The Ising Model and Its Dualities

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Abstract

Critical phenomena and phase transitions is explored through the 2d Ising model as an example of its universality class. Critical temperatures T_C for the 2- and 3-dimensional models was estimated to be $T_C^{2d} \approx 2.885J$ and $T_C^{3d} \approx 1.243J$ respectively. Focus was made on the dualities exhibited in the square bi-dimensional lattice, and their use, including the exact calculation of the critical temperature $T_C^{2d} = 2.269J$.

Contents

Acknowledgements

I dedicate this dissertation to those who helped with their discussions, diagrams, and proofreading, to those who helped me preserver, to those who made this enjoyable. Special thanks to Ali, Benjo, Sinead, Giuseppe, Roman, Julian, Agnes, Elodie, to my family. I thank my supervisor Dr. Arttu Rajantie.

Penses particulires vers ma mre, dans cette difficile preuve.

Alas, how strong a family likeness runs through blind and persecuting humanity in all Dimensions! Points, Lines, Squares, Cubes, Extra-Cubes — we are all liable to the same errors, all alike the Slaves of our respective Dimensional prejudices ...

Flatland, J. Abbott

Introduction

Materials can be classified into roughly two behaviour groups: fluids — including liquids, gas and plasmas — and solids. It is then clear that solids require particular attention, and its study has not disappointed researchers in the industry to the most fundamental kind. The first instinct is then to investigate the simplest kind of solids, and crystals are modelled by lattice structures, the Ising model being the simplest of the interacting kind.

The Ising model, despite being the simplest non-trivial statistical physics model, has proven to be useful in a wide range of subjects. Not only has it provided accurate description of certain very anisotropic crystals, such as $FeCl_2$ and $FeCO_3$ [5], but is also a prime example of the importance of space dimensions. It exhibits very different behaviours in low dimensions than in higher (3d and above). An example is Mean-field theory making false predictions in 1d whereas it is the only tool accessible in 4d. Its wealth extends when one dimension is complexified, and the Ising model is explored from a Quantum Field theory point of view, where tools such as Renormalisation Flow become available.

The bi-dimensional Ising model presents non-trivial critical phenomena resulting in a lot of interesting physics at one's disposal without requiring any added complexity. This is the subject of this dissertation. Historically, we can see it is a tricky subject, just by looking at how slow progress was. Starting in 1925, when Ernst Ising's attempted to model ferromagnets in one dimension, in his thesis [6] Despite the disappointing result of having a model with no critical phenomena, interest was pursued by Heisenberg who

developed a model of his own, then by Onsager. The latter calculated the free energy for the 2-dimensional Ising Model [13] two decades after Ising's thesis. Onsager went on to announces his result for the magnetisation in a null background field, in a conference in Florence. C.N. Yang then publishes the calculation in terms of the temperature, that Onsager had announced 3 years earlier [20]. It is Kogut's 1979 review [8], that finalised the progress made so far. No analytical expression for the partition function has ever been computed for 2d in an external field, nor higher dimensions.

In this dissertation, we propose to explore the Ising model and its physical consequences in one and two dimension. Particular focus made on the critical phenomena that arises, as well as the dualities exhibited in the square lattice, and their use.

1 Preliminary tools and motivation

1.1 Conventions

Let us first of all establish the conventions used here. We work in natural units where the speed of light, Boltzmann's constant and Planck's reduced constant are all unity

$$c = k_B = \hbar = 1$$

The signature chosen follows the condensed matter convention

$$\eta_{00} = -\eta_{ij} = 1$$

1.2 Statistical mechanics

This section is aimed at introducing the basic tools we will use when a physical quantity is sought for, and a thermodynamical limit taken. As we will see throughout, the partition function contains all the information of a given system.

Key equations to statistical physics are summed in following table.

