IMPERIAL COLLEGE LONDON

Blackfolds

Francisco Hernando Quintanilla
under the supervision of Dr. Toby Wiseman

Department of Physics
Blackett Laboratory
Imperial College London
London SW7 2BZ
United Kingdom

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In memoriam
Professor Pedro M. Mejías Arias,
whose unconditional support made this possible.

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Chapter 1

Introduction

The aim of this dissertation is to provide a detailed and exhaustive review of what has been agreed to name *Blackfolds* and the techniques used to study their physical properties, later on we will give a definition and exaplain what is meant by that term. Needless to say, these solutions to the Einstein Equations (EE) are closely related to the well known Black Holes of General Relativity (GR), in fact, they are their higher dimensional generalizations.

We will have a quick look at how the concept of a Black Hole emerged and developed. The idea of a massive body whose gravitational attraction is so strong that not even light would be able to escape its gravitational field dates as far back as Laplace's work *Exposition du Systéme du Monde*. There, Laplace stated that idea without proof. The latter would come only a year later in an essay whose translation can be found in the Appendix of Hawking and Ellis' book [1]. Later, at the begining of the 20th century, German physicist Karl Schwarzschild published a paper (1916) where he put forward his solution of the vacuum EE in $3 + 1$ spacetime dimensions leading to the metric named after him. However, the term *Black Hole* is due to American physicist John Archibald Wheeler who coined it on 1967. Ever since, there has been a great deal of work on the area and, from the advent of String Theory and Quantum Gravity onwards, solutions in higher dimensional spacetimes, with more than 3 space dimensions, have been studied. This is what we shall review and in what follows, we will present and summarize some of the reasons for the study of Black Holes in higher dimensions.

1.1 Large Extra Dimensions and Black Holes

Gravity in higher dimensions was proposed as a possible solution to the *hierarchy problem*, namely, why the scale of gravity in 4-dimensional spacetime: $M_p \sim 10^{19}$ GeV is 16 orders of magnitude larger than the Electro-Weak (EW) scale: $M_{EW} \sim 1$ TeV. This could be accounted for by considering the Universe as a higher dimensional *bulk* spacetime into which our 4-dimensional "world" (technically referred to as *brane*) is embedded. In this way, the above Planck scale appears as an effective energy scale derived from the underlying fundamental one which is taken to be of the order of the EW scale. The so-called *Large Extra Dimensions* (LED) model allowed the size of these extra dimensions to be as large as 1mm., unfortunately, this is the minimum scale to which Newtonian Gravity has been measured to hold, nothing is known about how gravity works at smaller scales. Nevertheless, we know that the EW interaction is sensitive to extra dimensions. In case the Gauge Bosons were able to propagate through these extra dimensions, in the *bulk*, we would obtain significantly different phenomenological results, therefore it has been postulated that these and the other components of ordinary matter are constrained to move in the $3+1$-dimensional *brane* which constitutes our "world" and only gravity (read gravitons) are allowed to propagate through these extra dimensons as well.

The LED paradigm arises when we postulate that the Planck scale is of the order of the EW scale, 1 TeV., from this, one can see that extra dimensions are required for the theory of gravity to be correct as we show below. Let us now find the relation between the effective Planck scale M_p and the fundamental one which we will denote by M_*. For 4-dimensional spacetimes the Einstein-Hilbert action takes the form:

$$S_{EH}^{(4)} \sim M_p^2 \int dx^4 \sqrt{g^{(4)}} \mathcal{R}^{(4)} \tag{1.1}$$

In terms of dimensions we have: $M_p^2 \sim L^{-2}$, $dx^4 \sim L^4$ and $\mathcal{R}^{(4)} \sim L^{-2}$ which give a dimensionless action $S_{EH}^{(4)}$. An analogous equation holds in a $(4+n)$-dimensional spacetime:

$$S_{EH}^{(4+n)} \sim M_*^{2+n} \int dx^{4+n} \sqrt{g^{(4+n)}} \mathcal{R}^{(4+n)} \tag{1.2}$$

If the processes we are interested in, take place in the *brane*, the $\mathcal{R}^{(4+n)}$ can be taken to be independent of the extra dimensions' variables and thus we can integrate out these variables to get the following expression:

$$S_{EH}^{(4+n)} \sim M_*^{2+n}\ell^n \int dx^4 \sqrt{h^{(4)}}\hat{\mathcal{R}}^{(4)} \tag{1.3}$$

where ℓ^n is the volume of the extra dimensions (assuming that all of them have the same characteristic lenght ℓ) and $h^{(4)}$ and $\hat{\mathcal{R}}^{(4)}$ are, respectively, the induced metric and Ricci Scalar on the *brane*. Dimensional analysis again gives a dimensionless action. Comparing the above equations we arrive at the following relation:

$$M_p^2 \sim M_*^{2+n}\ell^n \tag{1.4}$$

Now, we can introduce some numbers to approximate the order of the characteristic lenght ℓ of the LED. Let us take a 10-dimensional spacetime ($n = 6$) with $M_* \sim M_{EW} \sim 1$ TeV. One easily gets: $\ell^3 \sim 10^{-20}\quad \text{eV}^{-3}$. Which yields:

$$\ell \sim 10^{-14}\text{m.} \sim \text{fm.} \tag{1.5}$$

This scale is far too small to be measured by any "classical" test of Gravity which can be measured down to 1mm. only. It is interesting to make the same computation for the $n = 1$ and $n = 2$ cases, which gives $\ell \sim 10^9$km. and $\ell \sim$ mm. respectively. One can easily confirm that, the higher the number of extra dimensions the smaller its characteristic scale becomes.

This notwithstanding, there are other means to test wether the LED model is right or wrong. Black Hole production by collisions at accelerators is one of these and we shall explain its basics briefly. Before that though, let us consider the following alternative: Could we not simply have a BH formed from, say, an electron? The answer is: No. When dealing with particles we need to take into account two scales which will determine wether or not a BH can be formed out of that particles/s, these are the *de Broglie* wavelength given by $\lambda_Q \sim 1/m_{BH}$ which determine the scale at which quantum effects begin to be important and the *event horizon* radius $R_{Hor} \sim m_{BH}$. For a BH to be created, we must have $R_{Hor} \ggg \lambda_Q$ for a given mass m_{BH}. This is not satisfied by the electron's mass, therefore we need far larger masses in order to make the *de Broglie* wavelength smaller while making the *event horizon* radius R_{Hor} large enough for the above inequality to hold. One could think of it as enlarging the R_{Hor} so that λ_Q fits inside.

Going back to collisions at accelerators, earth-based colliders such as the LHC will operate at energies of up to 14 Tev. At those energies, in LED scenarios, some of the products are likely to have masses large enough for the above condition to hold thus producing small blackholes. As an example let us consider the formation of a Schwarzschild BH. In principle, this is possible when particle collisions take place at centre-of-mass (CM) energies above the fundamental Planck scale $M_* \sim 1$TeV. and the impact parameter $b < R_{Sch}^{CM}$, where R_{Sch}^{CM} stands for the Schwarzschild radius associatd with the enrgy in the CM frame. Following Feng and Shapere [2], we can approximate the geometrical cross section for BH production from a collision between two particles i and j (gluon-gluon scattering could be a good instance) by the formula of the geometrical cross section of a black disc of radius $b(E_{CM})$. This is so because at super-Planckian energies, the process can be studied semiclassically, hence

$$\hat{\sigma}(ij \to BH)(E_{CM}^2) \approx \pi b^2(E_{CM}) \tag{1.6}$$

For this brief discussion, we have deliberately ommited the form factor $F(E_{CM})$ accounting for the E_{CM} distribution into the BH mass/es, corrections to the geometry of the black disc and other theoretical aspects of the process neglected in a first approximation, for a more detailed discussion we refer the reader to Cavagliá [3].

But we are interested in BHs living in more than 4 spacetime dimensions so, how can they be created out of 4-dimensonal processes? and, what are the conditions for them to exist? The first question has already been adressed, for higher dimensional gravity to manifest itself we must work at energies comparable to those of the fundamental plack scale $M_* \sim 1$TeV and this could be attained at the LHC. The second one is not as straightforward, when ordinary matter, trapped in the *brane*, undergoes gravitational collapse an *event horizon* will form extending into all the $(4 + n)$ spacetime dimensions. On the one hand, if the $R_{Hor} \ggg \ell$ the BH will be, effectively, a normal 4-dimensional one of the kind we are *used to.* On the other hand, if we have $R_{Hor} \lll \ell$ the BH will extend into the extra dimensions thus giving rise to a, strictly speaking, higher dimensional BH immersed into the *bulk* spacetime. This is one of the reasons why we are interested in the understanding of higher dimensional BH, they might serve as probes for the LED model. Let us illustrate what we have just explained in words and, for example, look at the radius R_{Hor} of a BH with $m_{BH} = 5$TeV and see wether or not it satisfies the condition $R_{Hor} \lll \ell$ for a higher dimensional BH to be formed. The radius goes like [14] :

$$R_{Hor} \sim \left(\frac{m_{BH}}{M_*}\right)^{\frac{1}{n+1}} \frac{1}{M_*} \tag{1.7}$$

For the previous case with $n = 6$, ℓ given by (1.5), $M_* \sim 1$TeV and the above value for m_{BH} we get $R_{Hor} \sim 10^{-19}$m. which safely satisfies $R_{Hor} \lll \ell$.

Summarizing, the inverse of the higher dimensional Planck scale M_* determines the size of the extra dimensions ℓ (provided all of them are of the same order). Also, the inverse of the mass of the particle or of the CM mass (energy), if we are dealing with a collisional process, determines the size of the event horizon R_{Hor}. By comparing these length scales we can infer wether a higher dimensional BH will be formed or not: if $R_{Hor} \lll \ell$ a higher dimensional BH will be formed, it can be fit into the LED so to speak, otherwise it will not.

Before proceeding towards the description of the signatures of higher dimensional BH production let us cite another important process which might lead to higher dimensional BH production. This is the collision of Ultra High Energy Cosmic Rays (UHECR) with other particles in the earth's atmosphere. The process is analogous to that described in the previous section but it takes place in the terrestrial atmosphere instead of at earth-based accelerators. Cosmogenic neutrinos with energies above 10^6GeV. are likely to create higher dimensional BHs in the terrestrial atmosphere thus serving as probes for the LED theories. How we can keep track of them is explained in the paragraph below.

We will close this section summarizing the signatures of higher dimensional BH production (either at earth-based accelerators or at the atmosphere by UHECR), for a more detailed account of this topic we refer the reder to Giddings and Thomas [20]. The most important signature is the emission of Hawking Radiation. Higher dimensional BHs will decay rapidly *via* emission of Hawking Radiation ([16], [17] and [18]). Therefore, we would like to know what the evidences of this process would be and how they can account for the extra dimensions we are looking for. To begin with, we must note that the process of radiation is studied semiclasically and this can only be done if certain constraints are met. First and foremost, the energy of the particles radiated must be such that $\omega_{part} \lll m_{BH}$ so that we can neglect the backreaction of the metric. This implies that the Hawking Temperature of the BH satisfies $T_H \lll m_{BH}$ which is equivalent to demanding that $m_{BH} \ggg M_*$. The temperature of the BH depends on the dimensionality n so that, the higher the n, the hotter the BH will be and therefore, the amount of radiation emitted will increase making it easier for us to detect it. For a more detailed discussion and some numbers one might see the Table 3 in [19]. Cosmogenic BHs are no better and will also decay by Hawking

Radiation producing a shower of SM particles and gravitons of which we can only observe the former. The cross section of this process is up to two or three times the cross section of other SM processes. This, taken together with the quasi-horizontal showers they produce at every depth in the atmosphere, is a signature of BH creation. More details might be found at [2] and [3].

The Hawking Radiation measured in an experiment depends on three factors: *energy*, *spin* and *dimensionality*. Thus, making use of the spectrum measured in an experiment and of the *grey body factors*, which measure the deviation of the spectrum from that of an ideal balck body, one can compute the number of LEDs. We present below the most striking signatures of this process, for more see [20]:

- Large multiplicity events of final state partons. This is correlated with the BH's mass in a way determined by the Hawking Radiation process. In particular, multiplicity is higher and average energy per parton's final state lower for higher mass events.
- 5:1 ratio of hadronic to leptonic activity.

Here, we have referred to Giddings and Thomas [20] as a source of information regarding the collision process and BH production. However, Volshin [21] has argued that the cross sections presented by Giddings and Thomas should be modified by an exponential suppression factor. One then wonders wether this supression factor would render the detection of BH production impossible or not. Fortunately, Rizzo [22] has shown that the cross sections for BH production are still large enough to be detected after inclusion of the suppression factor.

1.2 AdS/CFT Correspondence and Hawking-Bekenstein Entropy

Let us begin by briefly explaining what is meant by the term Conformal Field Theory (CFT). A field theory is said to be a CFT when it is invariant under conformal transformations (angle preserving transfomrations) of the coordinates. As a consequance of this invariance, they lack of a chracteristic length scale, a fact which will be exploited soon.

We continue by quoting Maldacena [23] who explains in a few words what the AdS/CFT Correspondence consists of:

"The AdS/CFT relation postulates that all the physics in an asymptotically anti-de-Sitter spacetime can be described by a local quantum field theory that lives on the boundary. The boundary is given by $\mathbb{R} \times S^{d-1}$. The isometries of AdS act on the boundary. They send points on the boundary to points on the boundary. This action is simply the action of the conformal group in d dimensions, $so(2,d)$. Thus, the quantum field theory is a conformal field theory."

Originally, the idea that the physics of a volume of spacetime were encoded in the boundary limitting it is due to Nobel laureate Gerard't Hooft and was coined *holographic principle* [67], the AdS/CFT Correspondence is inspired by this idea. The way to link what happens inside a volume in a theory of gravity to the volume's boundary is as follows: the amount of informationn that can be stored inside a BH is proportional to the volume enclosed by its event horizon. The amount of information goes like $\sim \log S_{BH}$ and the entropy is proportional to the surface area of the event horizon (see the computation perfomed below about the degrees of freedom). This is how we can relate what happens inside the volume enclosed by the event horizon to its boundary. Here, we content ourselves with that brief explanation due to Maldacena since a detailed description of the AdS/CFT Correspondence lies out of the scope of this work, we refer the reader to the literature on the subject. However, we would like to know how the need for higher dimensional BH arises in the context of this theory. In order to show this, we first ask how it is possible that a $(d+1)$-dimensional theory is equivalent to a d-dimensional one. Apparently, the former has one more degree of freedom than the latter and so, in the porcess, we would lost some physical information which would render the correspondence a failure.

To see that this is not the case we could perform a computation of the degrees of freedom in the high-energy limit by resorting to the microcanonical ensemble. First, we introduce an effective temperature T and its corresponding *inverse temperature* defined by $\beta \equiv \frac{1}{k_B T}$. According to the theory of statistical mechanics, for a theory with massless particles (or with no scale) the entropy goes like $S \sim \frac{V_{d-1}}{\beta^{d-1}}$. For a CFT on the boundary $\mathbb{R} \times S^{d-1}$ and at temperature much larger than the radius of S^{d-1} we can take the volume V_{d-1} to be of order one (as we said at the begining of this section, there is no characteristic length scale so we can fix it at will), the radius of S^{d-1} must be $r \sim 1$, and hence, the entropy would go like:

$$S_{(d-1)} \propto c \frac{1}{\beta^{d-1}} \tag{1.8}$$

where c is a dimensionless constant which takes into account the number of effective fields in the theory. Let us now turn to the *bulk* point of view. In this case we also have massless particles: the gravitons. There are others but let us focus on these since they impose a lower bound to the entropy and for our purposes this is more than enough. The gravitons have larger entropy than the entropy of the above-mentioned region where $r \sim 1$ and therefore:

$$S_{gas\ of\ gravitons} > \frac{1}{\beta^d} \tag{1.9}$$

We can see at once that, for small enough β, we have:

$$S_{gas\ of\ gravitons} > S_{(d-1)} \tag{1.10}$$

In contradiction with what we would expect, that is, to have equal degrees of freedom in both cases and hence equal entropy so that a correspondence between the two theories could be stablished.

