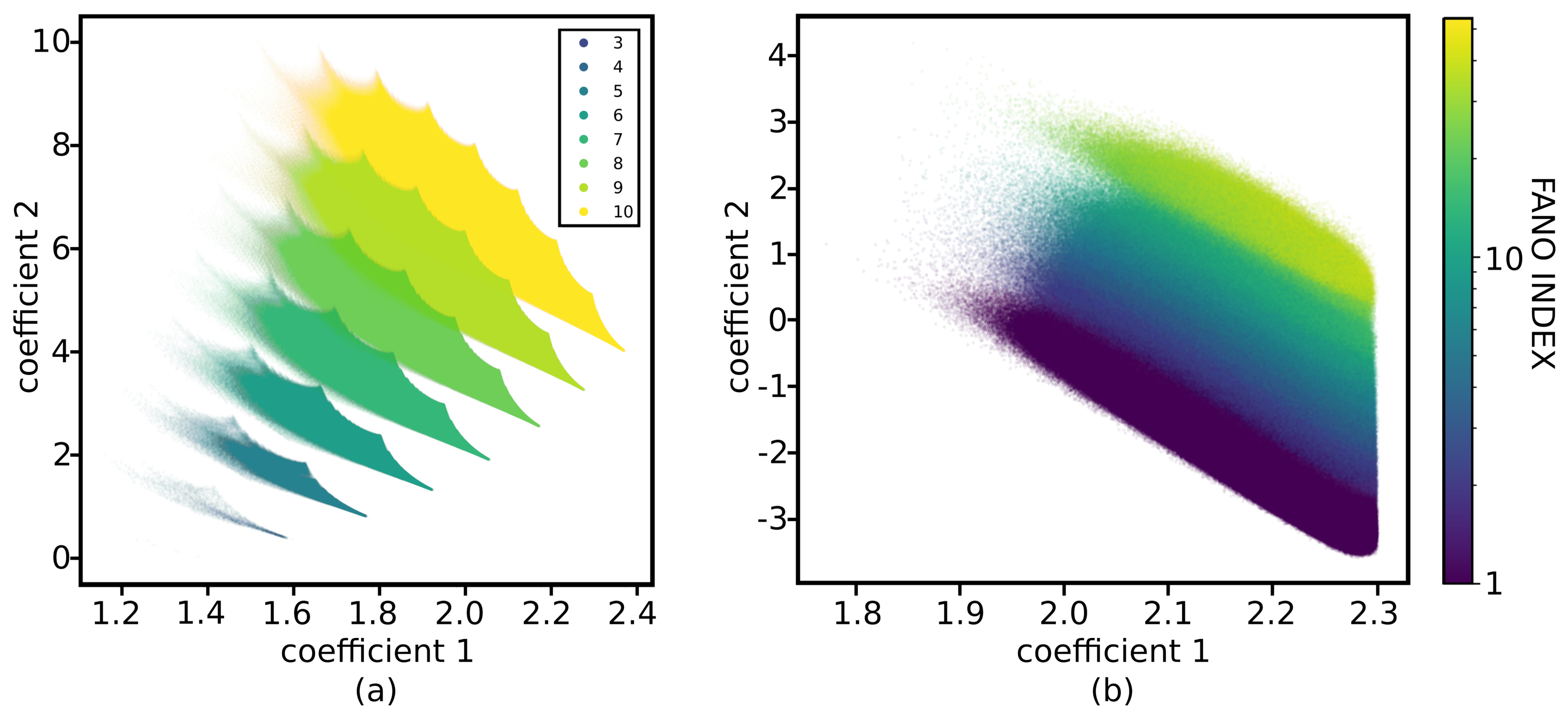


Figure 2: Visualising the $\mathbb{Q}$-Fano landscape in $\mathbb{R}^2$ using two of the growth coefficients of their **quantum period**, a conjectural invariant of $\mathbb{Q}$-Fano varieties, [1, 2]: (a) visualises $\mathbb{Q}$-Fano varieties with $\rho = 1$ for dim $= 3, \ldots, 10$, [2]; (b) visualises probable $\mathbb{Q}$-toric Fano varieties in dimension 8 and $\rho = 2$, coloured by Fano index, [3].

FUTURE DIRECTIONS

- We can use this framework to explore the landscape for different dimensions for $\rho = 2$.
- We should explore different Picard ranks, and non-toric varieties, but we lack a *standard form*: we need more powerful equivariant tools!

REFERENCES

[1] *Mirror Symmetry and Fano Manifolds*, T. Coates, A. Corti, S. Galkin, V. Golyshev, A. Kasprzyk. Proceedings of the 6th European Congress of Mathematics (2014).
[2] *Machine learning the dimension of a Fano variety*, T. Coates, A. Kasprzyk, S. Veneziale. Submitted (2023).
[3] *Machine learning detects terminal singularities*, T. Coates, A. Kasprzyk, S. Veneziale. Submitted (2023).

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