Ensemble	**Microcanonical**	**Canonical**	**Macrocanonical**
Macrostates	$E,\ V,\ N, \dots$	$T,\ V,\ N, \dots$	$T,\ V,\ \mu, \dots$
Probability	$p_i = \frac{1}{\Omega}$	$p_i(E_i) = \frac{e^{-E_i/T}}{\mathcal{Z}}$	$p_i(E_i) = \frac{e^{-(E_i-\mu N_i)/T}}{\Xi}$
Counting Number of States	$\Omega = \Omega(E, V, N, \cdots)$	$\mathcal{Z}(T, V, N, \cdots) = \sum_{\{i\}} e^{-E_i/T}$	$\Xi(T, V, \mu, \cdots) = \sum_{\{i\}} e^{-(E_i-\mu N_i)/T}$
Thermodynamical Quantities	$S = \ln(\Omega)$	$F = -T\ \ln(\mathcal{Z})$	$J = -T\ \ln(\Xi)$

Table 1: *Statistical ensembles and the physical quantities derivable*

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2 Statistical Physics of the 1d and 2d Ising Model

Lattice models are common in condensed matter. Not only are many systems easily represented on a lattice, but they are also simple to express both analytically and numerically. They consist of studying the relationship between the fundamental elements of the system, be it molecules, atoms or elementary particles, arranged as a fixed array. Here we will assume a square type array, but there exist triangular and hexagonal lattices, each coming with their own symmetries and (self)dualities.

The Ising model is the simplest of lattice models, where we consider nearest neighbour interactions only. Each point on the lattice is associated to a discrete parameter, representing the states of the elements. Historically, it was brought to explain ferromagnets, in which case the parameter is the spin taking values in $\mathbb{Z}_2$[1] for "up" and "down". Strictly speaking, they are the z components of the spin, allowing for all individual parts of Ising's Hamiltonian to commute. The direct consequence of this show the power of the simplicity of H's expression, as it is diagonal in the representation where the spins are. This reveals the Hamiltonian's eigenvalues and eigenfunction instantly, which proves to be very useful when obtaining the partition function.

Interactions between spins are determined by the coupling constant J, here taken to be positive $J > 0$ such that aligned-spin configurations are

[1]Note that $\mathcal{Z}_2$ is only an example. We could just as well consider $\mathbb{Z}_n$ symmetries or even continuous as $\mathbb{R}$. The latter has a name of its own right: the Gaussian Model. [[[These models will be rapidly explored in later sections.]]]

energetically favoured [2]. We shall see more of the role that $\mathbb{Z}_2$ plays, later.

In this section we will content ourselves with presenting examples of solving exactly for the partition function in 1d and discuss the exact solution of the 2d case. Although it no longer considered a realistic model of magnetism, for simplicity of language we will keep the vocabulary of a lattice of interacting spins for the rest of the section.

2.1 A Construction of the 1-Dimensional Exact Solution

Let us solve the 1-dimensional case by finding the explicit expression of $\mathcal{Z}_N$ using a recursion method chosen for its intuitiveness and discuss the macroscopic consequences.

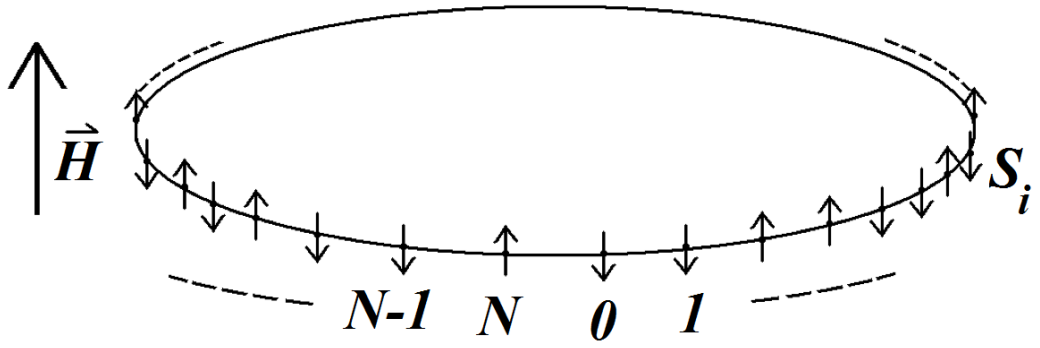

Figure 1: *A periodic 1-dimensional lattice model with $N+1$ sites, identical to the linear chain when N tends to infinity.*

The Hamiltonian of a chain N sites long, has the following internal and

[2] In the $J < 0$ case we have opposite-spin configuration favoured, resulting in antiferromagnets

external components respectively the first and second terms:

$$\mathcal{H} = -\frac{J}{2}\sum_{\langle i,j\rangle} s_i s_j - mH\sum_{i=0}^{N} s_i \tag{2.1}$$

Here $\langle i, j\rangle$ is for all pairs of spin, m is the magnetic moment of each lattice point [3] and plays the role of the coupling between the external field H and spin s_i at lattice point i. This should draw the reader's attention to the similarity with the Lagrangian formalism of field theories.