But we have missed something important: *gravity.* Our theory in the *bulk* must also include gravity, and from gravity arise BHs which impose constraints in the entropy. It is this feature that will unveil the apparent paradox above and illustrate why higher dimensional BHs are of interest for the AdS/CFT Correspondence. We refer the reader to Maldacena's paper [23] to see the explicit expression of the metric for higher dimensional BH in AdS spacetime, here we will simply give the results which lead to the agreement between degrees of freedom of both theories. The gravitons have a mass of the order $m \sim 1/\beta^{d+1}$. For small β the radius of the Schwarzschild BH is approximately: $r_s^d \sim gm \sim g/\beta^{d+1}$, where g is the Newton constant in units of the AdS radius and gives the characteristic scale of the system. This shows that, for large enough temperatures, the Schwarzschild radius is bigger than the characteristic scale of the system $1/\beta > 1/g$. Therefore, the computation which led to (1.8) is not valid at those high temperatures and we must resort to BH entropy in order to obtain a meaningful and correct result.

As we know, the entropy of a BH grows with the area of the horizon, so for d spatial dimensions the area goes like $\sim r_s^{d-1}$ and hence we have $S_{BH} \sim \frac{r_s^{d-1}}{g}$. Since the Hawking-Bekenstein entropy of big BH is given by $\beta \sim 1/r_s$ we arrive at the expression for the BH entropy in terms of β:

$$S_{BH} \propto \frac{1}{g}\frac{1}{\beta^{d-1}} \tag{1.11}$$

which, upon identification of g and c, is exactly of the same form as the expression found in (1.8). So, AdS/CFT matches the entropy of a higher dimensional BH with the ordinary thermal entropy of a field theory. Thus there is no contradiction between the number of degrees of freedom of the two theories.

The point of using higher dimensional BHs is that, when working with field theories at finite temperature we cannot perform any computations perturbatively for being strongly coupled theories and therefore it is very hard to obtain results. But we can resort to the AdS/CFT Correspondence and use the results obtained for higher dimensional BHs and gravity in order to gain knowledge of the corresponding field theory. This also works the other way round, we can, for example, proceed as we did above and compute the entropy in the field theory and, in the light os the AdS/CFT Correspondence, we can infer that of the corresponding higher dimensional BH.

To close this section we give an outline of a computation which also rised the interest in higher dimensional BHs, the Hawking -Bekenstein (simply known also by Hawking Entropy). The statistical counting of a BH's entropy was first carried out within the framework of String Theory [10]. In that paper, the authors chose an extremal 5-dimensional BH for being the simplest case giving non-trivial results. First, an expression for the entropy of a BH with electric charge Q_F and axion charge Q_H is derived from the low-energy limit. Then a counting of microscopic BPS states is carried out to derive another expression for the entropy of a BH of the same chracteristics. Finally, it is shown that the latter expression in the low-energy limit agrees with the former. This can serve as a test bench for the predictions of the microscopic String Theory of BHs and provides us with one more reason to study higher dimensional BHs.

1.3 Foundations of General Relativity

Finally, apart from all the experimental as well as theoretical reasons aforementioned, the study of higher dimensional BH is interesting in itself for the insights it can provide us with of the foundations of GR. The question of wether or not the theorems found to hold in 4 spacetime dimensions [1] could be generalized to higher dimensional spacetimes has already been studied, a review of the various theorems togerther with a discussion and an outline of the proofs can be found at [11]. In addition, and from a purely mathematical point of view, BHs constitute one of the most important Lorentzian Ricci-flat manifolds.

Chapter 2

Black Hole Motion

2.1 Posing the problem

In this chapter we are going to give an outline of the effective theory for BH motion in a background spacetime as well as derive its intrinsic and extrinsic equations of motion. This will be later generalized to the case of higher dimensional BHs but it is easier and more illustrative to look first at the usual 4-dimensional BH or 0-brane. We will begin with a few words about notation. In the following, and unless otherwise stated, when we consider a Blackfold of p spatial dimensions (or a p-brane) with world volume $\mathcal{W}_{p+1}$ embedded in a D-dimensional background spacetime we can define:

$$n = D - p - 3 \tag{2.1}$$

where $n+2$ is the codimension of the worldvolume of the p-brane. Spacetime indices $\mu, \nu ... = 0, ..., (D-1)$ and the covariant derivative compatible with the background metric $g_{\mu\nu}(x)$ is ∇_μ with the connection given by the usual $\Gamma^{\alpha}_{\mu\nu}$. Similarly, worldvolume indices $a, b... = 0, ..., p$ and the covariant derivative compatible with the worldvolume induced metric $\gamma_{ab}(\sigma)$ is D_a with connection given by the symbol $\left\{ {a \atop bc} \right\}$.

Let us consider the motion of a BH of mass m in a given D-dimensional background spacetime which is asymptotically flat. There are two scales to be considered in this problem:

- The scale associatd with the BH's mass m, which can also be expressed in terms of the horizon radius r_o.
- The scale associated with the background spacetime in which the BH moves and which will be characterized by the radius of curvature $\mathcal{R}$. Provided there are no singularities, the components of the Riemann

curvature tensor of the background $R_{\mu\nu\sigma\gamma}$ will be, up to a multiplicative factor of order unity, equal to $1/\mathcal{R}^2$. In the following, we will asssume a background spacetime with no matter so that the associated metric $g_{\mu\nu}(x)$ is a solution of the vacuum EE.

These two scales will play a crucial role in the method of *matched asymptotic expansions* which we will be presenting here. We only require them to satisfy: $m \lll \mathcal{R}$. For a more thorough discussion with explicit solutions and derivations see [12] and [13]. We will rely heavily on the approach of E. Poisson here. Nonetheless, there is no need for us to specify what the background metric is dependent on, wether it is a bigger BH around which the small one is orbitating or a Galaxy is completely immaterial for our purposes, it will suffice with knowing the functions $g_{\mu\nu}(x)$ whose derivation is given in [12].

The idea of the method is to match the expression for both metrics of the internal and external zones in the intermediate zone or buffer zone which lies inbetween them. Apart from finding the expression for the metric in all regions of the spacetime we also find the equations of motion of the BH in the background by imposing the matching. Let us begin by making clear what we mean by giving a proper definition of the different zones which play a role in the problem.

Let $r \geq 0$ be an adequate measure of distance from the BH, in our case, the radial coordinate of the Minkowski metric when expressed in spherical coordinates. Also, let r_i be a constant such that $m \lll r_i \lll \mathcal{R}$. The *internal zone* is defined by $r < r_i$. In this region, the metric is mainly fixed by the BH itself and we can account for the contribution of the background by simply adding terms in powers of $1/\mathcal{R}$ and $1/\mathcal{R}^2$:

$$\mathfrak{g}(internal\ zone) = g(blackhole) + H_1/\mathcal{R} + H_2/\mathcal{R}^2 + ..., \tag{2.2}$$

where $g(blackhole)$ is the metric of the BH in isolation, say the Schwarzschild solution (Schwarzschild-Tangherlini if we are dealing with its higher dimensional generalization), and the constants H_1 and H_2 are determined by the external universe and are functions of m and the spatial coordinate r found by solving EE. They must be regular at the horizon $r_o = 2m$ and must agree with the metric of the background universe for $r \ggg m$. Note that the angular dependence is included in $g(blackhole)$ and, possibly, as cofactors of the terms of the expansion H_1, H_2 and so on. See [12].

Now consider a constant r_e such that $\mathcal{R} \ggg r_e \ggg m \sim r_o$. The *external zone* is defined by $r > r_e$. Here, the metric is, in turn, mainly determined by the background and the effects of the moving BH are taken into account in the extra terms of order m and m^2:

$$\mathfrak{g}(external\ zone) = g(background) + mh_1 + m^2h_2 + ..., \qquad (2.3)$$

where $g(background)$ is the metric of the background without taking into account the moving BH and the other terms account for the BH motion and distortion of the background. In this region the influence of the BH is the same as that of a point particle of the same mass and so, the metric, will depend on the trajectory γ of the BH moving in the background. This dependence on γ is what will allow us to find the equations of motion for the BH when doing the matching. However, note that this description of the BH as a point particle is only possible in the external zone, in the internal zone we must regard it as an extended object and hence, the notion of trajectory doesn't make sense any longer.

When $m \lll \mathcal{R}$, the *buffer zone* is defined by $m \lll r_e < r < r_i \lll \mathcal{R}$, in this zone both m/r and $r/\mathcal{R}$ are small. It will be the aim of the next section to derive the general form of the metric in this zone starting from the ones we already know. However, we will not be concerning ourselves with a detailed derivation of the explicit solution, this can be found in [12].

2.2 Matching in the Buffer Zone

We will now show that a matching of the internal and external metrics is possbile in the *buffer zone*. As pointed out above, this will neccesarily determine the motion of the BH in the background. Let us begin by looking at the formal structure of the metric in the internal zone when we take r sufficiently large as to achieve $r \ggg m$. The angular dependece is omitted throughout in order to make the expressions neater, since they have no length dimension, angular variables will tipically appear as cofactors of the various terms of the expanssions, we refer to [12] for further details. The metric of the unperturbed BH, $g(blackhole)$ can be expanded in power series of m/r where the first contribution comes from Minkowski in order to have the appopriate asymptotic behaviour when $r \to \infty$: $g(blackhole) = \eta + m/r + m^2/r^2 + ...$ We whish to find out what the structure of the functions H_i is. For H_1, we must bear in mind that the terms must be functions of r and m such that they have dimensions of length in order to give a dimensionless contribution when multiplied by the factor $1/\mathcal{R}$. Besides, we know that the following inequalities hold in the buffer zone, in this case: $r \ggg m$ and $\mathcal{R} \ggg r$. Taking all of this into account, we can construct a power series whose terms ($\lll 1$ always) will have a power of m in the numerator and an appropriate power of r in the numerator or denominator. The first terms, up to a constant factor of order unity, are given by: $H_1/\mathcal{R} = r/\mathcal{R} + m/\mathcal{R} + m^2/(r\mathcal{R}) + ...$, the third term is a product of $m/\mathcal{R}$ and m/r which are both less than one and hence is also less than one. For H_2, analogous considerations apply, but now one of the factors in each term is $1/\mathcal{R}^2$ so the combinations of the powers of r and m will differ from the previous case: $H_2/\mathcal{R}^2 = r^2/\mathcal{R}^2 + mr/\mathcal{R}^2 + m^2/\mathcal{R}^2 + ...$ So, collecting all the terms we have a metric in the *buffer zone* of the form:

$$\begin{aligned}\mathfrak{g}(buffer\ zone) \qquad &= \eta + m/r + m^2/r^2 + \\ &r/\mathcal{R} + m/\mathcal{R} + m^2/(r\mathcal{R}) + \\ &r^2/\mathcal{R}^2 + mr/\mathcal{R}^2 + m^2/\mathcal{R}^2 + ... \end{aligned} \tag{2.4}$$

Now, let us turn to study the structure of the metric in the *buffer zone* from the point of view of the *external zone*. Now, however, h_i are functions of r and $\mathcal{R}$. All the considerations mentioned above about dimensionality and the formal structure of the terms apply here equally. Note that, in this case, we must expand the metric of the background in power series of $r/\mathcal{R}$ so: $g(background) = \eta + r/\mathcal{R} + r^2/\mathcal{R}^2 + ...$ In a fashion similar to that in the previous paragraph we find for h_1: $mh_1 = m/r + m/\mathcal{R} + mr/\mathcal{R}^2 + ...$ Also, for h_2 we have: $m^2h_2 = m^2/r^2 + m^2/\mathcal{R}^2 + m^2/r\mathcal{R} + ...$ Again, collecting terms, we arrive at:

$$\begin{aligned}\mathfrak{g}(buffer\ zone) \qquad &= \eta + r/\mathcal{R} + r^2/\mathcal{R}^2 + \\ &m/r + m/\mathcal{R} + mr/\mathcal{R}^2 + \\ &m^2/r^2 + m^2/\mathcal{R}^2 + m^2/(r\mathcal{R}) + ... \end{aligned} \tag{2.5}$$

We can see at once that, though differently arranged, the terms are exactly the same as in the expression obtained from the point of view of the *internal zone*. We could have brought the analysis of the expansion further but we would have encountered the same structure for both, hence a matching must be possible between both metrics in the *buffer zone*. Practically, before carrying out this matching we must transform from internal coordinates to external coordinates or *vice versa*. The calculation is extraordinarily tedious and lies out of the scope of this dissertation, the interested reader can find it in [12]. The key result of that calculation, after the desired coordinate transformation to perform the matching, is that we obtain an expression for the frame components of the acceleration vector a_b of the trajectory followed by the BH in the background and for the rotation tensor ω_{ab} which accunts for the rotation of the frame basis $\{e_a\}$ when it is Fermi-Walker transported along that trajectory.

The expression for the acceleration found by the above procedure, as given in the section 19.5 of [12], allows us to compute the trajectory γ of the BH provided we give it the adequate initial conditions. Now that we know the motion of the BH we can proceed further and chracterize it in terms of physical parameters.

2.3 BH's Equations of Motion

The first step in order to obtain the equations of motion is to adequately characterize the BH. So we need to find its stress-energy tensor. Note that the BH is being considered as a pointlike particle and therefore defining an energy density for a point particle does not make much sense, a pointlike particle is charactrized by its proper mass, therefore, to leading order:

$$T^{\mu\nu} \equiv T^{\mu\nu}_{BH} = m u^\mu u^\nu \delta^{(D)}(\vec{r} - \gamma) \tag{2.6}$$

where $m = m(\tau)$ is the mass of the BH, $u^\mu = \partial_\tau x^\mu_\gamma(\tau)$ is the velocity of the BH following the trajectory γ and τ is the proper time. Higher order terms would account for the finite size of the BH. It is apparent that it has the same structure as that of a pressureless perfect fluid upon identifying $\varepsilon = m\ \delta^{(D)}(\vec{r} - \gamma)$. We also have $g_{\mu\nu} u^\mu u^\nu = -1$. Note that this is an effective stress-energy tensor which accounts for the effects of the BH in regions far enough from it, if we were to study it in its vicinity we would have to modify the stress-energy tensor since it could no longer be regarded as a point particle but as an extended object.

