Available online at www.sciencedirect.com

ScienceDirect

Comput. Methods Appl. Mech. Engrg. 197 (2008) 2692–2700

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

A hierarchical multiscale framework for problems with multiscale source terms

Arif Masud [a,*], Leopoldo P. Franca [b]

[a] *Department of Civil and Environmental Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801-2352, USA*
[b] *Department of Mathematical Sciences, University of Colorado at Denver, Denver, CO 80217, USA*

Received 20 September 2007; received in revised form 16 December 2007; accepted 25 December 2007
Available online 26 January 2008

Abstract

This paper presents a hierarchical multiscale framework for problems that involve multiscale source terms. An assumption on the additive decomposition of the source function results in consistent decoupling of the fully coupled system and constitutes the new method. The structure of this decomposition is investigated and its mathematical implications are delineated. This method results in variational embedding of fine-scale information that is derived from the fine-scale equations, in the corresponding coarse-scale equations. It therefore provides a mathematically consistent way of bridging information between disparate spatial scales in the response function that are induced by multiscale forcing functions.

Keywords: Hierarchical multiscale framework; Multiscale source terms; Stabilized methods; Bridging scales

1. Introduction

Multiscale problems exist in all fields of engineering and sciences. Therefore, design of numerical methods that can model multiple material, spatial and temporal scales has been an area of active interest. A general trend in engineering practice toward miniaturization of mechanical and electronic components leading to micro-electro-mechanical systems (MEMS) and nano-electro-mechanical systems (NEMS) has highlighted the need for computational techniques that can bridge disparate spatial and temporal scales [20,40,42]. Concurrently, there has been a drive in the mechanics and materials community to develop micromechanics based constitutive models that can represent the micro-scale behavior of materials [39]. Full potential of these constitutive models can only be realized if they are integrated in variational formulations that can recognize and therefore model disparate scales in the problem. In addition, there are many problems all across the various engineering disciplines that are driven by scale dependent source terms. For example, in the field of solid/structural mechanics, elastic bodies can undergo deformations due to body forces that may be operational over the entire domain, as well as by applied point forces and moments, the effects of which are more localized. Likewise, in the field of reservoir simulation typical problems span a range of scales from kilometers (size-scale of the reservoir) down to micro-meters (size-scale of the grains). The flow of multiphase hydrocarbons is induced not only by the mean natural pressure in the reservoir but also by the induced pressure at the injection and production wells that appear as point sources and sinks on the physical length scale of the problem.

In the last two decades there have been concentrated efforts in the development of improved discretization methodologies. There have been several attempts to work within the variational structure of the problem to produce reliable and accurate formulations. In the context of mixed finite

* Corresponding author. Tel.: +1 217 244 2832; fax: +1 217 265 8039.
E-mail addresses: amasud@uiuc.edu (A. Masud), Leo.Franca@cudenver.edu (L.P. Franca).

0045-7825/$ - see front matter
doi:10.1016/j.cma.2007.12.024

element methods, an impressive number of finite element formulations have been developed that abide critical stability conditions (i.e., the Babuska–Brezzi condition and kernel ellipticity) thereby producing reliable approximations that can achieve high coarse-mesh accuracy at a relatively reasonable cost (see [5] and references therein). However, these elements are generally not convenient from the implementational standpoint. Their main drawback is the lack of a general strategy when one switches from one set of equations to another.

An attempt to develop a general methodology that could result in finite elements with enhanced accuracy and stability properties lead to the development of stabilized finite element methods in 1980s. The underlying philosophy of the stabilized methods is to strengthen the classical variational formulations so that discrete approximations, which would otherwise be unstable, become stable and convergent. In general stabilized methods are developed by adding perturbation terms to the standard Galerkin method [31], and these terms are usually of a least-squares form. Perturbation terms being residuals themselves imply that the exact solutions are satisfied by these terms, and therefore consistency of the method is preserved. Stabilized finite element methods, originally proposed as SUPG [8,30] or streamline diffusion methods for advection dominated problems, have been generalized, set in abstract frameworks, analyzed and denoted by Petrov–Galerkin or Galerkin/least-squares methods (GLS) or stabilized finite element methods (see e.g., [4,9,10,12–19,23–27,33,41] and references therein). If we revisit the original variational formulations as a starting point, enriching the Galerkin method with some "bubble" functions, can reproduce some versions of the methods developed above [1,3] or inspire alternative forms [11]. However, an open question has been whether either of the approaches had ultimately resulted in more stable and accurate methods, and in case they had, then how could one systematically build the ideal bubbles and/or develop the right stabilization parameters. The stabilized methods viewed from either point-of-view seem to be an unsuitable framework because of the loss of accuracy that can be attributed to the inadequacy of these methods to deal for example with large zero order terms in the differential equations. This issue was later addressed in [34] and extended to discontinuous-Galerkin methods in [7,28].

In mid 1990s Hughes revisited the stabilized finite element methods from a variational viewpoint and presented the Variational Multiscale method [21,22,29]. This paper draws inspiration from [21] and presents a hierarchical variational framework for application to problems subjected to scale dependent forcing functions. Objective in the present work is to develop a hierarchical multiscale framework that can help in decoupling of the scales that are induced by scale dependent source terms. A condition on the unique additive decomposition of the source terms results in a variationally consistent decoupling of the coupled system into two sub-systems: (i) a system that is driven by coarse-scale forcing functions, and (ii) a system that is driven by fine-scale forcing functions. This decomposition facilitates the modeling of different scales independently and their combined effect yields a physically and mathematically sound solution to the problem.

An outline of the paper is as follows. Section 2 presents ideas underlying the multiscale method with the help of a model problem. Section 3 highlights the significant features of the proposed formulation. Section 4 presents an extension of the ideas to a hierarchical multiscale framework, and conclusions are drawn in Section 5.

2. The multiscale computational framework

This section presents a mathematically consistent multiscale theory for problems that are driven by scale dependent forcing functions.

2.1. The model problem

Let $\mathscr{L}$ be the differential operator of the partial differential equation of interest

$$\mathscr{L}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

with prescribed boundary conditions on the response function, given as $\boldsymbol{u} = \boldsymbol{g}$ on Γ_g. The corresponding variational form obtained via the standard procedure can be expressed as

$$(\boldsymbol{w}, \mathscr{L}\boldsymbol{u}) = (\boldsymbol{w}, \boldsymbol{f}), \tag{2}$$

where $(\cdot,\cdot)$ is the L_2 inner product. It is important to realize that (2) can also represent the linearized form of the corresponding non-linear partial differential equation, in which case the right hand side would represent the out-of-balance residual force for the problem.