Considering only the internal term in the general expression (i.e setting $H = 0$), we write the partition function as given in table (1),

$$\mathcal{Z}_N = \sum_{\{s_i=\pm 1\}} \prod_{i=0}^{N-1} e^{\beta J(s_i s_{i+1})} \tag{2.2}$$

where β is defined as $\beta = \frac{1}{T}$. The special case of spin N, as an end point we explicitly know to contribute to the energy as

$$\begin{aligned}\sum_{\{s_N=\pm 1\}} e^{\beta J(s_{N-1}s_N)} &= e^{-\beta J} + e^{+\beta J} \\ &\equiv 2\cosh(\beta J)\end{aligned} \tag{2.3}$$

the first term being for the opposite spin configuration, and the second aligned. All $(N-1)$ intermediary points would have twice that same contribution. We can then establish a geometrical recursion relation as

$$Z_N = 2\cosh(\beta J) Z_{N-1}$$

[3] Possible generalisation to include variable individual magnetisation m_i

in terms of the end points

$$Z_N = 2\cosh(\beta J)^{N-2} Z_2$$

where $Z_2 = 4\cosh(\beta J)$ from (2.3). We can therefore obtain the exact partition function:

$$\boxed{\mathcal{Z}_N = 2\left(2\cosh(\beta J)\right)^{N-1}} \tag{2.4}$$

As mentioned earlier, we can now derive physical proprieties of the system from our new found result. Making use of the expression for the Helmholtz free energy (given in table (1)) we arrive at

$$F = -T\ln 2 - T(N-1)\ln\left(2\cosh\left(\beta J\right)\right)$$

Since we are interested in the macroscopic outcome, we take the large N limit giving

$$F_{N\longrightarrow\infty} = -TN\ln\left(2\cosh\left(\beta J\right)\right) \tag{2.5}$$

One noteworthy feature is the total dependance of the overall magnetisation M of the 1-dimensional ferromagnet, to the applied external field, as defined by Kramers and Wannier [9]:

$$M = m\frac{\partial\ln(\mathcal{Z}_N)}{\partial(m\frac{H}{T})} = \frac{NJ}{H}\tanh(\beta J) \tag{2.6}$$

As expected, the total magnetisation is proportional with the individual magnetic moments of each lattice points. Also, it depends on how much the partition function changes with respect to the external applied field H, with

coupling strength m again. In other words, there is no spontaneous magnetisation, only one phase exists for this case and as such, no critical phenomenon is observed.

Another physical quantity now derivable is the susceptibility per site χ, defined by

$$\chi = \frac{\partial M}{\partial H}|_{H=0} \tag{2.7}$$

It is interesting to note here, that the mean-field approximation predicts a phase transition, which is completely erroneous.

2.2 The 2-Dimensional Exact Solution

In the 2-dimensional problem, magnetisation appears as a different phase requiring a second order phase transition, hence making it a more interesting case. Let us use the eigenvalue method deployed by Kramers and Wannier [9] to find the partition function of a cylinder with finite height n and infinite circumference $m \longrightarrow \infty$.

We start by looking for the probability $P(s'_i)$ of the spin s'_i to be ± 1. Let us adapt our expression for the Boltzman energy to our infinitely long band

$$E = -\frac{J}{2}\sum_{i=0}^{n-1} s'_i s'_{i+1} - \frac{1}{2}mH\sum_{i=0}^{n} s'_i \tag{2.8}$$

The $s'_i s'_{i+1}$ interaction term is typical to the Ising model as it is nearest neighbour only.