The next step in order to find the stress energy tensor is to find the expression for the, in general, function $m(\tau)$. For our study we will consider a Schwarzschild-Tangherlini BH in $D = N + 1$-dimensional spacetime. The generalization of the Schwarzschild metric to higher dimensions is:

$$ds^2 = -f^2 dt^2 + g^2 dr^2 + r^2 d\Omega^2_{D-2} \tag{2.7}$$

where $d\Omega^2_{D-2}$ is the solid angle differential element of a $(D-2)$-dimensional sphere and the functions f and g are given by:

$$f = g^{-1} = \left(1 - \frac{C_D}{r^{D-3}}\right)^{\frac{1}{2}} \tag{2.8}$$

The next step is finding an expression for the mass of the BH in D spacetime dimensions. We will follow the argument given in [14] which is more straightforward but we present an alternative derivation of it following the ADM method in the corresponding appendix. We will work to leading order throughout. Let us begin by splitting the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.9}$$

We further require the perturbation to satisfy the *harmonic gauge condition*,

$$(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h^{\alpha}_{\alpha})_{,\nu} = 0 \tag{2.10}$$

The EE to leading order yield:

$$\Delta h_{\mu\nu} = -16\pi G \left(T_{\mu\nu} - \frac{1}{N-1}\eta_{\mu\nu} T^{\alpha}_{\alpha} \right) = -16\pi G \quad \bar{T}_{\mu\nu} \tag{2.11}$$

where the trace is dominated by the energy density $T^{\alpha}{}_{\alpha} \sim -T_{00}$ and $\Delta \equiv \nabla^2 = \nabla^{\mu}\nabla$ is the Laplacian operator in N spatial dimensions. Note also that, the fact that the BH's motion is not relativistic implies that the stress-energy tensor components may be ordered $|T_{00}| \ggg |T_{i0}| \ggg |T_{ij}|$ consistently with the energy dominant condition above. Therefore, to leading order, we can take $T_{i0} = T_{ij} \approx 0$ in accordance with (2.6) which gives the stress-energy tensor in the point particle approximation. Higher order terms would account for the finite size of the BH and would therefore include non vanishing off-diagonal terms such as the momentum density and angular moentum. We can use the Green's function for the N-dimensional Laplacian to solve for $h_{\mu\nu}$. Whereupon we expand it to leading order in the region where $r = |\bar{x}| \ggg |\bar{y}|$. The integrals range over a hypervolume of constant time x^0:

$$\begin{aligned} h_{\mu\nu}(x^i) &= \frac{16\pi G}{(N-2) \quad \mathcal{A}_{(N-1)}} \int d^N y \frac{\bar{T}_{\mu\nu}(y^i)}{|\bar{x} - \bar{y}|^{N-2}} \\ &\approx \frac{16\pi G}{(N-2) \quad \mathcal{A}_{(N-1)}} \frac{1}{r^{N-2}} \int d^N y \quad \bar{T}_{\mu\nu}(y^i) \end{aligned} \tag{2.12}$$

where $\mathcal{A}_{(N-1)}$ is the surface area of an $(N-1)$-dimensional sphere. Taking into account the definition of the mass of the BH (Note that plugging in (2.6) into (2.13) we get the expected result that $M = m$ since the m factors out and the integrated delta gives unity):

$$M = \int d^N y \quad T_{00}(y^i) \tag{2.13}$$

and

$$\bar{T}_{00} \approx T_{00} \left(\frac{N-2}{N-1} \right) \tag{2.14}$$

We get for h_{00}:

$$h_{00} \approx \frac{16\pi G}{(N-1) \quad \mathcal{A}_{(N-1)}} \frac{M}{r^{N-2}} \tag{2.15}$$

From (2.8) and (2.9) we can see that

$$-1 + \frac{C_{N+1}}{r^{N-2}} = \eta_{00} + h_{00} = -1 + h_{00} \Longrightarrow h_{00} = \frac{C_{N+1}}{r^{N-2}} \tag{2.16}$$

Comparison with (2.15) finally gives the expression for the mass of the BH:

$$M = \frac{C_{N+1}(N-1)\mathcal{A}_{(N-1)}}{16\pi G} = \frac{C_D(D-2)\mathcal{A}_{(D-2)}}{16\pi G} \tag{2.17}$$

This result is in agreement with the one obtained by using the ADM method. See the derivation of (A.15) for details of this calculation.

We are now ready to compute the equations of motion for the BH. For the leading order we can use conservation principles instead of solving the full set of EE which can be very awkward. In the *external zone*, the stress-energy tensor must be conserved along the trajectory of the moving BH and therefore:

$$-u_\mu u^\sigma \nabla_\sigma T^{\mu\nu} \equiv \bar{\nabla}_\mu T^{\mu\nu} = 0 \tag{2.18}$$

This equation, however, has components in both, parallel and orthogonal, directions to the BH's velocity. Let us first project it onto the velicity vector. Note that, since $m(\tau)$ is a scalar function, we have $\bar{\nabla}_\mu m(\tau) = \partial_\tau m(\tau)$, so:

$$\begin{aligned} u_\nu \bar{\nabla}_\mu T^{\mu\nu} = 0 \Longrightarrow \\ \partial_\tau m(\tau) = 0 \end{aligned} \tag{2.19}$$

(See A.16). For this case, and in the light of (2.6), the condition is trivially satisfied. In general, it states the conservation of the BH's mass along the trajectory.

Now let us look at the orthogonal component of the equations of motion. First we need to define the projector onto the orthogonal subspace. If $-u_\mu u^\sigma$ projects onto the tangent space to the world volume, the orthogonal operator will be given by:

$$\perp_{\mu\sigma} \equiv g_{\mu\sigma} + u_\mu u^\sigma \tag{2.20}$$

So the orthogonal component of the equations of motion is:

$$\begin{aligned} (g_{\sigma\nu} + u_\sigma u_\nu)\bar{\nabla}_\mu T^{\mu\nu} = 0 \Longrightarrow \\ m(\tau) a_\sigma = 0 \end{aligned} \tag{2.21}$$

(See A.17) These, (2.19) and (2.21), are the equations of motion of the BH. The first states the conservation of the mass along its trajectory and the second is the generalization of Newton's Law. We will see that the same approach is used when studying Blackfolds.

Chapter 3

Blackfold Dymanics

3.1 Effective Worldvolume Theory

We begin by giving a scheme of how the 4-dimensional solutiosn, when they exist, relate to the higher dimensional ones:

- Schwarzschild $\Longleftrightarrow$ Schwarzschild-Tangherlini
- Kerr $\Longleftrightarrow$ Myers-Perry
- $\nexists \Longleftrightarrow$ 5-dimensional Black Ring

However, the Blackfold approach will support the possibility that higher dimensional (D> 5) BR solutions exist.

There are two characteristic scales for high dimensional, neutral, vacuum BHs in asymptotically flat spacetimes, these are associated with the mass of the BH and its angular momentum:

$$\ell_M \sim (GM)^{\frac{1}{D-3}}; \qquad \ell_J \sim \frac{J}{M} \tag{3.1}$$

With such scales we can split GR degrees of freedom into two separate classes:

$$g_{\mu\nu} = \{g_{\mu\nu}^{(short)}, g_{\mu\nu}^{(long)}\} \tag{3.2}$$

Higher dimensional BHs are known to exist in regimes where $\ell_J \ggg \ell_M$, these are called *ultra spinning*, and this separation of scales makes it possible to effectively describe the dynamics of long wavelengths of the Blackfold. When the limit $\ell_M/\ell_J \to 0$ exists, the Blackfold results in a flat black p-brane.

The geometry of the flat black p-brane is simply given by adding p cartesian coordinates to the $(D-p)$-dimensional Schwarzschild solution, thus:

$$ds^2 = -\left(1-\frac{r_o^n}{r^n}\right)dt^2 + \sum_{i=1}^{p}(dz^i)^2 + \left(1-\frac{r_o^n}{r^n}\right)^{-1}dr^2 + r^2 d\Omega_{n+1}^2 \tag{3.3}$$

where r_o is called the *thickness* of the p-brane in the $n+2$ spatial tranversal directions. Let us introduce coordinates $\sigma^\mu = \{t, z^i\}$ which span the worldvolume with Minkowskian metric. We can boost along the worldvolume to get a more general expression of the metric. Let the velocity be $u_a = (1, u_i)$, where u_i are the spatial components of the worldvolume velocity. We also require that $u^a u^b \eta_{ab} = -1$, so:

$$ds^2 = \left(\eta_{ab} + \frac{r_o^n}{r^n}u_a u_b\right)d\sigma^a d\sigma^b + \left(1-\frac{r_o^n}{r^n}\right)^{-1}dr^2 + r^2 d\Omega_{n+1}^2 \tag{3.4}$$

As the collective coordinates of the Blackfold we choose the set:

$$\phi(\sigma^a) = \{x_\perp(\sigma^a), r_o(\sigma^a), u^i(\sigma^a)\} \tag{3.5}$$

where $x_\perp(\sigma^a)$ are the $(D-p-1)$ spatial coordinates which denote the position of the p-brane in the directions transverse to its worldvolume spanned by the set $\{r_o(\sigma^a), u^i(\sigma^a)\}$ which is equivalent to the set σ^a defined before.

In the effective theory we let $\partial x_\perp$, $ln r_o$ and u^i vary very slowly along the worldvolume in scales much larger than the smallest intrinsic or extrinsic curvature radius of the worldvolume denoted by R. Let us look briefly at how the method of *matched asymptotic expansions* may arise in this new context. We can define also an *internal zone*, where the short wavelength modes live, for $r \lll R$ and an *external zone*, where the long wavelength modes live, for $r \ggg r_o$. When $R \ggg r_o$ we can also define a *buffering zone* for $r_o \lll r \lll R$. It is in this latter zone where we must match both metrics $\{g_{\mu\nu}^{(short)}, g_{\mu\nu}^{(long)}\}$. For later reference the metric in the *internal zone* takes the form:

$$\begin{aligned} ds^2 = &\left(\gamma_{ab}(x_\perp(\sigma)) + \frac{r_o^n(\sigma)}{r^n}u_a(\sigma)u_b(\sigma)\right)d\sigma^a d\sigma^b + \\ &\left(1-\frac{r_o^n(\sigma)}{r^n}\right)^{-1}dr^2 + r^2 d\Omega_{n+1}^2 + \ldots \end{aligned} \tag{3.6}$$

where the dots account for the missing terms of the metric so that it is a solution of the EE. The new metric $\gamma_{ab}(x_\perp(\sigma))$ is nothing but the pull-back of the metric $g_{\mu\nu}^{(long)}$ onto the worldvolume (for details on the differential geometry of embedded manifolds and some formulae we will be using here without proof one can look at [26], [27] and references therein). Its dependence on the background transverse coordinates $x_{\perp}$ endowes it with an extrinsic curvature. From now on we will be following what is known as the *perfect fluid* and *generalized geodesic* approximation. For more rigorous methods valid for higher orders we refer the reader to [12], [29] and [33].

Now we proceed to find the stress-energy tensor of the black p-brane. The EE have been solved and a metric found (3.6) for the *internal zone*. However, the effects of the Blackfold in the *external zone* are encoded in a stress-energy tensor that only depends on the collective coordinates since, being far away enough from the BH, we can characterize it as a fluid described by the collective variables introduced before.

We have used the metric of the flat p-brane (3.4) and the prescription given in [30] for computing the stress-energy tensor within the ADM formalism ([31]). We give the result and refer the reader to the appendix for the details of the computation (See derivation of (B.11)). The procedure is analogous to that carried out in the previous chapter for the derivation of the BH mass.

$$T_{ab} = \frac{r_o^{D-p-3}\mathcal{A}_{(D-p-2)}}{16\pi G}\left(u_a u_b (D-p-3) - \eta_{ab}\right) \tag{3.7}$$

where, we recall that $\mathcal{A}_{(D-p-2)}$ is the surface area of a $(D-p-2)$-dimensional unit sphere and, according to the previous notation, $r_o^{D-p-3} \equiv C_{D-p}$. Note that, had we computed the stress-energy tensor in the whole spacetime we would have needed to include a delta function as we did before. Reexpressing it in terms of the parameter n defined in (2.1) we get:

$$T_{ab} = \frac{r_o^n \mathcal{A}_{(n+1)}}{16\pi G}\left(u_a u_b n - \eta_{ab}\right) \tag{3.8}$$

Allowing for variations of the collective coordinates we get, analogously to the metric, to first order:

$$T_{ab}(\sigma) = \frac{r_o^n(\sigma)\mathcal{A}_{(n+1)}}{16\pi G}\left(u_a(\sigma) u_b(\sigma) n - \gamma_{ab}(\sigma)\right) + ... \tag{3.9}$$

where the dots stand for terms with gradients of the collective coordinates which are taken to be small and can be safely neglected for our purposes.

Having thus derived the stress-energy tensor, we can make an analogy with fluid dynamics which will elucidate many aspects of Blackfold Thermodynamics and will provide us with a good test bench for our results and computations. For more detailed accounts of fluid dynamics and gravity one may want to have a look at [32] and [33].

Let us recall the generic form of the stress-energy tensor of an isotropic perfect fluid:

$$T_{ab} = (\varepsilon + P)u_a u_b + \gamma_{ab} P \tag{3.10}$$

where ε is the energy density, P is the pressure and u_a is the velocity satisfying $u^a u^b \gamma_{ab} = -1$. Comparing (3.10) with (3.9) we can see at once that the stress-energy tensor of the p-brane has the form of a perfect isotropic fluid when we take the following definitions:

$$\varepsilon \equiv \frac{r_o^n \mathcal{A}_{(n+1)}}{16\pi G}(n+1); \qquad P \equiv -\frac{1}{(n+1)}\varepsilon \tag{3.11}$$

The Hawking-Bekenstein entropy density can be computed in the rest frame (where $S = s$ since in the rest frame we have $\gamma(v^2) = 1$) and gives:

$$s_{HB} = \frac{\mathcal{A}_{Hor}}{4G} = \frac{\mathcal{A}_{(n+1)}}{4G} r_o^{n+1} \tag{3.12}$$

Now recall the second law of thermodynamics:

$$\mathcal{T} ds = d\varepsilon + P dV \tag{3.13}$$

When applied to BHs it reads:

$$\mathcal{T} ds = d\varepsilon - \Omega\, dJ - \phi\, dQ \tag{3.14}$$

However, here, we are concerned with non-spinning neutral Blackfolds so the last two terms drop and therefore we get:

$$\mathcal{T} ds = d\varepsilon \tag{3.15}$$

where $\mathcal{T}$ is the Hawking temperature of the BH. By plugging the differentials of ε and s_{BH} with respect to the collective coordinate r_o we obtain the expression for the Hawking temperature:

$$\mathcal{T} = \frac{n}{4\pi r_o} \tag{3.16}$$

Note also the general relation:

$$-P = \frac{1}{n}\mathcal{T} s \tag{3.17}$$

By definition, we have $\mathcal{T} \equiv \frac{\kappa}{2\pi}$. Thus, we also obtain the value of the surface gravity of the Blackfold at the horizon:

$$\kappa = \frac{n}{2r_o} \tag{3.18}$$

Very nice. We see that the higher the dimensionality the stronger the surface gravity is.

3.2 Embeddings and Geometry of the Worldvolume

We shall digress a little now and spend some time studying a few notions on embeddings which will be useful to dearive and deal with the equations of motion of the Blackfold. Given the induced metric on the $\mathcal{W}_{(p+1)}$, γ_{ab} to get the first fundamental form of the submanifold we can push-forward the contravariant induced metric γ^{ab} to get the contravariant fisrt fundamental form of the submanifold $\mathcal{W}_{(p+1)}$:

$$h^{\mu\nu} = \gamma^{ab}\partial_a x^\mu \partial_b x^\nu \tag{3.19}$$

We can raise indices μ, ν with the metric $g_{\mu\nu}$ and worldvolume indices a, b with the metric γ_{ab}. Now, we decompose the metric into projectors $h_{\mu\nu}$ for worldvolume subspace and $\perp_{\mu\nu}$ for orthogonal subspaces:

$$g_{\mu\nu} = h_{\mu\nu} + \perp_{\mu\nu} \tag{3.20}$$

Let us now see that $h^\mu{}_\nu$ is in fact a projector. By making use of the following identity:

$$h^\mu{}_\nu \partial_a x^\nu = \partial_a x^\mu \tag{3.21}$$

(See (B.12)) we see that $h^\mu{}_\nu$ satisfies the identity which characterizes a projector:

$$h^\mu{}_\nu h^\nu{}_\rho = h^\mu{}_\rho \tag{3.22}$$

which can be easily derived using techniques similar to those used for deriving (3.21). We also need to properly define the covariant differentiation of tensors living in the worldvolume, for this, we project the operator for the covariant differentiation on the tangential space to the worldvolume by using the projector above:

$$\bar{\nabla}_\nu \equiv h_\nu{}^\mu \nabla_\mu \tag{3.23}$$

The covariant derivative of the stress-energy tensor expressed as function of worldvolume coordinates T_{ab} is related to that expressed in bulk coordinates $T_{\mu\nu}$ by the following expression whose derivation can be found in the appendix of [26]:

$$h^\rho{}_\mu \bar{\nabla}_\nu T^{\nu\mu} = \partial_b x^\rho D_a T^{ab} \tag{3.24}$$

This relation will be useful later when we derive the equations of motion, this is why we present it here. We can define the extrinsic curvature tensor:

$$K_{\mu\nu}{}^\rho = h_\mu{}^\sigma \bar{\nabla}_\nu h_\sigma{}^\rho \tag{3.25}$$

which is tangent to $\mathcal{W}_{(p+1)}$ along its lower symmetric indices, and orthogonal to $\mathcal{W}_{(p+1)}$ along its upper index. Its trace is the *mean curvature vector*:

$$K^\rho \equiv h^{\mu\nu} K_{\mu\nu}{}^\rho = \bar{\nabla}_\mu h^{\mu\rho} \tag{3.26}$$

3.3 Blackfold Equations of Motion

Under the assumptions that:

- The stress-energy tensor derives from conservative dynamics, in our case GR, even if the macroscopic dynamics are dissipative (this would require higher terms in derivatives of the collective coordinates, not in our case).
- Spacetime diffeomorphism invariance holds, that is, the long wavelengh gravitational field $g_{\mu\nu}$ can be coupled to the worldvolume $\mathcal{W}_{(p+1)}$ theory.