2.2. Multiscale decomposition

We consider the bounded domain Ω discretized into n_{el} non-overlapping regions Ω^e (element domains) with boundaries $\Gamma^e, e = 1, 2, \ldots n_{\text{el}}$, such that

$$\Omega = \overline{\bigcup_{e=1}^{n_{\text{el}}} \Omega^e}. \tag{3}$$

We denote the union of element interiors and element boundaries by Ω' and Γ', respectively.

$$\Omega' = \bigcup_{e=1}^{n_{\text{el}}} (\text{int})\Omega^e \quad \text{(element interiors)}, \tag{4}$$

$$\Gamma' = \bigcup_{e=1}^{n_{\text{el}}} \Gamma^e \quad \text{(element boundaries)}. \tag{5}$$

We assume an additive decomposition of the total solution into resolvable scales $\tilde{\boldsymbol{u}}$ (i.e., scales that can be modeled via standard finite element methods) and unresolvable scales $\boldsymbol{u}'$, also called the fine scales or the sub-grid scales that are normally filtered out by the standard approaches

$$\boldsymbol{u} = \underbrace{\tilde{\boldsymbol{u}}}_{\text{coarse scale}} + \underbrace{\boldsymbol{u}'}_{\text{fine scale}} . \tag{6}$$

Likewise, we assume an additive decomposition of the weighting function

$$\boldsymbol{w} = \underbrace{\tilde{\boldsymbol{w}}}_{\text{coarse scale}} + \underbrace{\boldsymbol{w}'}_{\text{fine scale}} , \tag{7}$$

where $\tilde{\boldsymbol{w}}$ represents the weighting functions for the coarse scales and $\boldsymbol{w}'$ represents the weighting functions for the fine scales, respectively. To keep the presentation of ideas simple, we further assume that the fine-scales vanish at inter-element boundaries, i.e. $\boldsymbol{u}' = \boldsymbol{w}' = \boldsymbol{0}$ on Γ'.

We now consider an additive decomposition of the forcing function into coarse-scales $\tilde{\boldsymbol{f}}$ (mean or homogenized force) and fine scales $\boldsymbol{f}'$ (high modes and/or localized force) components

$$\boldsymbol{f} = \underbrace{\tilde{\boldsymbol{f}}}_{\text{coarse scale}} + \underbrace{\boldsymbol{f}'}_{\text{fine scale}} . \tag{8}$$

Remark 1. Various scale separations of the weighting functions are possible in Eq. (7). However they are subject to the restriction imposed by the stability of the formulation that requires the spaces for the coarse-scale and fine-scale functions to be linearly independent. Consequently, in the discrete case the space of coarse-scale weighting functions can be identified with the standard finite element spaces, while the fine-scale weighting functions can contain various finite dimensional approximations, e.g., bubble functions or *p*-refinements or higher order NURBS functions.

Remark 2. The assumption that fine-scales vanish at the inter-element boundaries helps in keeping the presentation of the ideas simple and concise. Relaxing this assumption in fact leads to a more general framework. This however requires Lagrange multipliers to enforce the continuity of the fine-scale fields across Γ'. It is important to note that Lagrange multipliers can be accommodated in the present hierarchical framework as well.

Remark 3. This method can be viewed as a procedure for taking singularity out of the system. For example, for elasticity problems driven by body forces as well as point forces, $\tilde{\boldsymbol{f}}$ represents the body forces while $\boldsymbol{f}'$ represents the point forces.

Remark 4. For a dynamics problem $\tilde{\boldsymbol{f}}$ represents the low frequency components and $\boldsymbol{f}'$ represents the high frequency components in the source terms.

Substituting $\boldsymbol{u}$, $\boldsymbol{w}$ and $\boldsymbol{f}$ in (2) we get

$$(\tilde{\boldsymbol{w}} + \boldsymbol{w}', \mathscr{L}(\tilde{\boldsymbol{u}} + \boldsymbol{u}')) = (\tilde{\boldsymbol{w}} + \boldsymbol{w}', \tilde{\boldsymbol{f}} + \boldsymbol{f}'). \tag{9}$$

In a linear problem (or a linearized problem in the case of non-linear partial differential equations) the proposed additive decomposition of the forcing function gives rise to a further decomposition of the coarse- and fine-scale solutions.

$$\tilde{\boldsymbol{u}} = \underbrace{\tilde{\boldsymbol{u}}_{\tilde{f}}}_{\text{coarse coarse}} + \underbrace{\tilde{\boldsymbol{u}}_{f'}}_{\text{coarse fine}} \tag{10}$$

$$\boldsymbol{u}' = \underbrace{\boldsymbol{u}'_{\tilde{f}}}_{\text{fine coarse}} + \underbrace{\boldsymbol{u}'_{f'}}_{\text{fine fine}} , \tag{11}$$

wherein $\tilde{\boldsymbol{u}}_{\tilde{f}}$ and $\boldsymbol{u}'_{\tilde{f}}$ are the coarse- and fine-scale components of the solution that arise because of $\tilde{\boldsymbol{f}}$. Similarly, $\tilde{\boldsymbol{u}}_{f'}$ and $\boldsymbol{u}'_{f'}$ are the coarse- and fine-scale components of the solution that are induced by $\boldsymbol{f}'$. Substituting (10) and (11) in (9) we get

$$(\tilde{\boldsymbol{w}} + \boldsymbol{w}', \mathscr{L}((\tilde{\boldsymbol{u}}_{\tilde{f}} + \tilde{\boldsymbol{u}}_{f'}) + (\boldsymbol{u}'_{\tilde{f}} + \boldsymbol{u}'_{f'}))) = (\tilde{\boldsymbol{w}} + \boldsymbol{w}', \tilde{\boldsymbol{f}} + \boldsymbol{f}'). \tag{12}$$

We assume that the forcing function $\boldsymbol{f}$ admits a unique additive decomposition as given in (8). For linear problems we employ linearity of the solution slot in (12), and for non-linear problems we consider the linearity of the solution slot in a linearized setting, which leads to two sub-problems as follows:

$$\text{Sub-system 1}: \quad (\tilde{\boldsymbol{w}} + \boldsymbol{w}', \mathscr{L}(\tilde{\boldsymbol{u}}_{\tilde{f}} + \boldsymbol{u}'_{\tilde{f}})) = (\tilde{\boldsymbol{w}} + \boldsymbol{w}', \tilde{\boldsymbol{f}}), \tag{13}$$

$$\text{Sub-system 2}: \quad (\tilde{\boldsymbol{w}} + \boldsymbol{w}', \mathscr{L}(\tilde{\boldsymbol{u}}_{f'} + \boldsymbol{u}'_{f'})) = (\tilde{\boldsymbol{w}} + \boldsymbol{w}', \boldsymbol{f}'). \tag{14}$$

Remark 5. The restriction on (12) that leads to (13) and (14) is that the decomposition of $\boldsymbol{f}$ in (8) should be unique. It is important to note that if we sum (13) and (14), we recover Eq. (12).

2.2.1. Coarse-scale problems

Linearity of the weighting function slot in (13) and (14) leads to two problems for the *resolvable scales*:

$$(\tilde{\boldsymbol{w}}, \mathscr{L}(\tilde{\boldsymbol{u}}_{\tilde{f}} + \boldsymbol{u}'_{\tilde{f}})) = (\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}}), \tag{15}$$

$$(\tilde{\boldsymbol{w}}, \mathscr{L}(\tilde{\boldsymbol{u}}_{f'} + \boldsymbol{u}'_{f'})) = (\tilde{\boldsymbol{w}}, \boldsymbol{f}'). \tag{16}$$

2.2.2. Fine-scale problems

Likewise, linearity of the weighting function slot in (13) and (14) leads to two problems for the *unresolvable or the sub-grid scales*:

$$(\boldsymbol{w}', \mathscr{L}(\tilde{\boldsymbol{u}}_{\tilde{f}} + \boldsymbol{u}'_{\tilde{f}})) = (\boldsymbol{w}', \tilde{\boldsymbol{f}}), \tag{17}$$

$$(\boldsymbol{w}', \mathscr{L}(\tilde{\boldsymbol{u}}_{f'} + \boldsymbol{u}'_{f'})) = (\boldsymbol{w}', \boldsymbol{f}'). \tag{18}$$

The general idea at this point is to solve the fine-scale problems locally, either using analytical methods or numerical methods, and extract the fine-scale components $\boldsymbol{u}'_{\tilde{f}}$ and $\boldsymbol{u}'_{f'}$. These can then be substituted in the corresponding coarse-scale problems given in (15) and (16), respectively, thereby eliminating the fine scales, yet modeling their effects.