We must now realise that our recursion relation depends on the previous chain s'_i (vertical dependance) and the previous spin s_{i-1} (horizontal depen-

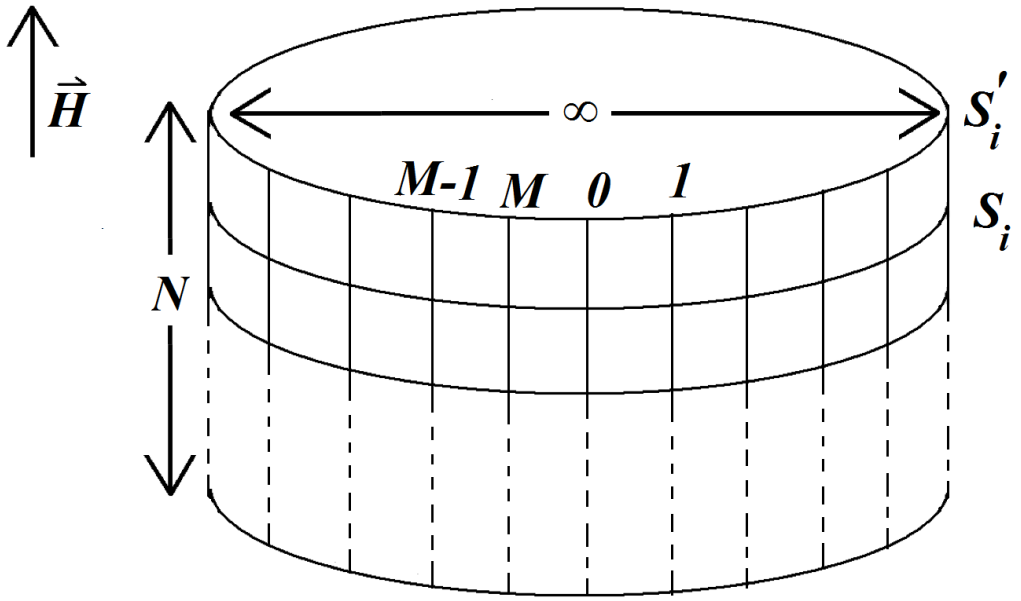

Figure 2: *Lattice of infinite circumference and finite width. The M ($M \longrightarrow \infty$) spins on the top row are labelled s'_i, the next row are labelled s_i. There are N rows.*

dance). The probability is proportional to the exponential of the Boltzman energy. Using the adapted E, we can write

$$P(s_i) \propto \sum_{s_i=\pm} e^{K\sum_{i=0}^{n-1} s'_i s'_{i+1} + K\sum_{i=0}^{n-1} s_i s_{i+1} + C\sum_{i=0}^{n} s_i} \tag{2.9}$$

where $K = \frac{1}{2}\frac{J}{T}$ is the self-interacting coupling and $C = \frac{mH}{T}$ is the coupling to the external magnetic field.

Next, we look for the probability $P(s_i)$ of the spin s_i to be ± 1 given the state of s'_i. Using the same intuition as previously to compute the recursion

relation, we expect

$$\lambda P(s_i, s_i') = P(s_i') e^{K \sum_{i=0}^{n-1} s_i' s_{i+1}' + K \sum_{i=0}^{n-1} s_i' s_i + C \sum_{i=0}^{n} s_i} \tag{2.10}$$

The λ factor comes in from the the fact that the probabilities are only proportional to the Boltzman energy.

The following steps are not detailed due to the difficulty to write and compute the eigenvalues and eigenvectors of said transfer matrix. The leading eigenvalue is

Yang demonstrated [20] the magnetisation per crystal area to be

$$M = \begin{cases} \frac{Nm}{2} \left(1 - \frac{1}{\sinh^4(\beta J)}\right)^{1/8} & T < T_C \qquad \text{ferromagnetic phase} \\ 0 & T > T_C \qquad \text{paramagnetic phase} \end{cases}$$

The magnetisation's divergence characteristic of critical phenomena, goes as $\beta = 1/8$, as we shall confirm in the next section 2. We have here an example of a proof that the system has distinct phases. This would is one of the starting assumptions in mean-field theory.

Another method was developed by M. Kac and J. C. Ward [11], using combinatorix and refined by N. V. Vdovichenko [17].

We shall stop there, as no exact expression of the partition function $\mathcal{Z}$ has been found for greater number of dimensions. Now that we have seen the exercise of obtaining the partition function, and demonstrated it's utility, allow us build upon the Ising model's symmetries to explore critical phenomena.