The stress-energy tensor must obey the conservation equations:

$$\bar{\nabla}_{\mu} T^{\mu\rho} = 0 \tag{3.27}$$

Also, since the stress-energy tensor lives in the worldvolume, regardless of the possibility of expressing it in terms of the background coordinates x_{μ} because of the embedding, its projection onto the orthogonal subspace to $\mathcal{W}_{(p+1)}$ must vanish, this is called the *transversality condition*:

$$\perp^{\nu}{}_{\mu} T^{\mu\rho} = 0 \tag{3.28}$$

Equations (3.27) are, indeed, the equations of motion for the whole set of collective variables $\phi(\sigma^a)$, both intrinsic and extrinsic. We will detail here how they can be decomposed in these two components:

$$\begin{aligned} \bar{\nabla}_{\mu} T^{\mu\rho} = \bar{\nabla}_{\mu}(T^{\mu\nu} h_{\nu}{}^{\rho}) = T^{\mu\nu} \bar{\nabla}_{\mu} h_{\nu}{}^{\rho} + h_{\nu}{}^{\rho} \bar{\nabla}_{\mu} T^{\mu\nu} = \\ T^{\mu\sigma} h_{\sigma}{}^{\nu} \bar{\nabla}_{\mu} h_{\nu}{}^{\rho} + h_{\nu}{}^{\rho} \bar{\nabla}_{\mu} T^{\mu\nu} = \\ T^{\mu\sigma} K_{\mu\sigma}{}^{\rho} + \partial_b x^{\rho} D_a T^{ab} = 0 \end{aligned} \tag{3.29}$$

So the D equation (3.27) are split into $D - p - 1$ equations orthogonal to $\mathcal{W}_{(p+1)}$ and another $p + 1$ equations parallel to the worldvolume $\mathcal{W}_{(p+1)}$, therefore both terms must vanish independently:

$$T^{\mu\sigma} K_{\mu\sigma}{}^{\rho} = 0 \qquad \textit{extrinsic equations} \tag{3.30}$$

$$D_a T^{ab} = 0 \qquad \textit{intrinsic equations} \tag{3.31}$$

Now we apply equations (3.27) to the stress-energy tensor of a perfect fluid and we get:

$$u^{\mu} u^{\nu} \bar{\nabla}_{\nu} \varepsilon + (\varepsilon + P)(a^{\mu} + u^{\mu} \bar{\nabla}_{\nu} u^{\nu}) + (h^{\mu\nu} + u^{\mu} u^{\nu}) \bar{\nabla}_{\nu} P + P K^{\mu} = 0 \tag{3.32}$$

where $a^\mu \equiv u^\nu \nabla_\nu u^\mu$ is the acceleration of the worldvolume's velocity u^μ. The projection of this equation along u^μ gives the *intrinsic equations*:

$$u^\nu \bar{\nabla}_\nu \varepsilon + (\varepsilon + P)\bar{\nabla}_\nu u^\nu \tag{3.33}$$

(See (B.13)) So the rest of the terms live in the orthogonal subspace, this yields the *extrinsic equations*:

$$(\varepsilon + P)a^\mu + (h^{\mu\nu} + u^\mu u^\nu)\bar{\nabla}_\nu P + PK^\mu = 0 \tag{3.34}$$

These equations are valid in general for any perfect fluid stress-energy tensor. We have seen that the stress-energy tensor of a neutral, non-spinning Black p-brane might be expressed in the same form, so using (3.11) we can derive the Blackfold equations of motion by simple substitution:

$$a^\mu + \frac{1}{n+1}u^\mu \bar{\nabla}_\nu u^\nu = \frac{1}{n}K^\mu + \bar{\nabla}^\mu \ln r_o \tag{3.35}$$

This set of equations describe the motion of the collective variables of the Blackfold in both, parallel and orthogonal, directions to the worldvolume $\mathcal{W}_{(p+1)}$. We can use the projectors we defined in order to split them into their *intrinsic* and *extrinsic* components:

$$h_{\rho\mu}a^\mu + \frac{1}{n+1}u_\rho \bar{\nabla}_\nu u^\nu = \bar{\nabla}_\rho \ln r_o \qquad \textit{intrinsic} \tag{3.36}$$

(See (B.14)) Which reads in worldvolume indices:

$$a_b + \frac{1}{n+1}u_a D_b u^b = \partial_a \ln r_o \qquad \textit{intrinsic} \tag{3.37}$$

Now, upon projection with the orthogonal operator defined in (3.20):

$$K^\rho = n\perp^\rho{}_\mu a^\mu \qquad \textit{extrinsic} \tag{3.38}$$

A final remark on these equations: The complete set of equations for a Blackfold must also include the backreaction of the metric, that is, how the metric in long scales is modified due to the presence of the Blackfold, these simply take the form of the EE with the effective stess-energy tensor of the Blackfold as source term:

$$R^{(long)}_{\mu\nu} - \frac{1}{2}R^{(long)}g^{(long)}_{\mu\nu} = 8\pi G T^{\text{eff}}_{\mu\nu} \tag{3.39}$$

$T^{\text{eff}}_{\mu\nu}$ is, to leading order, the same as the ADM stress-energy tensor we computed. But we only need to take into account the *intrinsic* and *extrinsic* equations to consistetly match the stress-energy tensor to long wavelength modes of the grvitational field so we can, in a first approximation, neglect the backrection equations. Note, however, that the set of equations (3.30) and (3.31) are only useful for test branes.

3.4 Stationary Blackfolds

We begin by giving a characterization of stationary configurations and some geometrical and thermodynamical relations satisfied by such fluids needed in order to solve the Blackfold equations, for more details we refere the reader to [34] which we follow here and detail some of the computations outlined there. We assume that the background is satationary and therefore there exist a timelike Killing vector field which generates the time translation symmetries denoted by ξ and a set of commuting, linearly independent, spacelike Killing vector fields denoted by χ_i which generate the symmetries of the background. In this latter set, the ones which commute with the fluid velocity, $\mathcal{L}_{\chi_i} u = 0$, generate the symmetries of the fluid also. The corresponding symmetries will be the energy E as well as the lineal/angular momenta $p_i \backslash J_i$ of the fluid symmetries.

For an isolated system to be stationary we require that the velocity is expansion-free and that the shear tensor of the fluid vanishes, this is equivalent to requiring that dissipative effects are absent. We will see that under this assumptions, we can find a Killing vector field proportional to the fluid's velocity. For such a fluid, the covariant derivative of the velocity is given by:

$$\nabla_\mu u_\nu = \omega_{\mu\nu} - u_\mu a_\nu \tag{3.40}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is the vorticity tensor of the fluid and $a_\nu = u^\mu \nabla_\mu u_\nu$ is the acceleration. For a fluid with local temperature and entropy density, $\mathcal{T}$ and s respectively, the Euler realtion is satisfied:

$$\rho + P = s\mathcal{T} \Longrightarrow \text{differentiate and use TD } 1^{st} \text{ Law} \Longrightarrow dP = s d\mathcal{T} \tag{3.41}$$

The latter is the Gibbs-Duhem relation. Also, we know that for a stationary configuration the gradient of the pressure must be orthogonal to the velocity of the fluid $u^\mu \nabla_\mu P = 0$. Recalling the Navier-Stokes equation for a relativistic fluid [34]: $(\rho + P) u^\mu \nabla_\mu u^\nu = -(g^{\mu\nu} + u^\mu u^\nu)\nabla_\mu P$ we see that, taking into account the condition on the gradient of the pressure above, we finally get:

$$a_\nu = -(\rho + P)^{-1} \nabla_\nu P = -\nabla_\nu \ln\mathcal{T} \tag{3.42}$$

where we have used the two previous Thermodynamic identities. Let us see that the fluid's velocity is proportional to a Killing vector field, we use the ansatz $K_\nu \equiv \alpha u_\nu$ and we will find a value of α such that the Killing equations are satisfied for the vector field K_ν. By making use of the ansatz together with the above equations, it is shown in the corresponding appendix that the Killing vector equations yield:

$$\nabla_{(\mu}[\alpha u_{\nu)}] = \alpha u_{(\mu} \nabla_{\nu)} \ln(\alpha \mathcal{T}) \tag{3.43}$$

(See (B.16)) We can see at once that, if we choose the function α in such a way that it is proportional to the local temperature: $\alpha = T/\mathcal{T}$ with T an integration constant, the Killing equations are satisfied for the vector field $K_\nu \equiv \alpha u_\nu$. Therefore, K_ν is a Killing vector field. Since α is non vanishing we can solve for u_ν to get $u_\nu = \frac{K_\nu}{\alpha}$. Thus, we have proved that the velocity vector field of the fluid is proportional to a Killing vector field of the worldvolume, there might be others though. In general, this Killing field will be a linear combination of background Killing fields, however, not all of the latter need to be present because some symmetries of the background might not be symmetries of the fluid.

Having proved that the velocity vector field is proportional to a Killing vector field for stationary conditions we proceed to solve the Blackfold equations explicitly. Denoting the worldvolume Killing vector field by $\mathbf{k} = k^a \partial_a$ we have, for the velocity u:

$$u = \frac{\mathbf{k}}{|\mathbf{k}|} \tag{3.44}$$

where

$$|\mathbf{k}| = \sqrt{-\gamma_{ab} k^a k^b} \tag{3.45}$$

k^a is the pullback of the timelike Killing vector field k^μ of the background which is needed to preserve the stationarity condition. They satisfy, respectively the following Killing equations:

$$D_{(a} k_{b)} = 0 \tag{3.46}$$

$$\nabla_{(\mu} k_{\nu)} = 0 \tag{3.47}$$

Now we have the set up, so let us derive some results. Contracting (3.47) with $k^\mu k^\nu$ gives:

$$k^\mu \partial_\mu |\mathbf{k}| = 0 \tag{3.48}$$

and this yields:

$$a^\mu = \partial^\mu \ln|\mathbf{k}| \tag{3.49}$$

(See (B.17) and (B.18)) Recall that one of the conditions for stationarity was that the velocity vector field was expanssion free. Taking this into account, the latest result implies, together with (3.37), that:

$$\partial^\mu \ln|\mathbf{k}| = \partial^\mu \ln r_o \tag{3.50}$$

so that:

$$\frac{r_o}{|\mathbf{k}|} = constant \tag{3.51}$$

In order to fix the proportinality constant we recall equation (3.16) which expresses the relation between the surface gravity and the local temperature.

Also, and from a thermodynamical point of view, we know that the local temperature $\mathcal{T}$ and the global temperature T are related by the redshift factor [35]:

$$\mathcal{T} = \frac{T}{|\mathbf{k}|} = \frac{n}{4\pi r_o} \Longrightarrow \frac{r_o}{|\mathbf{k}|} = \frac{n}{4\pi T} \tag{3.52}$$

Using the definition given after (3.17) we conclude that:

$$\frac{r_o}{|\mathbf{k}|} = \frac{n}{2\kappa} \Longrightarrow \kappa = \frac{n|\mathbf{k}|}{2r_o} \tag{3.53}$$

which, in the light of (3.51), means that the surface gravity is a constant over the horizon, a standard result which can be found at [25]. The solution to the intrinsic Blackfold equations is provided by equations (3.44) and (3.53) but we can give them a more explicit expression by expressing the Killing vector field $\mathbf{k}$ in terms of background Killing independent vector fields:

$$\mathbf{k} = \xi + \sum_{i=1}^{p} \Omega_i \chi_i \tag{3.54}$$

where Ω_i is a constant. ξ and χ_i can be chosen to be the Killing vectors of the background but this not need be so and we can simply require the linear combination to satisfy Killing's equations but not its components separately. We define worldvolume functions R_a by their values on the worldvolume:

$$R_0 = \sqrt{-\xi^2}|_{\mathcal{W}_{p+1}}; \qquad R_i = \sqrt{\chi_i^2}|_{\mathcal{W}_{p+1}} \tag{3.55}$$

If the vector field ξ is a Killing vector field of the background which generates time translations, its module is the redshift factor between infinity and the Blackfold worldvolume. Similarly, the modules of the vector fields χ_i are the radii of the orbits they generate, provided they are background Killing vectors for spatial symmetries. The Ω_i are the horizon angular velocities relative to observers which follow the orbits of ξ. The vector fields (no sum over the index i):

$$\frac{\partial}{\partial t} = \frac{1}{R_0}\xi; \qquad \frac{\partial}{\partial z^i} = \frac{1}{R_i}\chi_i \tag{3.56}$$

are orthonormal with respect to the metric γ_{ab} but it is convenient for us to regard them as extended over the whole manifold. Now, let us define some new quantities which will allow us to derive more compact expressions. We first introduce a worldvolume spatial velocity field defined by:

$$V_i(\sigma) = \frac{u \cdot \partial_{z^i}}{-u \cdot \partial_t} = \frac{\Omega_i R_i(\sigma)}{R_0} \tag{3.57}$$

so that, in virtue of the previous expressions, we have the following identities:

$$\mathbf{k} = R_0\left(\frac{\partial}{\partial t} + \sum_{i=1}^{p} V_i \frac{\partial}{\partial z^i}\right) \tag{3.58}$$

$$|\mathbf{k}| = \left(-\xi^2 - \sum_{i=1}^{p} \Omega_i^2 \chi_i^2 \right)^{\frac{1}{2}} = R_0 \sqrt{1 - V^2} \tag{3.59}$$

where

$$V^2 = \sum_{i=1}^{p} V_i^2 = \frac{1}{R_0^2} \sum_{i=1}^{p} \Omega_i^2 R_i^2 \tag{3.60}$$

From (3.59), we see that $|\mathbf{k}|$ can be thought of as a Loretz factor at some point in $\mathcal{W}_{p+1}$ slightly modified by a possible redshift factor, all relative to the static observers in the orbits of ξ. With (3.59) and (3.53) it is straightforward to derive:

$$r_o(\sigma) = \frac{nR_0(\sigma)}{2\kappa} \sqrt{1 - V^2(\sigma)} \tag{3.61}$$

So, for given κ and Ω_i the above equation gives an expression for the horizon thickness in terms of the R_a. The requirement that κ and Ω_i should be constant throughout the Blackfold worldvolume are known as the *blackness conditions* and could have been imposed from the very begining, by using general theorems for stationary blackholes, in order to have a regular horizon. Here, we have derived this property, regularity, by simply following fluid dynamical arguments, where κ and Ω_i appear as simple integration constants. See [35] for an alternative derivation.