2.3. The solution procedure

The solution of the uncoupled system of equations is accomplished in two steps as follows.

2.3.1. First step

Let us first consider Eq. (17) which can be written in a residual form as

$$(\boldsymbol{w}', \mathscr{L}\boldsymbol{u}'_{\tilde{f}}) = (\boldsymbol{w}', \tilde{\boldsymbol{f}} - \mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}). \tag{19}$$

Motivated by [6,21] and without loss of generality we assume that: (a) $\boldsymbol{w}'$ and $\boldsymbol{u}'_{\tilde{f}}$ are represented via bubble functions, and (b) we consider a constant projection of $[\tilde{\boldsymbol{f}} - \mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}]$ over the sub-domain Ω' which in the finite element setting is the sum over element interiors. With these two assumptions we can solve the fine problem (19) and extract the fine-scale component $\boldsymbol{u}'_{\tilde{f}}$ as

$$\boldsymbol{u}'_{\tilde{f}} = \frac{b^e\left(\int b^e \mathrm{d}\Omega\right)}{(b^e, \mathscr{L}b^e)}\left[\tilde{\boldsymbol{f}} - \mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}\right]. \tag{20}$$

This expression can be simplified for ease of presentation in terms of a parameter τ_1 which, in the context of standard stabilized methods is commonly called the stabilization parameter

$$\boldsymbol{u}'_{\tilde{f}} = -\tau_1[\mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}} - \tilde{\boldsymbol{f}}], \tag{21}$$

where $\tau_1 = b^e(\int b^e \mathrm{d}\Omega)/(b^e, \mathscr{L}b^e)$.

Remark 6. The problem given in (19) is defined over the sum of element interiors. It can therefore be solved at the element level either analytically or numerically. It yields the solution $\boldsymbol{u}'_{\tilde{f}}$ over element interiors, which is the sub-grid scale, normally poorly approximated in the conventional finite element schemes unless very fine meshes are employed. A closer look reveals that the solution of the *Green's function problem* corresponding to the Euler–Lagrange equations of this sub-problem yields

$$\boldsymbol{u}'_{\tilde{f}}(y) = -\int_{\Omega'} \boldsymbol{g}(x,y)(\mathscr{L}(\tilde{\boldsymbol{u}}_{\tilde{f}} + \boldsymbol{u}'_{\tilde{f}}) - \tilde{\boldsymbol{f}})(x)\mathrm{d}\Omega_x, \tag{22}$$

where $\boldsymbol{g}(x,y)$ represents the fine-scale Green's function. Eq. (22) can also be expressed as $\boldsymbol{u}'_{\tilde{f}} = M(\mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}} - \tilde{\boldsymbol{f}})$ where $\mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}} - \tilde{\boldsymbol{f}}$ is the *residual* of the resolved scales or the coarse-scales over element sub-domains.

Remark 7. Depending on the partial differential equation being considered, the stabilization parameter τ can be a scalar valued function, or it can be a second-order stabilization tensor (see e.g., [32,35–38]). Furthermore, numerical implementation of a tensorial $\boldsymbol{\tau}$ leads to a full stabilization matrix for triangular elements as well as for distorted quadrilateral elements, which brings in the cross-coupling effects in the stabilization terms. A tensorial $\boldsymbol{\tau}$ leads to a diagonal stabilization matrix only for the quadrilateral elements in their rectangular configurations [36].

Considering (15), employing linearity of the solution slot, and rearranging terms leads to

$$(\tilde{\boldsymbol{w}}, \mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}) + (\mathscr{L}^*\tilde{\boldsymbol{w}}, \boldsymbol{u}'_{\tilde{f}}) = (\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}}) \tag{23}$$

where $\mathscr{L}^*$ is the adjoint operator. The boundary term appearing in (23) is annihilated due to the assumption that $\boldsymbol{u}'_{\tilde{f}} = 0$ on Γ'. Substituting (21) in (23) and taking the force term to the right hand side yields

$$(\tilde{\boldsymbol{w}}, \mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}) + (\mathscr{L}^*\tilde{\boldsymbol{w}}, -\tau_1\mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}) = (\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}}) - (\mathscr{L}^*\tilde{\boldsymbol{w}}, \tau_1\tilde{\boldsymbol{f}}), \tag{24}$$

wherein the second term in (24) is the stabilization term and is a function of the residual of coarse-scales over the sum of element interiors. Eq. (24) gives rise to a stabilized form for the sub-problem which is driven by the coarse-scale forcing function $\tilde{\boldsymbol{f}}$.

2.3.2. Second step

Now consider Eq. (18), which is the fine-scale problem driven by the fine-scale forcing function $\boldsymbol{f}'$. In a finite element framework this problem is also defined element wise. Employing linearity of the solution slot and rearranging terms it can be written in a residual form

$$(\boldsymbol{w}', \mathscr{L}\boldsymbol{u}'_{f'}) = (\boldsymbol{w}', \boldsymbol{f}' - \mathscr{L}\tilde{\boldsymbol{u}}_{f'}). \tag{25}$$

We again assume that $\boldsymbol{w}'$ and $\boldsymbol{u}'_{f'}$ are represented via bubble functions. However $[\boldsymbol{f}' - \mathscr{L}\tilde{\boldsymbol{u}}_{f'}]$ can not be considered constant over the finite element sub-domains, which simply means that we need a larger space of bubble functions to approximate $\boldsymbol{u}'_{f'}$. With these two assumptions we can solve (25) and extract the fine-scale component $\boldsymbol{u}'_{f'}$, which can be written in a functional form as a projection from a higher dimensional space to a lower dimensional space.

$$\boldsymbol{u}'_{f'} = -\tau_2[\mathscr{L}\tilde{\boldsymbol{u}}_{f'} - \boldsymbol{f}']. \tag{26}$$

In the context of stabilized methods τ_2 is the commonly termed stabilization parameter.

Remark 8. One can use the notion of residual-free bubbles to help in the selection of bubble functions. For example, consider the advection–diffusion operator $\mathscr{L} = \boldsymbol{a} \cdot \nabla + \kappa\Delta$. For the advection dominated case the appropriate bubble is a straight line from zero up until a small region before the outflow, and thereafter it has an exponential decay to zero. In the diffusive limit the right bubble is the usual cubic bubble. With this in view one can pre-set the choice as a linear combination of these two bubbles. This substituted in Eq. (25) would give an expression for τ_2.

Remark 9. Having selected the bubble functions one can derive the stability parameter τ_2. A derivation for τ_2 for Eq. (25) is presented in Appendix A.