3 Criticality and Critical Exponents

3.1 Critical Phenomena

Ideal cases (i.e. non-interacting) are in general well behaved and easy to solve. This is why we can increase dimensions and degrees-of-freedom, and we will only suffer more complicated equations but nothing radical. As interactions are allowed, other parameters are introduced with significant consequences. Introducing interactions allows for several phases to a given system. By phase, we understand a macroscopic statistical equilibrium defined by at least one expectation value. As in the above example, the phases of a ferromagnet would be whether or not spontaneous magnetisation occurs.

3.1.1 Order of phase transitions

Let us quickly distinguish between the two classes of transitions: the first and the second.

The prior involves a latent heat, which is solely temperature-dependant. Entropy is varied and thus temperature must be either absorbed or released during the transition. The main consequence is the subsequent coexistence of the two phases. The water-vapour transition is an example of first order.

Second order transitions, to which the Ising universality class belongs, are characterised by a spontaneous symmetry breaking: the two resulting phases are distinct. Mathematically speaking, the order parameter — in our case the magnetisation — is a continuous function of the temperature, in other words is null at $T = T_C$. Entropy tends to the same value, as temperature tend to its critical value from lower or higher regardless, hence there is no latent

heat. It is now clear that a spontaneous magnetised phase (ferromagnet) cannot coexist with the disordered phase (paramagnet).

3.1.2 Physical Evidence of Divergence and Critical Exponents

In this section, we will summarise what what we have understood so far.

where to put that ?? If we define a spin-spin correlation function

$$\Gamma(n) \propto e^{-\frac{|n|}{[\xi(T)]}} \qquad\qquad |n| \gg 1 \tag{3.1}$$

We distinguish three cases when comparing the temperature with the critical temperature.

$T \gg T_C$: At high temperature the thermodynamical fluctuations prevail, in other words the χ is large. The correlation length $\xi(T)$ is of the order of the lattice spacing $\xi(T \gg T_C) \approx a$ which means the Ising model is accurate. Most importantly, the global[4] $\mathbb{Z}_2$ symmetry of the action 2.1 is preserved. Macroscopically, this is seen as there being no preferred direction to the spin: if all were to be inverted, the overall state would be unchanged.

$T \ll T_C$: As the global $\mathbb{Z}_2$ symmetry is spontaneously broken, in favour of an imposed direction in the $H \neq 0$ case, spontaneous magnetisation occurs[5] Magnetisation can play the role of local order parameter.

[4]The action is written as locally as possible, leaving little space for seeking more local symmetries.

[5]As seen before, the 1d case is entirely dependent on the external field, magnetisation is not spontaneous.

$T \longrightarrow T_C$: In systems with second order transitions — of which the Ising model is an example — some physical variables diverge as the temperature approaches the critical temperature. This is the first instance we encounter here, of infinities arising in the very physical theory of statistical mechanics. The reader will have noticed another hint towards the interest of exploring the same phenomena with the Field Theory language. All such divergences obey a power law

$$\frac{1}{t^A} \quad \text{where} \quad t = \frac{T - T_C}{T_C} \tag{3.2}$$

and A is a given critical exponent. These exponents are summarised by [8] in table (2).

Relation	**Value of the critical index in the 2d Ising Model**
$M \propto (T_C - T)^{\beta}$	$\beta = \frac{1}{8}$
$\Gamma(n) \propto \lvert n \rvert^{-(d-2+\eta)}$	$\eta = \frac{1}{4}$
$\xi \propto (T_C - T)^{-\nu}$	$\nu = 1$
$\chi \propto (T_C - T)^{-\gamma}$	$\gamma = 1.75$
$C \propto (T_C - T)^{-\alpha}$	$\alpha = 0$
$M \propto B^{\frac{1}{\delta}} \quad T = T_C$	$\delta = 15$

Table 2: Critical exponents and exact values for the 2d Ising Model

3.2 Evolution of Critical Temperature As a Function of Dimension

An interesting question to raise is how do we expect the critical temperature to evolve as we increase in dimension ? The two opposing "forces" here are the thermal fluctuations versus the spin-spin interactions. Two questions

follow then. First we must consider the different cases of lattice points being scalars, spinors, or even higher spin particles. Only then can we count the internal and external degrees of freedom as a function in dimension. As we will demonstrate in the following section, there is a duality in the Ising model that we can utilise to help with the latter.