Let us close this section with a few words about Blackfold's boundaries. These are developed when branes or strings intersect, so it is useful to make some remarks. We focus ourselves in a p-brane with timelike boundary denoted by $\partial\mathcal{W}_{p+1}$ and given by a function $f(\sigma) = 0$. As usual, $\partial_a f(\sigma)$ is a normal 1-form to the worldvolume's boundary of the p-brane. From fluid dynamics, we know that, should the fluid remain within its boundaries, we must require the boundary function f to be Lie-dragged along the boundary, that is:

$$\mathcal{L}_u|_{\partial\mathcal{W}_{p+1}} f = 0 \tag{3.62}$$

which is equivalent to requiring the more intuitive constraint that the velocity is normal to the boundary:

$$(u^a \partial_a f)|_{\partial\mathcal{W}_{p+1}} \tag{3.63}$$

We can now project the stress-energy tensor of the worldvolume into the orthogonal direction to the boundary. Since, by construction, the stress-energy tensor lives in the worldvolume the projection will vanish, this is equivalent to the *intrinsic equations*:

$$[((\varepsilon + P)u_a u_b + \gamma_{ab} P)\partial^a f]|_{\partial\mathcal{W}_{p+1}} = 0 \tag{3.64}$$

By making use of (3.63) we are left with:

$$(P\partial^a f)|_{\partial\mathcal{W}_{p+1}} = 0 \Longrightarrow P|_{\partial\mathcal{W}_{p+1}} = 0 \tag{3.65}$$

which for a neutral Blackfold translates, in the light of (3.11), into:

$$r_o|_{\partial\mathcal{W}_{p+1}} = 0 \tag{3.66}$$

This is an importat result which tells us that the Baclfold's horizon thickness r_o must approach zero as we approach the edge of the blackfold. It will have some implications when we talk about the topology of the horizon. Equation (3.66) implies, by (3.61), that, either the redshift goes to infinity at the boundary $R_0 \to 0$ or, more likely, the speed of the fluid approaches that of light at the boundary:

$$V^2|_{\partial\mathcal{W}_{p+1}} = 1 \tag{3.67}$$

3.5 Extrinsic Equations and First Law of Stationary Blackfold Dynamics

We still need to deal with the *extrinsic equations* (3.38) which, taking into account all the consideration made above, we can recast as:

$$K^\rho = \perp^\rho{}_\mu \partial^\mu \mathrm{ln}|\mathbf{k}|^n \tag{3.68}$$

Knowing $|\mathbf{k}|$ already, we can easily solve them, at least formally. It is sometimes useful to resort to Action Principles such as those pointed out in the appendix of [26] in (A.40). Summarizing, the equation above can be derived from the action:

$$I[x^\mu(\sigma)] = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}|\mathbf{k}|^n \tag{3.69}$$

by performing variations in the coordinates $x^\mu_\perp(\sigma)$ transverse to the worldvolume $\mathcal{W}_{p+1}$. We can put it in a more convenient form for practical calculations. The differentiation with respect to the asymptotic time t is related to the vector ξ by a factor R_0, (3.56), so, if we perform the trivial integration over the asymtotic time and we take its interval to have length β we can write:

$$I[x^\mu(\sigma)] = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}|\mathbf{k}|^n = \beta\int_{\mathcal{B}_p} dV_{(p)} R_0 |\mathbf{k}|^n. \tag{3.70}$$

where $\mathcal{B}_p$ is a spatial section of $\mathcal{W}_{p+1}$. Using the result derived in the previous section we can easily derive by simple substitution:

$$I[x^\mu(\sigma)] = \beta \int_{\mathcal{B}_p} dV_{(p)} R_0(\sigma) \Big(R_0^2(\sigma) - \sum_{i=1}^{p} \Omega_i^2 R_i^2(\sigma) \Big)^{\frac{n}{2}} \tag{3.71}$$

The $R_a(\sigma)$ must be regarded also as part of the embeding coordinates $x^\mu(\sigma)$ and of course they enter also through $dV_{(p)}$.

We will now show how this action may be expressed in terms of the Blackfold parametrers β, κ, Ω_i, M, J_i and $\mathcal{A}_H$. Where the last three stand for the mass, angular momenta and horizon surface area respectively. For this, we need to derive first expresions for M, J_i and $\mathcal{A}_H$.

Let us then begin by computing the mass. The mass is the conjugate to the vector field ξ that generates asymptotic time translations and so is given by:

$$M = \int_{\mathcal{B}_p} dV_{(p)} T_{\mu\nu} n^\mu \xi^\nu = \frac{\mathcal{A}_{(n+1)}}{16\pi G} \left(\frac{n}{2\kappa}\right)^n \int_{\mathcal{B}_p} dV_{(p)} R_0^{n+1} (1-V^2)^{\frac{n-2}{2}} (n+1-V^2) \tag{3.72}$$

(See (B.21)) where $T_{\mu\nu}$ is the covariant form of the pushed-forward stress-energy tnesor T^{ab} of the worldvolume perfect fluid and so has the generic form:

$$T_{\mu\nu} = (\varepsilon + P) u_\mu u_\nu + h_{\mu\nu} P \tag{3.73}$$

For the angular momenta we have:

$$J_i = -\int_{\mathcal{B}_p} dV_{(p)} T_{\mu\nu} n^\mu \chi_i^\nu = \frac{\mathcal{A}_{(n+1)}}{16\pi G} \left(\frac{n}{2\kappa}\right)^n n\Omega_i \int_{\mathcal{B}_p} dV_{(p)} R_0^{n-1} (1-V^2)^{\frac{n-2}{2}} R_i^2 \tag{3.74}$$

See (B.22) for details. To compute $\mathcal{A}_H$ we will use a trick which will also give us the expression for the entropy. We know the expression for the entropy density of the Blackfold (3.12) which must be modified by a Lorentz relativistic factor to yield the total entropy of the Blackfold:

$$S = \int_{\mathcal{B}_p} dV_{(p)} \frac{s}{\sqrt{1-V^2}} = \frac{\mathcal{A}_{(n+1)}}{4G} \left(\frac{n}{2\kappa}\right)^{n+1} \int_{\mathcal{B}_p} dV_{(p)} R_0^{n+1} (1-V^2)^{\frac{n}{2}} \tag{3.75}$$

But by definition we have that:

$$S_{HB} \equiv \frac{\mathcal{A}_H}{4G} \tag{3.76}$$

This implies an expression for the surface area of the horizon:

$$\mathcal{A}_H = \int_{\mathcal{B}_p} dV_{(p)} a_H = \mathcal{A}_{(n+1)} \left(\frac{n}{2\kappa}\right)^{n+1} \int_{\mathcal{B}_p} dV_{(p)} R_0^{n+1} (1-V^2)^{\frac{n}{2}} \tag{3.77}$$

from which we get:

$$a_H = \frac{\mathcal{A}_{(n+1)}}{\sqrt{1-V^2}} r_o^{n+1} = \mathcal{A}_{(n+1)} \left(\frac{n}{2\kappa}\right)^{n+1} R_0^{n+1} (1-V^2)^{\frac{n}{2}} \tag{3.78}$$

This could have been derived by following purely geometrical arguments as in [26].

The first law of Blackfold dynamics states a relation between the Blackfold parameters derived above and the action from which one can compute the extrinsic equations:

$$\hat{I} = \frac{\mathcal{A}_{(n+1)}}{16\pi G} \left(\frac{n}{2\kappa}\right)^n I = \beta \left(M - \sum_{i=1}^{p} \Omega_i J_i - \frac{\kappa}{8\pi G} \mathcal{A}_H \right) \tag{3.79}$$

See (B.23) for details. To get the equations of motion we leave the surface gravity and the angular velocities fixed (*blackness conditions*) and perform variations in the coordinates $x^\mu(\sigma)$:

$$\frac{\delta \hat{I}}{\delta x^\mu} = 0 \quad \Longleftrightarrow \quad \left(\frac{\delta M}{\delta x^\mu} - \sum_{i=1}^{p} \Omega_i \frac{\delta J_i}{\delta x^\mu} - \frac{\kappa}{8\pi G} \frac{\delta \mathcal{A}_H}{\delta x^\mu} \right) = 0 \tag{3.80}$$

where we can see that:

$$\frac{\delta \hat{I}}{\delta x^\mu} = \frac{\delta I}{\delta x^\mu} \tag{3.81}$$

since the overall factor vanishes upon variation.

Chapter 4

Myers-Perry Black Holes

We have already studied the simplest kind of higher dimensional BHs in section 2.3, the Schwarzschild-Tangherlini solution, where, following [14], we derived some of its properties, such as the stress energy tensor or the mass. In this section, we will study the higher dimensional generalization of the Kerr solution, the *Myers-Perry* BH (MPBH), there are more general solutions with several angular momenta in independent rotation planes but we refer the reader to the original papers [14] and [36] for the details of them. Here, we shall only focus on the case of a single angular momentum because it already exhibits the most characteristic features of the general solution and allows us to take the *ultraspinning limit*. In this limit, we can use the formalism developed for Blackfolds to describe the ultraspinning MPBH as a Black p-brane. The fact that this limit is well defined will play an important role when we come to study the stability of Blackfolds and hence of the BH solutions from which they are derived such as the corresponding MPBH.

4.1 Spinning Blackholes

In $D = N + 1$ spacetime dimensions the situation is richer than in the usual $3 + 1$ dimensions and the BH is characterized by a larger set of parameters, namely: the mass and the $\lfloor N/2 \rfloor$ independent angular momenta corresponding to the $\lfloor N/2 \rfloor$ Casimirs of the rotation group $SO(N)$. ($\lfloor N/2 \rfloor$ stands for the integer part of $N/2$). The solution of the vacuum EE in this case is found by using an ansatz [14] which, in the lower dimensional limit, reproduces the known Kerr solution, the generalization is thus:

$$ds^2 = -\beta^2(r,\rho)(du + a\ sin^2\theta\ d\phi)^2 + 2(du + a\ sin^2\theta\ d\phi)(dr + a\ sin^2\theta\ d\phi) \\ +\rho^2(d\theta^2 + sin^2\theta\ d\phi^2) + r^2\ cos^2\theta\ d\Omega^2_{D-4} \quad (4.1)$$

where $\rho^2 = r^2 + a^2\ cos^2\theta$ and $d\Omega^2_{D-4}$ is the squared solid angle element of the unit $D-4$ dimensional sphere. For the case, $D = 4$ the last term drops, $\beta^2 = 1 - 2GMr/\rho^2$ and we are left with the usual Kerr solution. For $D > 4$, the above metric is a solution of the vacuum EE with:

$$\beta^2 = 1 - \frac{\mu}{r^{D-5}\rho^2} \tag{4.2}$$

To examine the asymptotic behaviour and compute some quantities, it is better for us to use *Boyer-Lindquist* (BL) coordinates, performing the transformation:

$$\begin{aligned} dt &= du - \frac{(r^2 + a^2)\ dr}{r^2 + a^2 - \mu\ r^{5-D}} \\ d\varphi &= d\phi + \frac{a\ dr}{r^2 + a^2 - \mu\ r^{5-D}} \end{aligned} \tag{4.3}$$

and rearranging terms, we find the metric in BL coordinates:

$$\begin{aligned} ds^2 = -dt^2 + sin^2\theta(r^2 + a^2)d\varphi^2 + \Delta(dt + a\ sin^2\theta\ d\varphi)^2 + \\ \Psi\ dr^2 + \rho^2\ d\theta^2 + r^2\ cos^2\theta\ d\Omega^2_{D-4} \end{aligned} \tag{4.4}$$

where we have defined:

$$\begin{aligned} \Delta &\equiv \frac{\mu}{r^{D-5}\rho^2} \\ \Psi &\equiv \frac{r^{D-5}\rho^2}{r^{D-5}(r^2 + a^2) - \mu} \end{aligned} \tag{4.5}$$

we can take the limit $a \to 0$ to get the Schwarzschild solution in $N + 1$ spacetime dimensions. Following arguments analogous to those presented in section 2.3 to derive the mass of the Schwarzchild, we can derive the mass for this case. For the Spinning MPBH, we have in the asymptotic limit:

$$g_00 = -1 + \frac{\mu}{r^{N-2}} = \eta_{00} + h_{00} \Longrightarrow h_{00} = \frac{\mu}{r^{N-2}} \tag{4.6}$$

comparing with (2.16) and solving for M yields:

$$M = \frac{(N-1)\mathcal{A}_{(N-1)}}{16\pi G}\mu = \frac{(D-2)\mathcal{A}_{(D-2)}}{16\pi G}\mu \tag{4.7}$$

Now we must compute the expression for the angular momenta which is a new magnitude we did not have in the case of section 2.3. Following [14], the definition of the angular momenta second-rank tensor is:

$$J^{km} = 2\int x^k\ T^{m0}\ d^N x \tag{4.8}$$

Since we can expand the perturbation to the metric $h_{\mu\nu}$ as was shown in (2.13) we can easily conclude that, to first order, we must have:

$$h_{0i} \approx -\frac{8\pi G}{\mathcal{A}_{(N-1)}} \frac{x^k}{r^N} J_{ki} \tag{4.9}$$

Therefore, we need to study the off-diagonla terms of the metric in the asymptotic limit. We can perform the following transformation to cartesian coordinates when $r \to \infty$:

$$\begin{aligned} x &\approx r\ sin\theta\ cos\phi \\ y &\approx r\ sin\theta\ sin\phi \end{aligned} \tag{4.10}$$

so that, for the off-diagonal term we have:

$$\begin{aligned} 2\ \Delta\ a\ sin^2\theta\ dt\ d\varphi = 2\ \frac{\mu}{r^{N-4}\ \rho^2}\ a\ sin^2\theta\ dt\ d\varphi \approx 2\ \frac{\mu}{r^{N-2}}\ a\ sin^2\theta\ dt\ d\varphi = \\ 2\ \frac{a\ \mu}{r^N}\ dt(r^2\ sin^2\theta\ d\varphi) = -2\ \frac{a\ \mu}{r^N}\ dt(y\ dx - x\ dy) \quad \Longrightarrow \\ h_{xt}\ dx\ dt + h_{tx}\ dt\ dx = 2\ h_{tx}\ dx\ dt = -2\ \frac{a\ \mu}{r^N}\ y\ dx\ dt \Longrightarrow \\ h_{tx} = -\frac{a\ \mu}{r^N}\ y \end{aligned} \tag{4.11}$$

Analogously we get:

$$h_{ty} = -\frac{a\ \mu}{r^N}\ x \tag{4.12}$$

Comparison with (4.9) gives a single non-zero angular momentum:

$$\begin{aligned} -\frac{8\pi G}{\mathcal{A}_{(N-1)}} \frac{x}{r^N} J_{xy} &= -\frac{a\ \mu}{r^N}\ x \\ -\frac{8\pi G}{\mathcal{A}_{(N-1)}} \frac{y}{r^N} J_{yx} &= -\frac{a\ \mu}{r^N}\ y \Longrightarrow \\ J_{xy} = J_{yx} &= \frac{\mathcal{A}_{(N-1)}}{8\pi G} a\ \mu \end{aligned} \tag{4.13}$$

and, by using (4.7) we get:

$$J_{xy} = J_{yx} = \frac{2}{N-1} M\ a = \frac{2}{D-2} M\ a \tag{4.14}$$

For this metric, the horizon occurs when $g^{rr} = \Psi^{-1} = 0$. Let us study this case by case. We look for the largest root of the equation:

$$(r_+^2 + a^2) - \frac{\mu}{r_+^{D-5}} = 0 \tag{4.15}$$

For $D = 4$, we are in the well known Kerr case and therefore a regular horizon is known to exist for values of a up to the Kerr bund: $a = \mu/2$. In $D = 5$, we have a simpler equation which has real roots up to the value $a = \pm\sqrt{\mu}$, where the signs account for the different directions of rotation. However, this extremal solution has zero area and it is, in fact, a naked ring. Finally, for $D \geq 6$ we can take the limits at large and small values of r and we find:

$$\begin{aligned} \lim_{r\to\infty} \Psi^{-1} &= \quad 1 \\ \lim_{r\to 0} \Psi^{-1} &= \; -\tfrac{\mu}{r^{N-4}\,\rho^2} \end{aligned} \tag{4.16}$$

So $\Psi^{-1} = 0$ must cross zero at some intermediate radius and this result is independent of the value of a: there will always be an event horizon (positive real value of r_+) whatever the value of a provided $D \geq 6$. This is a crucial result for it will allow us to take a as large as we desire and go to the *ultraspinning limit*, this is what we shall do in the following section.