Now consider (16) which is the coarse-scale problem that is driven by the fine-scale forcing function. Employing linearity of the solution slot, applying integration by parts, and the condition that $\boldsymbol{u}'_{f'} = \boldsymbol{0}$ on Γ', leads to

$$(\tilde{\boldsymbol{w}}, \mathscr{L}\tilde{\boldsymbol{u}}_{f'}) + (\mathscr{L}^*\tilde{\boldsymbol{w}}, \boldsymbol{u}'_{f'}) = (\tilde{\boldsymbol{w}}, \boldsymbol{f}'), \tag{27}$$

where $\mathscr{L}^*$ is the adjoint operator. We now substitute $\boldsymbol{u}'_{f'}$ from (26) in (27) and take the force term to the right hand side.

$$(\tilde{\boldsymbol{w}}, \mathscr{L}\tilde{\boldsymbol{u}}_{f'}) + (\mathscr{L}^*\tilde{\boldsymbol{w}}, -\tau_2\mathscr{L}\tilde{\boldsymbol{u}}_{f'}) = (\tilde{\boldsymbol{w}}, \boldsymbol{f}') - (\mathscr{L}^*\tilde{\boldsymbol{w}}, \tau_2\boldsymbol{f}'). \tag{28}$$

It is important to realize that the solution of (28) gives $\tilde{\boldsymbol{u}}_{f'}$ which is the coarse-scale component of the solution field that arises because of the fine-scale component in the forcing function.

Eqs. (24) and (28) are a system of two sets of equations that provide the scale dependent response functions to the partial differential equation that is driven by the coarse-scale and fine-scale forcing functions, respectively.

3. Significant features of the proposed formulation

This section highlights the important features of the proposed formulation:

1. From (17) we obtain $\boldsymbol{u}'_{\tilde{f}}$ which when substituted in (15) gives rise to a stabilized form for coarse-scales $\tilde{\boldsymbol{u}}_{\tilde{f}}$. This is the part of the unknown field that arises because of $\tilde{\boldsymbol{f}}$ (i.e., coarse-scales or low-frequency components in the forcing function).
2. From (18) we obtain $\boldsymbol{u}'_{f'}$ which when substituted in (16) gives rise to a stabilized form for coarse-scales $\tilde{\boldsymbol{u}}_{f'}$. This component of the unknown field in fact arises because of $\boldsymbol{f}'$ (i.e., the fine-scales in forcing function).
3. For linear problems the total solution is then obtained via the principal of superposition as
$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_{\tilde{f}} + \tilde{\boldsymbol{u}}_{f'}. \tag{29}$$
4. The present method is also applicable to non-linear problems wherein additive decomposition leading to two sub-systems can be performed after carrying out linearization of the problem. However, principle of superposition is not directly applicable to the linearized states of the non-linear problem. Consequently, Lagrange multiplier methods for overlapping solutions are required (see e.g., Belytschko and coworkers [2,18]).
5. To give an example of scale dependent source terms, let as assume that $\tilde{\boldsymbol{f}}$ and $\boldsymbol{f}'$ are given by the following functions:
$$\tilde{\boldsymbol{f}} = \sin\left(\frac{2\pi\boldsymbol{x}}{L}\right) \quad \text{(coarse-scale function)},$$
$$\boldsymbol{f}' = A\sin\left(\frac{n\pi\boldsymbol{x}}{L}\right) \quad \text{(fine-scale function)},$$
where, as an example $A = 0.1$ and $n = 8$ (see Fig. 1).
6. For problems in mechanics that are driven by body forces as well as localized forces and point forces, $\tilde{\boldsymbol{f}}$ represents the body forces and $\boldsymbol{f}'$ represents the localized force fields as shown in Fig. 2.
7. We want to point out that for problems wherein $\boldsymbol{f}'$ is periodic, (28) can be solved over a smaller subdomain Ω_{sub} of Ω. In other words, one can have a representative domain or a *unit cell* with periodic boundary conditions. Once the fine problem is solved over the unit cell Ω_{sub}, cell periodic conditions can be employed to generate $\tilde{\boldsymbol{u}}_{f'}$ over the entire domain Ω. Consequently, the cost of solving (28) can be reduced substantially.
8. For application to dynamics problems this approach can help decouple the problem based on the additive decomposition of the forcing function into the high-frequency and the low-frequency components. Furthermore, higher the frequency of the fine-scale forcing function, smaller will be the representative unit cell Ω_{sub} over which Eq. (28) will need to be modeled. Consequently, a refined mesh will be required only over a smaller sub-domain and it can substantially reduce the cost of computation.
9. The proposed multiscale framework yields variational *bridging scale methods*. These methods can be used for bridging the scales in computational micro-

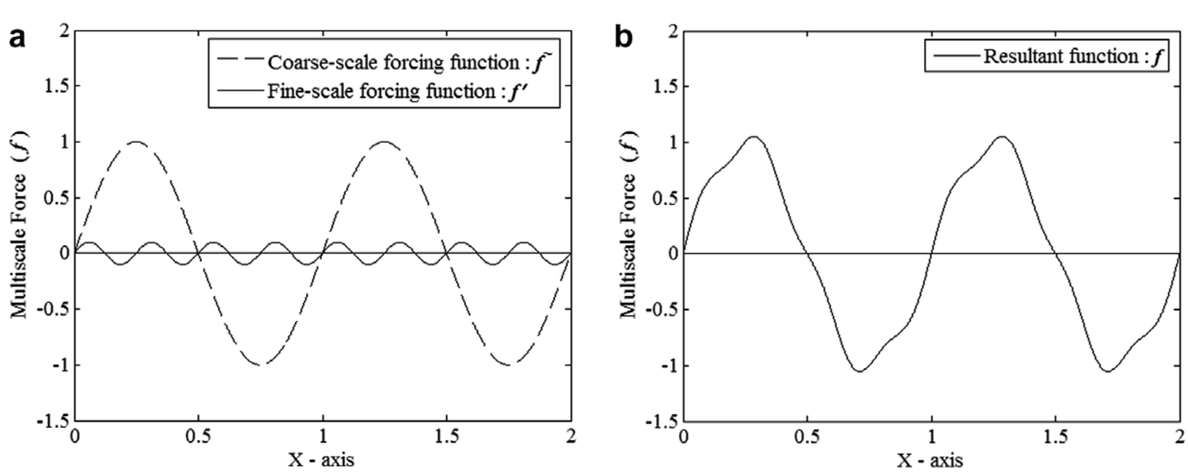

Fig. 1. (a) Coarse- and fine-scale forcing functions and (b) superposed function.