4 Dualities in the Ising Model

4.1 Estimation of the Critical Temperature in 2d and 3d

In 1936, Peierls with the help of H. Bethe [14], used a geometrical argument which provides us with a quick method to obtain an estimation for the critical temperature, as approached from below.The method followed here is an adaptation of Peierls'. He counts how many spins can be enclosed with known information from the partition function $\mathcal{Z}$, to finally obtain

$$e^{-E/T_c} \sim \frac{1}{3} \tag{4.1}$$

Instead, here we shall trace a random path in the dual lattice and use energy arguments. As detailed later, the dual to the square net of lattice points (here our spins) is also a square net (here considered to represent to spin-spin interaction). It is to be understood that the pictures are equivalent due to the square being its own dual. For our estimation, we start by assuming the lowest energy state. In the spin picture, all spin states are aligned, which

would correspond to a trivial interaction picture. If then, a pocket of anti-aligned spins is created, it requires energy ΔE proportional to the length L of the domain wall in the dual picture

$$\Delta E = 2JL \tag{4.2}$$

The unique dependence on L (i.e. the independence of the number of spin enclosed), allows the freedom of the shape of the pocket and its connectedness (cf. figures 3 and 4). As illustrated below, we see that

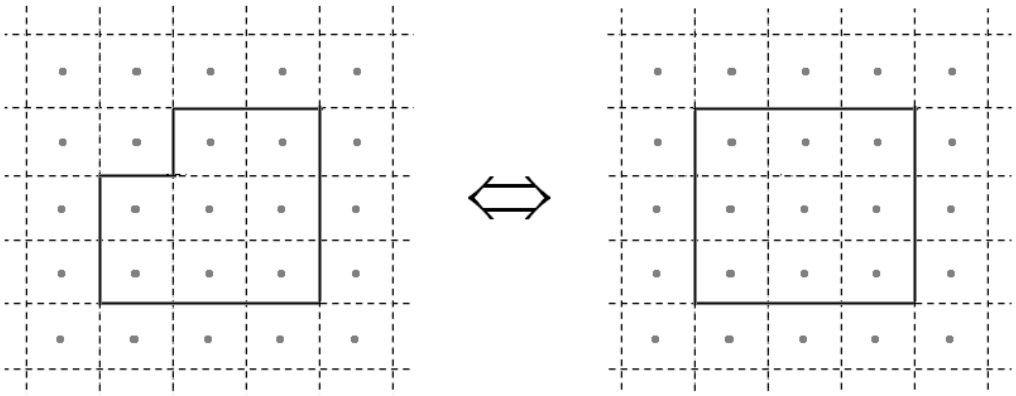

Figure 3: *The lattice of dots represents the spin lattice, and the dashed lines are the interaction lattice.* $L = 12$ *in both graphs, therefore both require the same energy, yet do not have the same number of spins flipped.*

A boundary of length L can be thought of as an L steps closed path along the array's links. At each site, a choice of two directions is sufficient to guarantee a closed path[6]. There are then 2^L ways to realise a boundary

[6]Four choices is most general, but excluding the back-stepping possibility, we are left with 3. But here we are looking for an upper bound, so 2 is sufficient