4.2 Ultraspinning Regime

We will now present a few results given by Emparan and Myers in [36] about the horizon's geometry in the ultraspinning regime which will serve us to confirm the approximation to be made in the next section. The ultraspinning regime is formally defined by the following conditions: $a \to \infty$ and $\mu =$ constant. In this limit, from (4.15), we can approximate the horizon radius by:

$$r_+ \simeq \left(\frac{\mu}{a^2}\right)^{\frac{1}{D-5}} \ll a \tag{4.17}$$

Now, the total $(D-2)$-dimensional surface area of the horizon is given by:

$$\mathcal{A}_{Hor} = \mathcal{A}_{D-2}\, r_+^{D-4}(r_+^2 + a^2) \tag{4.18}$$

where $\mathcal{A}_{D-2}$ is the surface area of the $(D-2)$-dimensional unit sphere. In the limit, we can approximate it by:

$$\mathcal{A}_{Hor} \simeq \mathcal{A}_{D-2}\, r_+^{D-4} a^2 \simeq \mathcal{A}_{D-2}\left(\frac{\mu^{D-4}}{a^2}\right)^{\frac{1}{D-5}} \tag{4.19}$$

Notice that the surface area *decreases* as the angular momentum parameter *increases* with fixed mass parameter. We can also consider the surface area of different spatial sections of the $(D-2)$-dimensional horizon. Let us begin by the plane of rotation, the 2-dimensional surface area is given by:

$$\begin{aligned} \mathcal{A}^{(2)}_{\parallel Hor} &= \mathcal{A}_{D-2}(r_+^2 + a^2) \simeq \mathcal{A}_{D-2}a^2 \\ \mathcal{A}^{(D-4)}_{\perp Hor} &= \mathcal{A}_{D-4}(r_+ \; cos\theta)^{D-4} \end{aligned} \tag{4.20}$$

We can see at once that the parallel area grows with the angular momentum parameter whilst the perpendicular section area decreases when a grows. Generally, we can define characteristic lenghts for both surface areas and we would have:

$$\begin{aligned} \mathcal{A}^{(D-2)}_{Hor} &\propto \ell_\parallel^2 \ell_\perp^{D-4} + ... \\ \mathcal{A}^{(2)}_{\parallel Hor} &\propto \ell_\parallel^2 + ... \\ \mathcal{A}^{(D-4)}_{\perp Hor} &\propto \ell_\perp^{D-4} + ... \end{aligned} \tag{4.21}$$

Comparison with (4.20) suggests the following identifications:

$$\begin{aligned} \ell_\parallel &\sim a \\ \ell_\perp &\sim r_+ \; cos\theta \end{aligned} \tag{4.22}$$

4.3 Black Brane Limit

When we take the ultraspinning limit, we saw that the horizon area shrinked as the parameter a grew. We would like to keep a finite and non-zero value for the horizon area so we must also take $\mu \to \infty$ kepping the following parameter finite: $\hat{\mu} \equiv \mu/a^2$. We will also define a new variable: $\sigma = a \; sin\theta$ which will be kept fixed as $a \to \infty$ by making $\theta \to 0$. This is equivalent to approaching the pole as we approach the horizon. We also kept r fixed. Taking this limit in (4.4) we get:

$$\begin{aligned} ds^2 = -\left(1 - \frac{\hat{\mu}}{r^{D-5}}\right)dt^2 + \frac{dr^2}{1 - \frac{\hat{\mu}}{r^{D-5}}} + r^2 \; d\Omega^2_{D-4} + \\ d\sigma^2 + \sigma^2 \; d\varphi^2 \end{aligned} \tag{4.23}$$

Here we can recognize the metric (3.3) for a $(p = 2)$-brane with the cartesian coordinates of the brane plane expressed in polar coordinates $\{\sigma, \varphi\}$. Restricting the analysis to the Blackbrane plane, we are going to derive various quantities and confirm that the results agree, making the appropriate identifications, with what we have just seen in the previous sections. So, the Blackbrane worldvolume is given, to first order, by the $2 + 1$ Minkowskian metric:

$$ds^2 = -dt^2 + dr^2 + r^2 \; d\varphi^2 \tag{4.24}$$

where we have just renamed: $\sigma = r$. The Blackfold is therefore rotating in the polar plane and the condition of stationarity requires that the fluid rotates rigidly and that disipative effects are absent. In this case, the Killing vectors of the metric we are interested in are:

$$\xi = \partial/\partial t, \quad \chi = \partial/\partial\varphi \tag{4.25}$$

recalling (3.44) and (3.54) we have:

$$u = \frac{1}{\sqrt{1-\Omega^2\, r^2}}\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right) \tag{4.26}$$

In our case, according to (2.1), we have $n = 1$. So, knowing the Killing vectors above, we can solve the intrinsic equation at once, from (3.52) and using (3.45) with the metric (4.24), we have:

$$r_o(r) = \frac{1}{4\pi T}\sqrt{1-\Omega^2 r^2} \tag{4.27}$$

from which we can see that, in order to have a real r_o, the radial coordinate must lie in the interval $0 \leq r \leq \Omega^{-1}$. Further, the condition $r_o(\Omega^{-1}) = 0$ implies, according to (3.66), that $r = \Omega^{-1}$ is the boundary of the Blackfold.

In order to compute the Blackfold parameters, we introduce the new useful quantities: the disk radius a (not to be confused with the angular momentum parameter which here is not used) and the horizon thickness r_+ at the axis of rotation:

$$a = \Omega^{-1}, \quad r_+ = r_o(0) = \frac{1}{4\pi T} \tag{4.28}$$

To compute the mass and angular momentum of the Blackfold we use (3.72) and (3.74), but first, we need to find the expression for the stress-energy tensor. This is given by (3.8) and it is straightfoward to see that it gives:

$$T_{ab} = \frac{r_o}{4G}\left(u_a u_b - \eta_{ab}\right) \tag{4.29}$$

Using equations (3.72) and (3.74) we find:

$$\begin{aligned} T_{ab}n^a\xi^b = T_{tt} &= \frac{r_+}{4G}\frac{2-r^2/a^2}{\sqrt{1-r^2/a^2}} \\ -T_{ab}n^a\chi^b = T_{t\varphi} &= \frac{r_+}{4G}\frac{r^2/a^2}{\sqrt{1-r^2/a^2}} \end{aligned} \tag{4.30}$$

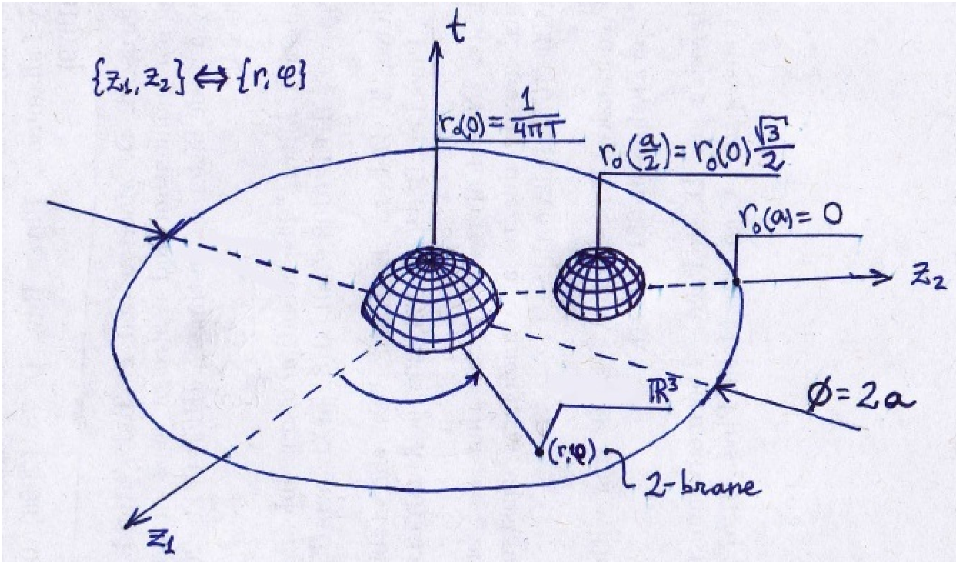

Figure 4.1: Above, is schematically drawn a $D = 6$ Blackbrane. Note that the size of the Blackfold is finite and limited by a. At each point of the 2-brane worldvolume (the disc in the picture) dwells a fibered $\mathbb{R}^3$. We have drawn, at three different points along the 2-brane's radius, the sphere (living in its corresponding $\mathbb{R}^3$) respresenting the event horizon surface. The radius of the event horizon is shown in the picture, it degenerates into a point at the edge of the Blackfold as we have seen.

since $n^a = \xi^a$. Then, plugging this in into the aforementioned equations, we find:

$$\begin{aligned} M &= \int_0^{2\pi} d\varphi \int_0^a dr\ r\ T_{tt} = \frac{2\pi}{3G} r_+ a^2 \\ J &= -\int_0^{2\pi} d\varphi \int_0^a dr\ r\ T_{t\varphi} = \frac{\pi}{3G} r_+ a^3 \end{aligned} \tag{4.31}$$

where we have used the formula for the surface area of the $(m-1)$-dimensional sphere, in our case we must take $m = 5$:

$$\mathcal{A}_{(m-1)} = m \frac{2^{\frac{m+1}{2}} \pi^{\frac{m-1}{2}}}{m!!}; \quad (2k-1)!! \equiv \frac{2k!}{2^k\ k!} \tag{4.32}$$

the integrals above and the one below are straightforward with the change of variables $r^2 = x$. The entropy-density current is given by $s^a = s\ u^a$, where s is given by (3.12):

$$-s^a n_a = \frac{\pi}{G} r_+^2 \sqrt{1 - r^2/a^2} \tag{4.33}$$

and to obtain the entropy we integrate to get:

$$S = -\int_0^{2\pi} d\varphi \int_0^a dr\ r\ s^a n_a = \frac{2\pi^2}{3G} r_+^2 a^2 \tag{4.34}$$

Up to this point, we have been using the Blackfold approach to derive these magnitudes, now let us look at the exact values of them for the MPBH, see [14]. They are given by:

$$M = \frac{2\pi}{3G}\mu, \quad J = \frac{1}{2}aM, \quad S = \frac{2\pi^2}{3G} r_+ \mu \tag{4.35}$$

where, now, a is the angular momentum parameter and μ the mass parameter. We saw that the horizon radius for the MPBH with $D = 6$ was given by (4.15) which, in the limit of *ultraspinning regime* ($a \to \infty$) gives:

$$\mu \to a^2 r_+ \tag{4.36}$$

One can readily see that, taking this limit in the expressions (4.35), they yield (4.31) and (4.34) upon identifying the parameters r_+ and a on both sides.

We can extend this correspondence further. For the Blackfold, we have just seen that the radii of the horizon in both, parallel and transverse (here called *thickness*), directions are:

$$r_{\parallel}^{bf} = a, \quad r_{\perp}^{bf} = r_o(r) \tag{4.37}$$

which can be recast defining $\theta = \arcsin(r/a)$ as:

$$r_{\parallel}^{bf} = a, \quad r_{\perp}^{bf} = r_+ \cos\theta \tag{4.38}$$

Again, upon identification of the Blackfold and MPBH parameters, we get (4.22). This further confirms that the Blackfold approach can be used for describing the *ultraspinning regime* of MPBH. As we have pointed out already, the existence of this limit will be a key factor when studying instability.

Chapter 5

Black Rings

In this section, we will present another set of solutions for the vacuum EE in higher dimensions, we will follow the lines of the previous chapter and first describe the solution in a number of dimensions which makes it manageable. Also, we will digress a little on the violation of BH uniqueness. Then, we will take an appropriate limit which enables us to describe the Black Ring as a curved Black String. Finally we will compare the results with those obtained with the Blackfold approach and we will confirm that they are equivalent as in the previous case. We will work to first order in r_o/R, we refer to [37] for a much more detailed discussion to higher orders.

5.1 The Black Ring

We will focus on the neutral Black Ring, see for example [38] to find a discussion of the charged Black Ring. For the details of the basic structure of a ring in 4 spatial dimensions see the first section [39], here we will be following largely the next one.

The solution has also been given in slightly different, but related, forms in [40] and [41]. The metric for the 5-dimensional spacetime Black Ring is:

$$ds^2 = -\frac{F(y)}{F(x)}\left(dt - C\,R\frac{1+y}{F(y)}d\psi\right)^2$$
$$+\frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right] \tag{5.1}$$

where R is the radius of the ring and

$$F(\xi) = 1 + \lambda\xi, \quad G(\xi) = (1-\xi^2)(1+\nu\xi), \quad C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}} \tag{5.2}$$

and the dimensionless parameters λ and ν must lie in the range:

$$0 < \nu \leq \lambda < 1 \tag{5.3}$$

for the roots of G to be real, see [41] for a more detailed discussion of the parameters' ranges. When both λ and ν vanish, we recover flat spacetime. R sets the scale of the solution and λ and ν are related to the shape and rotation speed of the ring as we will see.

Also the ranges of the coordinates x and y are:

$$-\infty \leq y \leq -1, \quad -1 \leq x \leq 1 \tag{5.4}$$

with $y = -\infty$ corresponding to the location of the ring, $y = -1$ is the plane of rotation, $x = 1$ is the interior of the disk bounded by the ring and $x = -1$ is its complement outside the ring. Asymptotic infinity is recovered for $x, y \to -1$. The orbits of the vector fields $\partial/\partial\psi$ and $\partial/\partial\phi$ do not close smoothly for arbitrary values of the parameters λ and ν and have conical singularities at $x = y = -1$. In order to avoid this, we must make both coordinates ψ and ϕ cyclic:

$$\Delta\psi = \Delta\phi = 4\pi \frac{\sqrt{F(-1)}}{|G'(-1)|} = 2\pi \frac{\sqrt{1-\lambda}}{1-\nu} \tag{5.5}$$

To avoid another conical singularity at $x = 1$ we must have also:

$$\Delta\phi = 2\pi \frac{\sqrt{1+\lambda}}{(1+\nu)} \tag{5.6}$$

which, compared with the previous cyclic conditions means:

$$\lambda = \frac{2\nu}{1+\nu^2} \tag{5.7}$$

this leaves only two free parameters in the solution, namely R and ν. This has a physical explanation. Initially, we had three parameters R, ν and μ from which the last two could be thought of as being related to the mass and the angular momentum of the ring. Without spinning, the ring would tend to collapse under gravitational attraction so, lest it collapses, we set the ring in rotational motion at such a pace that the gravitational forces are balanced by the centrifugal ones thus achieving the desired equilibrium. The angular velocity we must give to the ring will depend on its mass and/or on its chracteristic dimension, in our case, the radius. So, we have three parameters and we need to impose one relationship amongst them in order to have an stable ring. Three parameters and one relation leaves only two parameters free which is what we have found above. The absence of conical singularities is equivalent to the requirement of having balance without any external forces whatsoever.

With the above conditions imposed on the parameters the solution is asymptotically flat when $x, y \to -1$. But the geometry is distorted by the presence of curvature so, in order to go to manifestly asymptotically flat coordinates,we must define the following new variables:

$$\tilde{r}_1 = \tilde{R}\frac{\sqrt{2(1+x)}}{x-y}, \quad \tilde{r}_2 = \tilde{R}\frac{\sqrt{-2(1+y)}}{x-y},$$
$$\tilde{R}^2 = R^2\frac{1-\lambda}{1-\nu}, \quad \left(\tilde{\psi}, \tilde{\phi}\right) = \frac{2\pi}{\Delta\psi}\left(\psi, \phi\right) \tag{5.8}$$

with this coordinates, the metric (5.1) asymptotes to:

$$d\mathrm{x}^2_{(4)} = d\tilde{r}_1^2 + \tilde{r}_1^2 d\tilde{\phi}^2 + d\tilde{r}_2^2 + \tilde{r}_2^2 d\tilde{\psi}^2 \tag{5.9}$$

which is, indeed, flat.