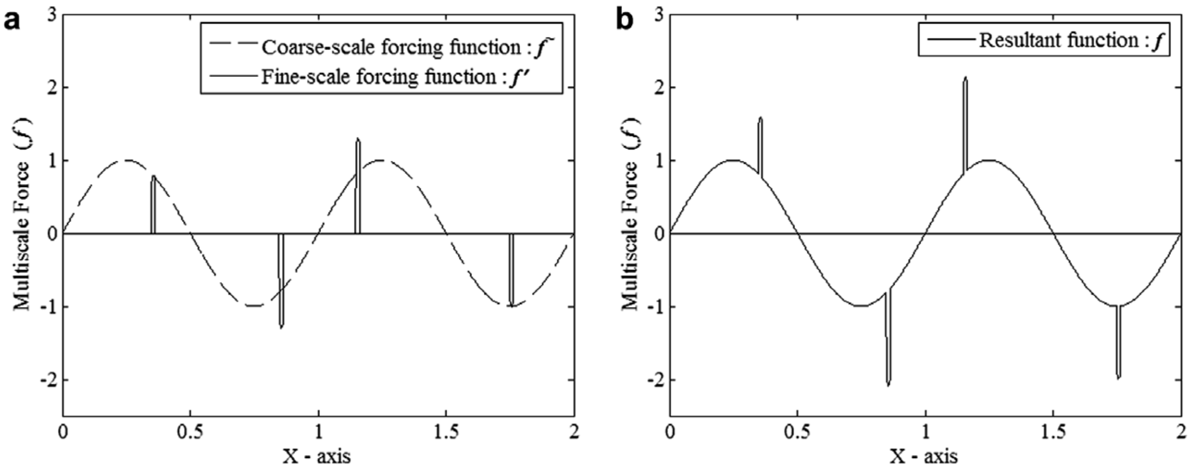

Fig. 2. (a) Coarse- and fine-scale forcing functions and (b) superposed function.

and nano-mechanics. In this context $\boldsymbol{u}'_{\tilde{f}}$ and $\tilde{\boldsymbol{u}}_{f'}$ can be considered as the bridging scales as they transfer information from one scale level to the next [37].

10. The proposed framework can be applied to problems that span a range of scales from micro-meters to kilometers, as are encountered in the modeling of petroleum reservoirs. Flow of multiphase hydrocarbons is induced by the mean pressure in the reservoirs in addition to the induced pressure at the injection and production wells, that, on the length scale of the reservoir, can be treated as localized or point sources and sinks in the medium. Mean pressure can be represented via $\tilde{\boldsymbol{f}}$ therefore providing the reservoir model, while the well pressure can be represented via $\boldsymbol{f}'$, and therefore providing a variationally consistent framework for the modeling of wells.

4. The hierarchical multiscale framework

This section presents an extension of the ideas presented in Section 2 to multi-level scale separation. We consider the governing equation given by (1), forcing function given by Eq. (8), and assume an additive decomposition of the solution and the weighting function in three-levels as follows:

$$\boldsymbol{u} = \underbrace{\tilde{\boldsymbol{u}}}_{\text{coarse}} + \underbrace{\boldsymbol{u}'}_{\text{fine}} + \underbrace{\hat{\boldsymbol{u}}}_{\text{very-fine}}, \tag{30}$$

$$\boldsymbol{w} = \underbrace{\tilde{\boldsymbol{w}}}_{\text{coarse}} + \underbrace{\boldsymbol{w}'}_{\text{fine}} + \underbrace{\hat{\boldsymbol{w}}}_{\text{very-fine}}. \tag{31}$$

Substituting the additively decomposed $\boldsymbol{u}$, $\boldsymbol{w}$ and $\boldsymbol{f}$ in (2) we get

$$(\tilde{\boldsymbol{w}} + \boldsymbol{w}' + \hat{\boldsymbol{w}}, \mathscr{L}(\tilde{\boldsymbol{u}} + \boldsymbol{u}' + \hat{\boldsymbol{u}})) = (\tilde{\boldsymbol{w}} + \boldsymbol{w}' + \hat{\boldsymbol{w}}, \tilde{\boldsymbol{f}} + \boldsymbol{f}'). \tag{32}$$

The proposed additive decomposition of the forcing function gives rise to a further decomposition of the coarse, the fine, and the very-fine scale solutions as follows:

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_{\tilde{f}} + \tilde{\boldsymbol{u}}_{f'}, \tag{33}$$

$$\boldsymbol{u}' = \boldsymbol{u}'_{\tilde{f}} + \boldsymbol{u}'_{f'}, \tag{34}$$

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}_{\tilde{f}} + \hat{\boldsymbol{u}}_{f'}. \tag{35}$$

Substituting (33)–(35) in (32), and assuming a unique decomposition of the forcing function leads to the split of (32) into two sub-problems

$$\text{Sub-Prob. 1}: (\tilde{\boldsymbol{w}} + \boldsymbol{w}' + \hat{\boldsymbol{w}}, \mathscr{L}(\tilde{\boldsymbol{u}}_{\tilde{f}} + \boldsymbol{u}'_{\tilde{f}} + \hat{\boldsymbol{u}}_{\tilde{f}})) = (\tilde{\boldsymbol{w}} + \boldsymbol{w}' + \hat{\boldsymbol{w}}, \tilde{\boldsymbol{f}}), \tag{36}$$

$$\text{Sub-Prob. 2}: (\tilde{\boldsymbol{w}} + \boldsymbol{w}' + \hat{\boldsymbol{w}}, \mathscr{L}(\tilde{\boldsymbol{u}}_{f'} + \boldsymbol{u}'_{f'} + \hat{\boldsymbol{u}}_{f'})) = (\tilde{\boldsymbol{w}} + \boldsymbol{w}' + \hat{\boldsymbol{w}}, \boldsymbol{f}'). \tag{37}$$

It is important to note that if we sum (36) and (37), we recover Eq. (32). We now employ the linearity of the weighting function slot, and it leads to the following three-sets of coupled equations.

Coarse problem: Coarse-scale weighting function component $\tilde{\boldsymbol{w}}$ in (36) and (37) leads to two systems of equations driven by $\tilde{\boldsymbol{f}}$ and $\boldsymbol{f}'$. These systems yield the *coarse-scales* arising from $\tilde{\boldsymbol{f}}$ and $\boldsymbol{f}'$.

$$(\tilde{\boldsymbol{w}}, \mathscr{L}(\tilde{\boldsymbol{u}}_{\tilde{f}} + \boldsymbol{u}'_{\tilde{f}} + \hat{\boldsymbol{u}}_{\tilde{f}})) = (\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}}), \tag{38}$$

$$(\tilde{\boldsymbol{w}}, \mathscr{L}(\tilde{\boldsymbol{u}}_{f'} + \boldsymbol{u}'_{f'} + \hat{\boldsymbol{u}}_{f'})) = (\tilde{\boldsymbol{w}}, \boldsymbol{f}'). \tag{39}$$

Fine problem: Similarly, the fine-scale weighting function component $\boldsymbol{w}'$ in (36) and (37) leads to two systems of equations driven by $\tilde{\boldsymbol{f}}$ and $\boldsymbol{f}'$. These systems yield the fine-scales that also play the role of the *intermediate scales* in the problem

$$(\boldsymbol{w}', \mathscr{L}(\tilde{\boldsymbol{u}}_{\tilde{f}} + \boldsymbol{u}'_{\tilde{f}} + \hat{\boldsymbol{u}}_{\tilde{f}})) = (\boldsymbol{w}', \tilde{\boldsymbol{f}}), \tag{40}$$

$$(\boldsymbol{w}', \mathscr{L}(\tilde{\boldsymbol{u}}_{f'} + \boldsymbol{u}'_{f'} + \hat{\boldsymbol{u}}_{f'})) = (\boldsymbol{w}', \boldsymbol{f}'). \tag{41}$$