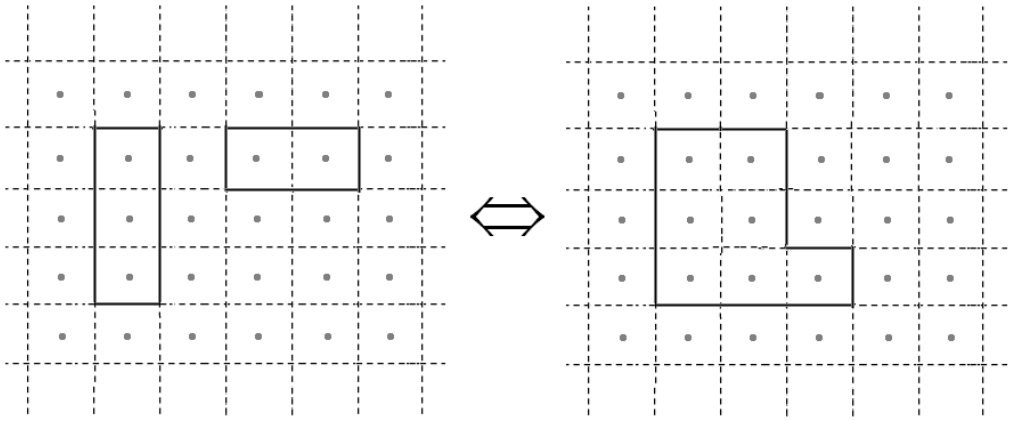

Figure 4: *Same convention as above. Here $L = 14$ in both cases, therefore both graphs are energetically equivalent, despite the different topologies*

given an L, all topologies included, thus according to statistical mechanics

$$\Delta S = \ln(2^L) = L \ln(2) \tag{4.3}$$

Combined with equation 4.2, we can finally express the variation in free energy

$$\begin{aligned} \Delta F &= \Delta E - T\Delta S \\ &= 2JL - TL\ln(2) = L\left(2J - T\ln(2)\right) \end{aligned} \tag{4.4}$$

At low temperature, where a ferromagnetic behaviour is observed, we can assume stability of the creation of such boundaries, i.e. $\Delta F \geqslant 0$. We can obtain a first approximation to the critical temperature T_C, as approached from below by plugging equation 4.4 into the inequality

$$T_C \approx \frac{2}{\ln(2)} J \leqslant T \tag{4.5}$$

which gives

$$\boxed{T_C \approx 2.885J} \tag{4.6}$$

We can expand this argument for 3d, if we suppose equation 4.2 is still holding. On the other hand, we should consider 5^L ways to close a path in the cubic array. The same calculations lead to an evaluation of the critical temperature of the 3d Ising model as

$$\boxed{T_C = \frac{2}{\ln(5)} J \approx 1.243J} \tag{4.7}$$

This appears to be a lower temperature than in 2d. One is tempted to conclude that thermal fluctuations are stronger than spin-spin interactions, as dimensions increase. This approach, although naïve, has the power to estimate a value for T_C despite not being able to solve the model exactly.

4.2 Kramers-Wannier (Self)-Duality in 2d

Let us investigate the behaviour of an M by M links square lattice[7], in the absence of an external field. To do so, and demonstrate another, yet more important, self-duality of the Ising's square lattice, we shall use the high, then low temperature series expansion. We will again use geometrical arguments to consolidate the expansion.

[7] In the thermodynamical limit, when $M \longrightarrow \infty$ the number of links and lattice points N are indistinguishable. The calculation is made in the interaction picture, and remains valid in the spin picture.

4.2.1 High-Temperature Expansion

Let us suppose two coupling constants for the vertical and horizontal link-link interactions to be identical $J_V = J_H = J$. We can then start with the partition function written as

$$Z_M = \sum_s e^{J\sum_{\langle i,j\rangle} s_i s_j + J\sum_{\langle i,k\rangle} s_i s_k} \tag{4.8}$$

Using $K = \beta J$ and the identity

$$e^{Ks} = \cosh(K) + \sinh(Ks) \tag{4.9}$$

we can rewrite Z_M

$$Z_M = (\cosh K)^{2M} \sum_s \left(\prod_{\langle i,j\rangle} (1 + s_i s_j \tanh K) \right) \tag{4.10}$$

At high temperature, the system is in the disordered phase. Because of the unique dependance of Z_M on $\tanh K$, let us observe that $\lim_{T\longrightarrow\infty} \tanh K = 0$. It is now natural to expand in the high temperature limit. In fact, expanding the product will produce 2^2M terms that can be thought of graphically. Returning to diagrams 3 and 4, we can now rigorously construct them, and they turn out to be high order terms. The process is to represent each of the three possible terms that can be produced: to every $s_i s_j$ products is associated a vertical or horizontal link whereas to each factor 1 produced, nothing is drawn. Repeating the process for all 2^2M terms becomes reminiscent of the random path created in the previous section. In effect, it corresponds

to the geometrical quantity we implicitly used with Peierls' argument, which outputs valid polygons (i.e. domain walls) given a length L:

$$\Phi = \sum_{P} (\tanh K)^{p+q} \tag{4.11}$$

where $p + q = L$ are the horizontal and vertical links drawn. Thus we have a pre-expansion partition function written as

$$Z_M = 2^N (\tanh K) M \ \Phi \tag{4.12}$$

Now remains to sum over all polygons. Taking into account $s \in \mathbb{Z}_2$, we can expect the conditions — listed in the previous section — for the polygons to be valid to be naturally satisfied: in effect, the sum is null unless is outputs an even L per polygon.