Let us turn back to the general analysis of the metric (5.1). From the definition of F in (5.2), one can see at once that it vanishes for $y = -1/\lambda$. At this point, the norm of the vector $\partial/\partial t$ becomes null, it changes from timelike to spacelike (an *ergosurface* so to speak) according to the change in the sign of $F(y)$. One can easily see that, for the parameters' ranges above stated, $F(x)$ cannot possibly vanish so the norm is well defined. We could worry about the regularity of the metric and its inverse at this point but one can check that they are smooth there.

Nevertheless, the metric blows up at $y = -1/\nu$ which is a root of $G(y)$. This can be shown to, merely, be a coordinate effect which can be cured by the change of coordinates $(t, \psi) \to (v, \psi')$:

$$dt = dv - C\ R\frac{1+y}{G(y)\sqrt{-F(y)}}dy, \quad d\psi = \psi' + \frac{\sqrt{-F(y)}}{G(y)}dy \tag{5.10}$$

the metric is now:

$$ds^2 = -\frac{F(y)}{F(x)}\left(dv - C\ R\frac{1+y}{F(y)}d\psi'\right)^2$$
$$+\frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi'^2 + 2\frac{d\psi' dy}{\sqrt{-F(y)}} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right] \tag{5.11}$$

which is regular at $y = -1/\nu$. Moreover, $y = -1/\nu$ is a Killing horizon with Killing vector:

$$V = \frac{\partial}{\partial v} + \Omega\frac{\partial}{\partial\hat{\psi}} \tag{5.12}$$

where $\hat{\psi} \equiv (2\pi/\Delta\psi)\psi'$ and Ω is the angular velocity of the horizon with respect to the observers in the asymptotic infinity and is given, in terms of the solution's parameters, by:

$$\Omega = \frac{1}{R}\sqrt{\frac{\lambda - \nu}{\lambda(1+\lambda)}} \tag{5.13}$$

The real spacelike singularity of the metric, where the invariant $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ diverges, lies at $y = -\infty$.

Now we will digress a little on a striking feature of higher dimensional BHs. This is the breaking of the uniqueness, or *no hair theorem*, which is known to hold in 4-dimensional spacetime cases. For this, we need to compute firstly the chractrerizing parameters of the BH such as the mass, angular momentum and so on. We can use the approach of comparison with linearized gravity given by [14] that has already been used in previous sections to find, for the Black Ring:

$$\begin{aligned} M &= \frac{3\pi R^2}{4G}\frac{\lambda}{1-\nu} \\ J &= \frac{\pi R^3}{2G}\frac{\sqrt{\lambda(\lambda-\nu)(1+\lambda)}}{(1-\nu)^2} \\ \mathcal{A}_{Hor} &= 8\pi^2 R^3 \frac{\nu^{3/2}\sqrt{\lambda(1-\lambda^2)}}{(1-\nu)^2(1+\nu)} \\ T &= \frac{1}{4\pi R}(1+\nu)\sqrt{\frac{1-\lambda}{\lambda\nu(1+\lambda)}} \end{aligned} \tag{5.14}$$

where, for the last two, we have made use of the definition of surface gravity (3.17). In order to bring to the surface the violation of the uniqueness, we need some dimensionless parameters with respect to which we can confront two solutions. These parameters are the reduced horizon area a_H and the square of the reduced spin j which are defined as:

$$j^2 \equiv \frac{27\pi}{32G}\frac{J^2}{M^3}, \quad a_H \equiv \frac{3}{16}\sqrt{\frac{3}{\pi}}\frac{\mathcal{A}_{Hor}}{(GM)^{3/2}} \tag{5.15}$$

We con express a_H in terms of j for the Black Ring solution as well as for MPBH. For the Black Ring we find:

$$a_H = 2\sqrt{\nu(1-\nu)}, \quad j^2 = \frac{(1+\nu)^3}{8\nu} \qquad \text{(Black Ring)} \tag{5.16}$$

with $0 < \nu \leq 1$. For the MPBH, we can use the expressions derived in the previous chapter and the ones found in [14] to find:

$$a_H = 2\sqrt{2(1-j^2)} \qquad \text{(MPBH)} \tag{5.17}$$

The plot of a_H against j^2 is shown below in Figure 5.1 and it has been taken from [39].

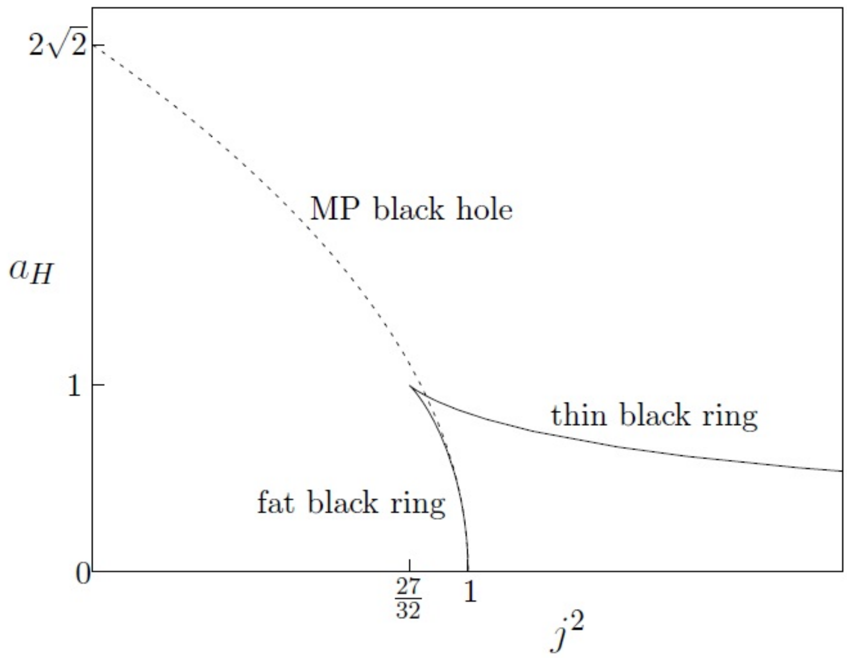

Figure 5.1: We can appreciate that, for the range $\frac{27}{32} \leq j^2 \leq 1$, there are up to three different solutions for a given value of j^2. There is also a point at which two different solutions, BR and MPBH, co-exist with the same j^2 as well as the same a^2. Also, note that, in this range, the reduced area parameter a_H is always larger for the MPBH than for the BR which makes the former a more stable solution then the latter.

A close examination of the exact solutions above tells us the ranges of j^2 where solutions of MPBH and BR are found:

$$\begin{aligned} j^2 &< 1 \quad \text{(MPBH)} \\ j^2 &\geq \frac{27}{32} \quad \text{(BR)} \end{aligned} \tag{5.18}$$

from which it is clear that in the range:

$$\frac{27}{32} \leq j^2 < 1 \tag{5.19}$$

we will find more than one solution for a given value of j^2 (same mass and angular momentum), in particular there will be three: one MPBH, one *fatty* Black Ring and one *skinny* Black Ring. However, these solutions have different values for the reduced area parameter a_H.

As pointed out below the figure, in the range $0 \leq j^2 \leq 1$, where both solutions coexist, there are only two points for which these solutions have exactly the same values of j^2 and a_H^2, namely: $j^2 = a_H^2 = 8/9$ and $j^2 = 1$ for which $a_H^2 = 0$, the latter corresponding to a naked singularity. Thus, violating BH uniqueness. Besides, we can see from the graphic that the MPBH solution is constantly above the fat BR solution. However, it is precisely at the point $j^2 = a_H^2 = 8/9$ where it changes from being *above* the thin BR solution to being *below* it. These two facts suggest that, depending on the value of j^2, one solution will dominate over the other in case we perturb the most unestable. As we will see later, the BR solution can develop an unstable perturbation leading to its fragmentation, but we postpone these study for the next section. Finally, we woul like to draw the reader's attention to the fact that this stability behaviour changes when we consider both solutions in higher dimensions, $D \geq 6$, and take the MPBH to the ultraspinning regime and compare it to the thin ultraspinning BR solution, the result is that the thin BR solution dominates over the MPBH over the whole range of j^2. See [37] for the details of the calculation. Sadly, it is impossible to compare this result to the 5-dimensional spacetime case because of the dynamical Kerr bound which exists for this case and prevents us from taking it to the ultraspinning regime. (See paragraph below (4.15)). In the figure, this bound corresponds to taking $j^2 = 1$ where both solutions become singular with zero area (naked singularity). Notwithstanding this, the thin BR branch avoids this point and might be taken to the ultraspinning limit.

5.2 From the Boosted Black String to the Black Ring

Let us begin by considering the metric (5.1) in the following coordinates:

$$r = -\frac{R}{y}, \quad \cos\theta = x \tag{5.20}$$

in the ranges:

$$0 \leq r \leq R, \quad 0 \leq \theta \leq \pi \tag{5.21}$$

We also rename the parameters:

$$\nu = \frac{r_o}{R}, \quad \lambda = \frac{r_o \cosh^2\sigma}{R} \tag{5.22}$$

It gives a very ugly expression for the metric (5.1):

$$ds^2 = -\frac{\hat{f}}{\hat{g}}\left(dt - r_o \sinh\sigma \cosh\sigma\sqrt{\frac{R + r_o \cosh^2\sigma}{R - r_o \cosh^2\sigma}}\frac{\frac{r}{R} - 1}{r\,\hat{f}}Rd\psi\right)^2 +$$

$$+\frac{\hat{g}}{\left(1 + \frac{r\,\cos\theta}{R}\right)^2}\left[\frac{f}{\hat{f}}\left(1 - \frac{r^2}{R^2}\right)R^2 d\psi^2 + \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)f} + \frac{r^2}{g}d\theta^2 + \frac{g}{\hat{g}}r^2 \sin^2\theta d\phi^2\right] \quad (5.23)$$

with:

$$f = 1 - \frac{r_o}{r}, \quad \hat{f} = 1 - \frac{r_o{}^2\sigma}{r}$$

$$g = 1 + \frac{r_o}{R}\cos\theta, \quad \hat{g} = 1 + \frac{r_o \cosh^2\sigma}{R}\cos\theta \quad (5.24)$$

When we take the limit:

$$r,\ r_o,\ r_o \cosh^2\sigma \lll R \quad (5.25)$$

we have $g, \hat{g} \approx 1$ and defining $z \equiv R\psi$ we obtain:

$$ds^2 = -\hat{f}\left(dt + \frac{r_o \sinh 2\sigma}{2r\hat{f}}dz\right)^2 + \frac{f}{\hat{f}}dz^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 \quad (5.26)$$

which we can rearrange to obtain the expression of a boosted Black String along the z axis [37] with $n = 1$ and $u^z \equiv v = \tanh\,\sigma$ the velocity of the boost along the z direction:

$$ds^2 = -\hat{f}dt^2 - 2\frac{r_o}{r}\cosh\sigma \sinh\sigma\ dtdz + \left(1 + \frac{r_o}{r}\sinh^2\sigma\right)dz^2$$
$$+\frac{dr^2}{f} + r^2 d\Omega_2^2 \quad (5.27)$$

(See (C.6)) The ADM stress-energy tensor for the bosted Black String is [30]:

$$T_{tt} = \frac{r_o}{16\pi G}\left(\cosh^2\sigma + 1\right)$$
$$T_{tz} = \frac{r_o}{16\pi G}\cosh\sigma \sinh\sigma$$
$$T_{zz} = \frac{r_o}{16\pi G}\left(\sinh^2\sigma - 1\right) \quad (5.28)$$

The characteristic parameters of the boosted Black String are given [37] by:

$$M = \int_{S^1\times S^2} T_{tt}$$
$$J = R\int_{S^1\times S^2} T_{tz}$$
$$\mathcal{A}_{Hor} = 2\pi R r_o^2 4\pi\cosh\sigma = 8\pi^2 r_o^2 R\,\cosh\sigma \quad (5.29)$$

Up to this point, we have been dealing with a boosted Black String, but we are interested in a Black Ring so we must impose some condition on the Black String in order to curve it into a Black Ring and prevent it from collapsing due to its own gravitational attraction. This equilibrium condition, in order to have a finite and constant radius of the R, is the absence of pressure tangential to the surface:

$$\frac{T_{zz}}{R} = 0 \Longrightarrow \sinh^2\sigma = \frac{1}{n} \tag{5.30}$$

Taking this into account, equations (5.29) yield:

$$\begin{aligned} M &= \frac{3\pi}{2G} r_o\, R \\ J &= \frac{\pi}{\sqrt{2}G} r_o\, R^2 \\ \mathcal{A}_{Hor} &= 8\sqrt{2}\pi^2 r_o^2 R \end{aligned} \tag{5.31}$$

5.3 Black 1-Brane Limit

Let us first derive the equilibrium condition for a Blackfold not to collapse under gravitational attraction when we curve it into a ring. This condition will be related to the tension of the Blackfold so we will first derive an expression for the total tensional energy of the Blackfold in terms of its stress-energy tensor:

$$\mathcal{T}_{tot} \equiv -\int_{\mathcal{B}_p} dV_p R_0 \left(\gamma^{ab} + n^a n^b\right) T_{ab} \tag{5.32}$$

which can be shown (See (C.2) and (C.4)) to be equal to:

$$\mathcal{T}_{tot} = (D-3)M - (D-2)\left(TS + \sum_{i=1}^{p} \Omega_i J_i\right) \tag{5.33}$$

The integrated Smarr generalized formula for asymptotically flat vacuum BH in a D-dimensional spacetime [14] is precisely:

$$(D-3)M - (D-2)\left(TS + \sum_{i=1}^{p} \Omega_i J_i\right) = 0 \tag{5.34}$$

and must be recovered when the extrinsic equations are satisfied for a Blackfold with spatial section $\mathcal{B}_p$ in a Minkowskian background where one has $R_0 = 1$. In the light of (5.33), this is equivalent to requiring the total tensional energy to vanish:

$$\mathcal{T}_{tot} = 0 \tag{5.35}$$

Let us consider now an stationary Black 1-Brane in a 5-dimensional spacetime Minkowskian background. The stationarity killing vector is $\xi = \partial/\partial t$ so $R_0 = 1$. The ring has an axis of symmetry with its corresponding isometry and therefore the integrand in (5.32) can be taken out of the integral so that the total tensional energy is simply equal to the integrand up to an irrelevant constant factor:

$$\mathcal{T}_{tot} \propto -\left(\gamma^{ab} + n^a n^b\right) T_{ab} \tag{5.36}$$

where we have taken into account that for the Black Ring we have $n_a = \xi_a$. Further, we denote $-\xi^a u_a = \cosh\sigma$, with σ being the rapidity of the boost with respect to the stationary observers in the asymptotic infinity which follow the orbits of ξ. If we plugg the results (3.10) and (3.11) for the Blackfold with $p = n = 1$ in (5.36) it is straightforward to obtain (recall that: $\gamma^{ab}\gamma_{ab} = \delta^a{}_a = p + 1$):

$$0 = \left(\gamma^{ab} + \xi^a \xi^b\right) \left(\left(\varepsilon + P\right) u_a u_b + P\gamma_{ab}\right) = \varepsilon \sinh^2\sigma + P \cosh^2\sigma \tag{5.37}$$

which is equivalent to:

$$\tanh^2\sigma = -\frac{P}{\varepsilon} \tag{5.38}$$

for the specific Blackfold we are dealing with, and with the aid of (3.11), we find that (5.38) gives:

$$\sinh^2\sigma = \frac{1}{n} \tag{5.39}$$

which is, precisely, (5.30), the equilibrium condition we found neccesary for the Black Ring not to collapse under gravitational attraction. Thus, we have recast the problem of the Black Ring by using the Blackfold approach. This can be further confirmed by computing the characteristic magnitudes of the Blackfold: mass, angular momentum and so on and comparing them to the exact solution for the Black Ring. Here we content ourselves with doing the most straightforward, the mass. A computation to higher orders in r_o/R can be found in [37]. Needless to say, the mass is found to be:

$$M = \frac{3\pi}{2G} r_o\, R \tag{5.40}$$

(See C.5)

Chapter 6

Gregory-Laflamme Instability

In this chapter we will study how an unstable perturbation might be developed, provided certain conditions hold, in higher dimensional Black Holes, in particular we will work out in detail the neutral BR case in $D = 5$. It is possible to study the same Gregory-Laflamme (GL) kind of instability in MPBHs by taking them to the membrane limit and using the Blackfold approach [36]. We leave this for the last sections of this chapter, where we illustrate how instabilities may also be accounted for by using the Blackfold paradigm we have been studying so far.