Very fine problem: Likewise, the very-fine-scale weighting function component $\hat{\boldsymbol{w}}$ in (36) and (37) leads to two systems of equations driven by $\tilde{\boldsymbol{f}}$ and $\boldsymbol{f}'$, that yield the *unresolved or the sub-grid-scales*. These scales will need to be *modeled* as explained below

$$(\hat{\boldsymbol{w}}, \mathscr{L}(\tilde{\boldsymbol{u}}_{\tilde{f}} + \boldsymbol{u}'_{\tilde{f}} + \hat{\boldsymbol{u}}_{\tilde{f}})) = (\hat{\boldsymbol{w}}, \tilde{\boldsymbol{f}}), \tag{42}$$

$$(\hat{\boldsymbol{w}}, \mathscr{L}(\tilde{\boldsymbol{u}}_{f'} + \boldsymbol{u}'_{f'} + \hat{\boldsymbol{u}}_{f'})) = (\hat{\boldsymbol{w}}, \boldsymbol{f}'). \tag{43}$$

At this point we make a simplifying assumption that is based on the notion of a clear separation between coarse and the very-fine-scales and was first presented in [17]. We assume that the projection of the very-fine-scales onto the coarse scales in (38) and (39) is approximately zero

$$(\tilde{\boldsymbol{w}}, \mathscr{L}\hat{\boldsymbol{u}}_{\tilde{f}}) \approx 0 \quad \text{and} \quad (\tilde{\boldsymbol{w}}, \mathscr{L}\hat{\boldsymbol{u}}_{f'}) \approx 0. \tag{44}$$

Consequently, in this framework the coarse scales are influenced by the very-fine-scales through the fine or the "intermediate" scales. Same argument leads to a restriction on the opposite projection in (42) and (43), respectively

$$(\hat{\boldsymbol{w}}, \mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}) \approx 0 \quad \text{and} \quad (\hat{\boldsymbol{w}}, \mathscr{L}\tilde{\boldsymbol{u}}_{f'}) \approx 0. \tag{45}$$

4.1. The solution procedure

The solution procedure for the above hierarchical, coupled, multiscale system of equations follows the following general steps. First consider the set of Eqs. (38), (40) and (42) that are driven by the coarse-scale forcing function $\tilde{f}$. Applying $(45)_1$ to (42) leads to

$$(\hat{\boldsymbol{w}}, \mathscr{L}(\boldsymbol{u}'_{\tilde{f}} + \hat{\boldsymbol{u}}_{\tilde{f}})) = (\hat{\boldsymbol{w}}, \tilde{\boldsymbol{f}}) \tag{46}$$

Writing (46) in the residual form leads to

$$(\hat{\boldsymbol{w}}, \mathscr{L}\hat{\boldsymbol{u}}_{\tilde{f}}) = (\hat{\boldsymbol{w}}, \tilde{\boldsymbol{f}} - \mathscr{L}\boldsymbol{u}'_{\tilde{f}}). \tag{47}$$

Due to the orthogonality condition assumed in $(45)_1$, Eq. (47) is driven by the residual of the fine or the "intermediate" scales. Solving for $\hat{\boldsymbol{u}}_{\tilde{f}}$ leads to the functional form for the very-fine solution as

$$\hat{\boldsymbol{u}}_{\tilde{f}} = -\tau_1[\mathscr{L}\boldsymbol{u}'_{\tilde{f}} - \tilde{\boldsymbol{f}}], \tag{48}$$

where τ_1 is a function of the operator $\mathscr{L}$ acting on $\hat{\boldsymbol{u}}_{\tilde{f}}$. Since the operator $\mathscr{L}$ is differential, so τ_1 is an integral operator in nature. Substituting (48) in (40) leads to

$$(\boldsymbol{w}', \mathscr{L}(\tilde{\boldsymbol{u}}_{\tilde{f}} + \boldsymbol{u}'_{\tilde{f}})) + (\mathscr{L}^*\boldsymbol{w}', -\tau_1[\mathscr{L}\boldsymbol{u}'_{\tilde{f}}]) = (\boldsymbol{w}', (1-\tau_1)\tilde{\boldsymbol{f}}). \tag{49}$$

Simplifying further we get

$$(\boldsymbol{w}', \mathscr{L}\boldsymbol{u}'_{\tilde{f}}) + (\mathscr{L}^*\boldsymbol{w}', -\tau_1[\mathscr{L}\boldsymbol{u}'_{\tilde{f}}]) = (\boldsymbol{w}', (1-\tau_1)\tilde{\boldsymbol{f}} - \mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}). \tag{50}$$

It is important to realize that the term $(1-\tau_1)\tilde{\boldsymbol{f}}$ is the total coarse-scale force $\tilde{\boldsymbol{f}}$ minus the component of $\tilde{\boldsymbol{f}}$ that is consumed in driving the very-fine-scale problem (42) that yielded $\hat{\boldsymbol{u}}_{\tilde{f}}$ given in (48). As such, the right hand side of (50) is the residual of the coarse-scales from where the effect of the force component that has been used in driving the very-fine-scale problem is subtracted away. Solution of (50) yields the following functional form:

$$\boldsymbol{u}'_{\tilde{f}} = -\tau_2[\mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}} - (1-\tau_1)\tilde{\boldsymbol{f}}]. \tag{51}$$

Consequently, the intermediate scale is driven by the residual of the coarse-scales with respect to a modified force function. τ_2 in (51) is a function of the inverse of the operator on $\boldsymbol{u}'_{\tilde{f}}$ and therefore it is an integral operator.

Now consider Eq. (38). Applying $(44)_1$ we get

$$(\tilde{\boldsymbol{w}}, \mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}) + (\mathscr{L}^*\tilde{\boldsymbol{w}}, \boldsymbol{u}'_{\tilde{f}}) = (\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}}). \tag{52}$$

Substituting (51) in (52) yields

$$(\tilde{\boldsymbol{w}}, \mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}) + (\mathscr{L}^*\tilde{\boldsymbol{w}}, -\tau_2[\mathscr{L}\tilde{\boldsymbol{u}}_{\tilde{f}}]) = (\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}}) - (\mathscr{L}^*\tilde{\boldsymbol{w}}, \tau_2(1-\tau_1)\tilde{\boldsymbol{f}}). \tag{53}$$

In (53) the term $\tau_2(1-\tau_1)\tilde{\boldsymbol{f}}$ is the total coarse-scale force $\tilde{\boldsymbol{f}}$ minus the parts that have been consumed in driving the very-fine-scale problem (47) and the fine-scale problem (50), yielding solution components $\hat{\boldsymbol{u}}_{\tilde{f}}$ and $\boldsymbol{u}'_{\tilde{f}}$, respectively. To see this, write the right hand side of (53) as $(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}} - \mathscr{L}[\tau_2(1-\tau_1)\tilde{\boldsymbol{f}}])$ in which $(1-\tau_1)\tilde{\boldsymbol{f}}$ is part of the force from which the component that is consumed in driving the very fine problem is subtracted away.

Remark 10. A solution procedure for (39), (41) and (43) can follow along similar lines. Important consideration would be that now one will need to work in a higher dimensional functional space involving denser meshes. However, following along the lines of Section 2 and employing periodic boundary conditions, the cost of computation for these problems can be substantially reduced.