The first term of the sum is clearly 1 corresponding to no domain wall. The first non-trivial term is the smallest domain wall: a square of length $L = 4$, which can be placed anywhere on the lattice, hence occur N times.

4.2.2 Low-Temperature Expansion

Not immediately obvious, is that we can draw a correspondence between the high-temperature limit in the interaction picture to the low temperature limit in the spin picture. One can see that in the first case, the first term corresponds to no domain walls, indeed at $T = 0$ all spins are aligned. One spin flipped is bounded by the next term in the polygon expansion: the $L = 4$ square.

Even without much details, this could be anticipated from the asymptotic bound behaviour and odd parity of $\tanh K$. This can be put highlighted, following Kogut's [8] method, by defining a new variable K^* as

$$\tanh K = e^{-2K^*} \tag{4.13}$$

with which we can rewrite the expansion of the partition function and draw the following relation

$$\frac{Z(K^*)}{(e^{K^*})^N} = \frac{Z(K)}{(2\cosh^2 K)^N} \tag{4.14}$$

By construction, we associated K with the high temperature and K^* with the low, such that they are inversely proportional to each other. Rearranging both equations, we can make the duality finally obvious:

$$\boxed{\frac{Z(K^*)}{\sinh^{N/2}(2K^*)} = \frac{Z(K)}{\sinh^{N/2}(2K)}} \tag{4.15}$$

The final argument is to suppose there are strictly two phases, hence one critical temperature. If this is the case, then $K = K^*$ or

$$\sinh(2K_C) = 1 \tag{4.16}$$

must be satisfied, leading to the final result

$$\boxed{T_C^{2d} = 2.269J} \tag{4.17}$$

4.2.3 Exact Critical Temperature in 2d

Now the low-high temperature self-duality made clear, we have acquired the power to switch from one end of the temperature scale to the other, by simply switching from lattice to the interaction picture. This is particularly useful when one calculation is hard or even impossible in one limit, but feasible in the other limit.

4.3 Note on Correspondence to String Theory

Based purely on the dimensional arguments, one can draw parallels between flipped spins and their domain walls, with particles and their world line.

Lattice Dimension	**Domain Wall Dimension**	**String Theory Correspondent**
2d	1d	kink particle and its (0+1)d world line
3d	2d	string and its (1+1)d world line
4d	3d	2-brane and its (2+1)d world line

Table 3: Lattice Points and Domain Walls Correspondence to String Theory

The high-low temperature Kramers-Wannier (self)-duality of the 2-dimensional Ising Model is a particular example of the more general weak-strong S-duality. Here the gauge group (and the dual group) is $\mathbb{Z}_2$, as we have seen earlier.

Conclusion

We have derived two exact solutions to the Ising model. In doing so, we have discovered that at low dimensions the mean-field approximation leads to erroneous result. As we increase in dimensions, and reach 4d, mean-field

theory is not only accurate, but our only way to provide insight into our model.

Ironically, the only thing the Ising model has failed to do is fulfil its original intent and accurately reproduce ferromagnets. On the other hand, it allowed us to explore exactly and analytically critical behaviours are insensitive to microscopic scales, hence the usefulness of universality classes. It is indeed sufficient to study the Ising model as a representative, to characterise the entire class. We had already noted the appearance of the divergences and the importance of the correlation length, as hinting the need to introduce a Quantum Field language. Indeed, using the correlation length as a variable instead of the lattice length, in other words coarse graining in the context of a renormalisation group transformation, could be grounding for future work. Further investigation into the conformality at critical point, of the Ising model and all models whose hamiltonian is $\mathbb{Z}_2$ invariant at same dimensions, would provide elegant solutions.

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