5. Conclusions

We have presented a hierarchical multiscale method for consistently decoupling multiscale systems that are subjected to scale dependent source terms. An assumption on unique additive decomposition of the multiscale source terms leads to two sub-systems: a sub-system that is driven by the coarse-scale forcing functions; and a sub-system that is driven by the fine-scale forcing functions. Accordingly, the proposed method can be viewed as a variationally consistent procedure for taking singularity out of the system. The variational decoupling of the problem facilitates independent modeling of phenomena at different scales. Contrary to the standard Galerkin approach where the fine scales of the problem can only be resolved via successive mesh refinements, in the present method fine scales are embedded in the corresponding coarse-scale problems via variationally consistent mathematical projections. The hierarchical multiscale framework presented here yields variational *bridging scale methods* that provide a mechanism for passing information between disparate scales that are induced by multiscale source terms.

Appendix A. Derivation of stabilization parameter τ_2

Let us expand the fine-scale trial solution $u'_{f'}$ as

$$u'_{f'} = c_1\phi_1 + c_2\phi_2, \tag{54}$$

where c_1 and c_2 are the coefficients and ϕ_1 and ϕ_2 are the interpolation functions for the fine-scale trial solutions. Likewise, we expand the fine-scale weighting functions as

$$w'_i = \gamma_i\phi_i. \tag{55}$$

Substituting the fine-scale trial solution and weighting function in the scalar version of Eq. (25)

$$(\gamma_i\phi_i, \mathscr{L}(c_1\phi_1 + c_2\phi_2)) = (\gamma_i\phi_i, f' - \mathscr{L}\tilde{u}_{f'}). \tag{56}$$

Eq. (56) is required to hold for arbitrary weighting functions for the fine-scales, and therefore it should hold for all γ_i. Furthermore, taking the constant coefficients for the fine-scale trial solutions out of the integral expression, we get

$$c_1(\phi_i, \mathscr{L}\phi_1) + c_2(\phi_i, \mathscr{L}\phi_2) = (\phi_i, f' - \mathscr{L}\tilde{u}_{f'}). \tag{57}$$

This leads to a system of linear equations

$$c_1(\phi_1, \mathscr{L}\phi_1) + c_2(\phi_1, \mathscr{L}\phi_2) = (\phi_1, f' - \mathscr{L}\tilde{u}_{f'}), \tag{58}$$
$$c_1(\phi_2, \mathscr{L}\phi_1) + c_2(\phi_i, \mathscr{L}\phi_2) = (\phi_2, f' - \mathscr{L}\tilde{u}_{f'}), \tag{59}$$

which can be simplified as follows:

$$ac_1 + bc_2 = f, \tag{60}$$
$$cc_1 + dc_2 = g. \tag{61}$$

We solve the system for coefficients c_1 and c_2

$$\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{Bmatrix} f \\ g \end{Bmatrix} = \frac{1}{(ad-bc)}\begin{Bmatrix} df - bg \\ -cf + ag \end{Bmatrix}. \tag{62}$$

To reconstruct the fine-scale field $u'_{f'}$, premultiply with $\{\phi_1\phi_2\}$

$$u'_{f'} = \frac{1}{(ad-bc)}\{\phi_1\phi_2\}\begin{Bmatrix} df - bg \\ -cf + ag \end{Bmatrix} = \frac{1}{(ad-bc)}\{\phi_1(df-bg) + \phi_2(-cf+ag)\}. \tag{63}$$

Substituting a, b, c, d, f and g, we get the expression for $u'_{f'}$

$$u'_{f'} = \frac{\phi_1[(\phi_2, \mathscr{L}\phi_2)(\phi_1, \delta) - (\phi_1, \mathscr{L}\phi_2)(\phi_2, \delta)] + \phi_2[-(\phi_2, \mathscr{L}\phi_1)(\phi_1, \delta) + (\phi_1, \mathscr{L}\phi_1)(\phi_2, \delta)]}{(\phi_1, \mathscr{L}\phi_1)(\phi_2, \mathscr{L}\phi_2) - (\phi_1, \mathscr{L}\phi_2)(\phi_2, \mathscr{L}\phi_1)}, \tag{64}$$

where $\delta = (f' - \mathscr{L}\tilde{u}_{f'})$. This expression provides a general definition of the stability parameter τ_2.

If we assume the projection of the residual of the coarse-scales to be constant over the sum of element interiors, we get a simplified expression for $u'_{f'}$

$$u'_{f'} = \frac{(\phi_1\int\phi_1(\phi_2, \mathscr{L}\phi_2) - \phi_1\int\phi_2(\phi_1 - \mathscr{L}\phi_2) - \phi_2\int\phi_1(\phi_2, \mathscr{L}\phi_1) + \phi_2\int\phi_2(\phi_1, \mathscr{L}\phi_1))}{(\phi_1, \mathscr{L}\phi_1)(\phi_2, \mathscr{L}\phi_2) - (\phi_1, \mathscr{L}\phi_2)(\phi_2, \mathscr{L}\phi_1)}(f' - \mathscr{L}\tilde{u}_{f'}), \tag{65}$$

which can be written in a concise form as

$$u'_{f'} = \tau_2(f' - \mathscr{L}\tilde{u}_{f'}), \tag{66}$$

where stabilization parameter τ_2 is defined as

$$\tau_2 = \frac{(\phi_1\int\phi_1(\phi_2, \mathscr{L}\phi_2) - \phi_1\int\phi_2(\phi_1 - \mathscr{L}\phi_2) - \phi_2\int\phi_1(\phi_2, \mathscr{L}\phi_1) + \phi_2\int\phi_2(\phi_1, \mathscr{L}\phi_1))}{(\phi_1, \mathscr{L}\phi_1)(\phi_2, \mathscr{L}\phi_2) - (\phi_1, \mathscr{L}\phi_2)(\phi_2, \mathscr{L}\phi_1)} \tag{67}$$

References

[1] C. Baiocchi, F. Brezzi, L. Franca, Virtual bubbles and the Galerkin-least-squares method, Comput. Meth. Appl. Mech. Engrg. 105 (1993) 125–141.

[2] T. Belytschko, S.P. Xiao, Coupling methods for continuum model with molecular model, Internat. J. Multiscale Comput. Engrg. 1 (1) (2003) 115–126.

[3] F. Brezzi, M. Bristeau, L. Franca, M. Mallet, G. Roge, A relationship between stabilized finite element methods and the Galerkin method with bubble functions, Comput. Meth. Appl. Mech. 96 (1992) 117–129.

[4] F. Brezzi, J. Douglas, Stabilized mixed methods for the Stokes problem, Numer. Math. 53 (1988) 225–236.

[5] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, New York, 1991.

[6] F. Brezzi, L.P. Franca, T.J.R. Hughes, A. Russo, $b = \int g$, Comput. Meth. Appl. Mech. Engrg. 145 (1997).

[7] F. Brezzi, T.J.R. Hughes, L.D. Marini, A. Masud, Mixed discontinuous Galerkin methods for Darcy flow, SIAM J. Scient. Comput. 22 (1) (2005) 119–145.

[8] A.N. Brooks, T.J.R. Hughes, Streamline upwind/Petrov–Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier–Stokes equations, Comput. Meth. Appl. Mech. Engrg. 32 (1982) 199–259.
[9] J. Douglas, J. Wang, An absolutely stabilized finite element method for the Stokes problem, Math. Comput. 52 (1989) 495–508.
[10] L.P. Franca, E.G.D. Do Carmo, The Galerkin gradient least-squares method, Comput. Meth. Appl. Mech. Engrg. 74 (1989) 41–54.
[11] L.P. Franca, C. Farhat, Bubble functions prompt unusual stabilized finite element methods, Comput. Meth. Appl. Mech. Engrg. 123 (1995) 299–308.
[12] L.P. Franca, S.L. Frey, Stabilized finite element methods: II. The incompressible Navier–Stokes equations, Comput. Meth. Appl. Mech. Engrg. 99 (1992) 209–233.
[13] L.P. Franca, S.L. Frey, T.J.R. Hughes, Stabilized Finite element methods: I. Application to the advective–diffusive model, Comput. Meth. Appl. Mech. Engrg. 95 (1992) 253–276.
[14] L.P. Franca, A. Nesliturk, On a two-level finite element method for the incompressible Navier–Stokes equations, Inter. J. Numer. Meth. Engrg. 52 (2001) 433–453.
[15] L.P. Franca, A. Nesliturk, M. Stynes, On the stability of residual-free bubbles for convection–diffusion problems and their approximation by a two-level finite element method, Comput. Meth. Appl. Mech. Engrg. 166 (1998) 35–49.
[16] L.P. Franca, R. Stenberg, Error analysis of some Galerkin least squares methods for the elasticity equations, SIAM J. Numer. Anal. 28 (1991) 1680–1697.
[17] V. Gravemeier, W.A. Wall, E. Ramm, A three-level finite element method for the in stationary incompressible Navier–Stokes equations, Comput. Meth. Appl. Mech. Engrg. 193 (2004) 1323–1366.
[18] P.A. Guidault, T. Belytschko, On the L2 and the H1 couplings for an overlapping domain decomposition method using Lagrange multipliers, Inter. J. Numer. Meth. Engrg. 70 (2007) 322–350.
[19] P. Hansbo, A. Szepessy, A velocity–pressure streamline diffusion finite element method for the incompressible Navier–Stokes equation, Comput. Meth. Appl. Mech. Engrg. 84 (1990) 175–192.
[20] T. Hou, X. Wu, A multiscale finite element method for elliptic problems in composite materials and porous media, J. Comput. Phys. 134 (1997) 169–189.
[21] T.J.R. Hughes, Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origin of stabilized methods, Comput. Meth. Appl. Mech. Engrg. 127 (1995) 387–401.
[22] T.J.R. Hughes, G. Feijoo, L. Mazzei, J.-B. Quincy, The variational multiscale method: a paradigm for computational mechanics, Comput. Meth. Appl. Mech. Engrg. 166 (1998) 3–24.
[23] T.J.R. Hughes, L.P. Franca, A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity/pressure spaces, Comput. Meth. Appl. Mech. Engrg. 65 (1987) 85–96.
[24] T.J.R. Hughes, L.P. Franca, M. Balestra, A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuska–Brezzi condition: a stable Petrov–Galerkin formulation of the Stokes problem accommodating equal-order interpolations, Comput. Meth. Appl. Mech. Engrg. 59 (1986) 85–99.
[25] T.J.R. Hughes, L.P. Franca, G.M. Hulbert, A new finite element formulation for computational fluid dynamics: VIII The Galerkin-least-squares method for advective–diffusive equations, Comput. Meth. Appl. Mech. Engrg. 73 (1989) 173–189.
[26] T.J.R. Hughes, L.P. Franca, M. Mallet, A new finite element formulation for computational fluid dynamics: I. Symmetric forms of the compressible Euler and Navier–Stokes equations and the second law of thermodynamics, Comput. Meth. Appl. Mech. Engrg. 54 (1986) 223–234.
[27] T.J.R. Hughes, L.P. Franca, M. Mallet, A new finite element formulation for computational fluid dynamics: VI Convergence analysis of the generalized SUPG formulation for linear time-dependent multidimensional advective–diffusive systems, Comput. Meth. Appl. Mech. Engrg. 63 (1987) 97–112.
[28] T.J.R. Hughes, A. Masud, J. Wan, A discontinuous-Galerkin finite element method for Darcy flow, Comput. Meth. Appl. Mech. Engrg. 195 (2006) 3347–3381.
[29] T.J.R. Hughes, L. Mazzei, K.E. Jansen, Large eddy simulation and the variational multiscale method, Comput. Visual. Sci. 3 (2000) 47–59.
[30] C. Johnson, U. Navert, J. Pitkaranta, Finite element methods for linear hyperbolic problem, Comput. Meth. Appl. Mech. Engrg. 45 (1984) 285–312.
[31] A. Masud, Preface to the special issue on stabilized and multiscale finite element methods, Comput. Meth. Appl. Mech. Engrg. 193 (2004) iii–iv.
[32] A. Masud, L.A. Bergman, Application of multiscale finite element methods to the solution of the Fokker–Planck equation, Comput. Meth. Appl. Mech. Engrg. 194 (2005) 1513–1526.
[33] A. Masud, T.J.R. Hughes, A space-time Galerkin/least-squares finite - element formulation of the Navier–Stokes equations for moving domain problems, Comput. Meth. Appl. Mech. Engrg. 146 (1997) 91–126.
[34] A. Masud, T.J.R. Hughes, A stabilized mixed finite element method for Darcy flow, Comput. Meth. Appl. Mech. Engrg. 191 (2002) 4341–4370.
[35] A. Masud, R.A. Khurram, A multiscale/stabilized finite element method for the advection–diffusion equation, Comput. Meth. Appl. Mech. Engrg. 193 (2004) 1997–2018.
[36] A. Masud, R. Khurram, A multiscale finite element method for the incompressible Navier–Stokes equations, Comput. Meth. Appl. Mech. Engrg. 195 (2006) 1750–1777.
[37] A. Masud, R. Kannan, A multiscale computational framework for the modeling of carbon nanotubes, in: E. Onate, D.R.J. Owen, (Eds.), the Proceedings of the VIII International Conference on Computational Plasticity (COMPLAS VIII), CIMNE Barcelona, Spain, September 5–8, 2005.
[38] A. Masud, K. Xia, A variational multiscale method for inelasticity: application to superelasticity in shape memory alloys, Comput. Meth. Appl. Mech. Engrg. 195 (2006) 4512–4531.
[39] M. Ortiz, Computational micromechanics, Comput. Mech. 18 (1996) 321–338.
[40] H.S. Park, W.K. Liu, An introduction and tutorial on multiple-scale analysis in solids, Comput. Meth. Appl. Mech. Engrg. 193 (2004) 1733–1772.
[41] T.E. Tezduyar, J. Liou, M. Behr, A new strategy for finite element computations involving moving boundaries and interfaces: the DSD/ST procedure: I. The concept and the preliminary numerical tests, Comput. Meth. Appl. Mech. Engrg. 94 (1992) 339–352.
[42] S.P. Xiao, T. Belytschko, A bridging domain method for coupling continua with molecular dynamics, Comput. Meth. Appl. Mech. Engrg. 193 (2004) 1645–1669.