6.1 Thermodynamical argument for instability

For the sake of brevity, we will present the result (it can be derived rigorously with all the machinery developed) that motivates the study of the instability and we refer the reader to the original papers by Gregory [42] and jointly with Laflamme [43] for a more detailed discussion and motivation which goes over some of the points we have seen.

Following a similar argument to that developed in the introduction when we discussed the *hierarchy problem*, we can find the relationship between Newton's constant in the 5 and 4 dimensional cases to be: $G_5 = G_4\ell = \ell$, where we have taken $G_4 = 1$ and ℓ is the characteristic length of the extra dimension. Bearing this in mind, we compute the entropy of the 5-dimensional Schwarzschild (MP non-rotating BH) and Black String by using (3.12) with $n = 1$ for the Black String and $n = 2$ for the Schwarzschild BH.

Also, use the relationship amongst the Newton's constants to finally obtain:

$$S_{Sch} = \frac{\pi^2 r_o^3}{2\,\ell}, \qquad S_{BS} = \pi r_+^2 \tag{6.1}$$

where r_o and r_+ are the horizon radii for the Schwarzschild BH and for the BS respectively. We now need to make use of the mass formulae for these BHs so that we can compare the entropy for the same mass. The masses are found by simply using the first of (3.11) for the BS and (2.18) for the Schwarzschild:

$$M_{Sch} = \frac{3\pi r_o^3}{2\ell}, \qquad M_{BS} = \frac{r_+}{2} \tag{6.2}$$

after setting the masses equal we can rearrange to finally get:

$$S_{Sch} = 4\pi M^2 \sqrt{\frac{8\ell}{27\pi M}}, \qquad S_{BS} = 4\pi M^2 \tag{6.3}$$

so it is clear that if we take ℓ large enough the entropy of the Schwarzschild solution will be larger than that of the Black String. Therefore, when perturbing the Black String (or a Black Ring which, as we saw, is nothing but a closed finite Black String) we could expect the solution to evolve into a more stable configuration such as the Schwarzschild.

6.2 Perturbation of the $D = 5$ Neutral Black String

We define the perturbation fo the Black String metric (3.3) with $n = p = 1$ by:

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \tag{6.4}$$

with this, the Ricci tensor gets an extra contribution:

$$R_{\mu\nu} \to R_{\mu\nu} - \frac{1}{2}\Delta_L h_{\mu\nu} \tag{6.5}$$

where Δ_L is the Lichnerowicz operator defined by:

$$\Delta_L h_{\mu\nu} \equiv \Box h_{\mu\nu} + 2R_{\mu\sigma\nu\rho}h^{\sigma\rho} - 2R^{\sigma}_{(\mu}h_{\nu)\sigma} - 2\nabla_{(\mu|}\nabla^{\sigma}h_{|\nu)\sigma} + \nabla_{\mu}\nabla_{\nu}h^{\sigma}{}_{\sigma} \tag{6.6}$$

Since we are working in the vacuum, the Ricci tensor still satisfies the vacuum EE and we conclude that the perturbation must obey $\Delta_L h_{\mu\nu} = 0$. The fact that the Riemann tensor do not have z components simplifies the equations considerably and, since we are in the vacuum, we can choose the *transverse trace-free* gauge for the perturbation which is set by the condition:

$$\nabla_{\mu}h^{\mu}{}_{\nu} = h^{\sigma}{}_{\sigma} = 0 \tag{6.7}$$

and further simplifies the perturbation equation to:

$$\Delta_L h_{\mu\nu} = 0 \rightarrow \Box h_{\mu\nu} + 2R_{\mu\sigma\nu\rho}h^{\sigma\rho} = 0 \tag{6.8}$$

One can find the complete expressions for the equations in the apendix of the original paper by Gregory and Laflamme [43]. Here we will content ourselves with writing down the most relevant ones for our calculation.

The Black String has both time and z-translation invariance as well as $SO(3)$ invariance due to the symmetry of the Schwarzschild piece of the solution. The main contribution to the perturbation will be spherically symmetric too so the cross terms with angular dependece vanish. The general form of the perturbation may be writen:

$$h_{\mu\nu} = \mathcal{R}e \left\{ e^{\Omega t} e^{i\mu z} \begin{pmatrix} h_{tt} & h_{tr} & 0 & 0 & h_{tz} \\ h_{tr} & h_{rr} & 0 & 0 & h_{rz} \\ 0 & 0 & h_{\theta\theta} & 0 & 0 \\ 0 & 0 & 0 & h_{\theta\theta}\sin^2\theta & 0 \\ h_{tz} & h_{rz} & 0 & 0 & h_{zz} \end{pmatrix} \right\} \tag{6.9}$$

where we have been able to factor out the t and z dependence because of the symmetries and the separation of variables method. The condition for instability is $\omega = -i\Omega$ with $\Omega > 0$ so that we have $e^{i\omega t} = e^{\Omega t}$.

For the angular part of the metric $h_{\theta\theta}$ we note that higher angular momenta modes are more stable than zero angular momentum ones (or s-modes) and therefore we will restrict our analysis to this s-mode so that $h_{\theta\theta} = K(r)$ where $K(r)$ is the radial function of an s-wave.

Let us look at the components with z dependence. We will illustrate the case for h_{zz} since the cases for $h_{\mu z}$ are analogous. The ODE satisfied by h_{zz} is:

$$\frac{\partial^2 h_{zz}}{\partial r^2} + \left(\frac{r + (r - r_+)}{r - r_+}\right)\frac{1}{r}\frac{\partial h_{zz}}{\partial r} - (\mu^2 r(r - r_+) + \Omega^2 r^2)\frac{h_{zz}}{(r - r_+)^2} = 0 \tag{6.10}$$

To find the behaviour of the solution at infinity we take the limit $r \rightarrow \infty$ and we get:

$$\frac{\partial^2 h_{zz}}{\partial r^2} - (\mu^2 + \Omega^2)h_{zz} = 0 \tag{6.11}$$

So the solution is:

$$r \rightarrow \infty \qquad h_{zz} \sim e^{\pm\sqrt{\mu^2+\Omega^2}r} \tag{6.12}$$

The other limit is $r \to r_+$, so taking the most divergent parts:

$$\frac{\partial^2 h_{zz}}{\partial r^2} - \frac{\Omega^2 r_+^2}{(r-r_+)^2} h_{zz} = 0 \tag{6.13}$$

We find that the solution is:

$$r \to r_+ \qquad h_{zz} \sim (r-r_+)^{\pm\Omega r_+} \tag{6.14}$$

One can see at once that the regular solution vanishes at both infinity and at the horizon. If h_{zz} is non-zero it will have a turning point in the range $r_+ \leq r \leq \infty$ and the sign of $\frac{\partial_r^2 h_{zz}}{h_{zz}}$ must also change in this interval. Let us solve for $\frac{\partial_r^2 h_{zz}}{h_{zz}}$ in the equation (6.10):

$$\frac{\partial_r^2 h_{zz}}{h_{zz}} = \frac{(\mu^2 r(r-r_+) + \Omega^2 r^2)}{(r-r_+)^2} - \left(\frac{r + (r-r_+)}{r-r_+}\right) \frac{1}{r} \frac{\partial_r h_{zz}}{h_{zz}} \tag{6.15}$$

But, at the turning point, the slope of the function is zero and so $\partial_r h_{zz} = 0$ which leaves us simply with:

$$\frac{\partial_r^2 h_{zz}}{h_{zz}} = \frac{(\mu^2 r(r-r_+) + \Omega^2 r^2)}{(r-r_+)^2} \tag{6.16}$$

which is positive for any $r \in [r_+, \infty)$. Therefore, $h_{zz} = 0$ throughout the whole interval. An analogous reasoning leads to the conclusion that $h_{tz} = h_{rz} = 0$, the complete equations may be found in the apendix of [43].

We are thus left with h_{tt}, h_{rr} and h_{tr} with which we can define new functions:

$$\begin{aligned} h_\pm &= \frac{h_{tt}}{V} \pm V\, h_{rr} \\ h &= h_{tr} \end{aligned} \tag{6.17}$$

where $V = (1 - \frac{r_+}{r})$. After imposing the gauge conditions the equations of motion for the three old functions reduce to a pair of ODEs and a constraint amongst the new functions:

$$\begin{aligned} h_+ &= \frac{h_-}{V} \frac{(2r^2\Omega^2 + r^2\mu^2 V - \frac{(1-V^2)}{2})}{(r^2\mu^2 + 1 - V)} - \frac{rh}{\Omega} \frac{(4\Omega^2 + \mu^2(1-3V))}{(r^2\mu^2 + 1 - V)} \\ \partial_r h &= \frac{\Omega(h_+ + h_-)}{2V} - \frac{(1+V)h}{rV} \\ \partial_r h_- &= \frac{\mu^2 h}{\Omega} + \frac{h_+}{r} + \frac{(1-5V)h_-}{2rV} \end{aligned} \tag{6.18}$$

The asymptotic behaviour is similar to the h_{zz} case but with different coefficients:

$$r \to \infty : \quad \begin{cases} h \sim \pm\sqrt{\mu^2+\Omega^2}\; e^{\pm\sqrt{\mu^2+\Omega^2}r} \\ h_- \sim \frac{\mu^2}{\Omega}\; e^{\pm\sqrt{\mu^2+\Omega^2}r} \end{cases} \tag{6.19}$$

$$r \to r_+ : \quad \begin{cases} h \sim (\pm\Omega r_+ - \frac{1}{2})(r-r_+)^{\pm\Omega r_+ - 1} \\ h_- \sim (\frac{\mu^2}{\Omega} \pm \frac{2}{r_+})(r-r_+)^{\pm\Omega r_+} \end{cases} \tag{6.20}$$

So an instability corresponds to a solution of the above equations together with the constraint and the asymptotic behaviour shown. Also, we must note that when $\Omega r_+ > 1$ we can use a similar argument to that used for the h_{zz} perturbation to show that the perturbations vanish and hence there is not instability. However, when $\Omega r_+ < 1$ it can be shown [43] that a turning point exist and we might integrate the equations numerically to find a solution, if it exists, which leads to an instability. Summarizing the results obtained so far, the perturbation looks like:

$$h_{\mu\nu} = \mathcal{R}e \left\{ e^{\Omega t} e^{i\mu z} \begin{pmatrix} \frac{V}{2}(h_+ + h_-) & h & 0 & 0 & 0 \\ h & \frac{1}{2V}(h_+ - h_-) & 0 & 0 & 0 \\ 0 & 0 & K(r) & 0 & 0 \\ 0 & 0 & 0 & K(r)\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \tag{6.21}$$

Now, we turn to the issue of boundary conditions which is a key ingredient of the problem. First of all, we need to work with a metric which is regular at the horizon since the Cauchy surface upon which we will impose the initial data neccesarily touches it. We will use Kruskal null coordinates to get a regular metric at the horizon. Formulae for the general case in D spacetime dimensions may be found in [43], here we restrict ourselves to the $D = 5$ case we are working with. Let us begin by defining the tortoise coordinate for the Black String. It turns out to be the same as for the 4-dimensional Schwarzschild BH ($D = 4$) since the z does not enter the expressions for the horizon. The procedure is standard and may be found in most books on GR but we will give the outline here for completeness. So we have the tortoise coordinate:

$$r^* = r + r_+ \log(r - r_+) \tag{6.22}$$

The Kruskal null coordinates are thus defined:

$$\begin{aligned} U &= e^{\frac{t+r^*}{2r_+}} \\ V &= -e^{-\frac{t-r^*}{2r_+}} \end{aligned} \tag{6.23}$$

So that the divergent part of the metric looks like:

$$ds^2 = \frac{4r_+^2 e^A}{UV} dU dV + ... \tag{6.24}$$

where A is an adequate constant chosen in such a way that the metric is finite at r_+. Now we perform a last change of coordinates given by:

$$\begin{aligned} R &= U - V \\ T &= U + V \end{aligned} \tag{6.25}$$

and the metric finally turns out to be:

$$ds^2 = \frac{16r_+^2 e^A}{T^2 - R^2}(dT^2 - dR^2) + ... \tag{6.26}$$

this metric is completely regular at the horizon and it is the one we should be working with when performing the calculations in the computer. The expressions of the perturbation $h_{\mu\nu}$ in these new coordinates is obtained by the standard procedure of second rank tensor transformation, the final result may be found at [43] but we omit it here for being irrelevant to the following considerations.

We are left only with the problem of specifying the Cauchy surface for the initial data. This surface must touch $\mathcal{I}^+$ as well as the future horizon $\mathcal{H}^+$, or even the neck of the Schwarzschild wormhole. The option that it touches the past horizon is ruled out by purely physical reasons since a past horizon is never formed in a process of gravitational collapse. We present a schematic picture in Figure 6.1 below.

Now, we could implement a programm in a computer to solve the equations (6.18) with the asymptotic behaviour given by (6.19) and (6.20). We pick out the exponentially decaying branch of them and the one with the positive sign in the exponent so that the perturbation is finite at infinity. The procedure is carried out by fixing μ and finding values of Ω for which a solution of the equations with the desired behaviour exists. We present the graph for the case under consideration below in Figure 6.2.

It is not hard to see that there is a threshold value μ^{GL} above which the instability does not exist. This value of the critical mass can be related [47] to a critical GL wave number which can de related in turn to a critical wavelength. Also, for $\mu \to 0$ it is apparent that Ω does not vanish. One can argue and show that the mode is pure gauge and we can simply ignore it since it is not physical, see [43] for a discussion. However, we will give a very brief idea of how one can show wether a mode is pure gauge or physical.

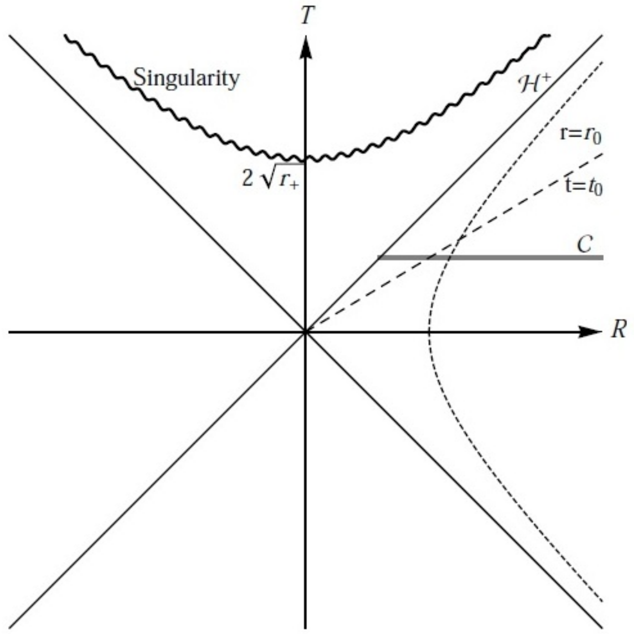

Figure 6.1: $\mathcal{C}$ represents the Cauchy surface upon which we impose the initial data. Figure courtesy of Gregory and Laflamme [43].

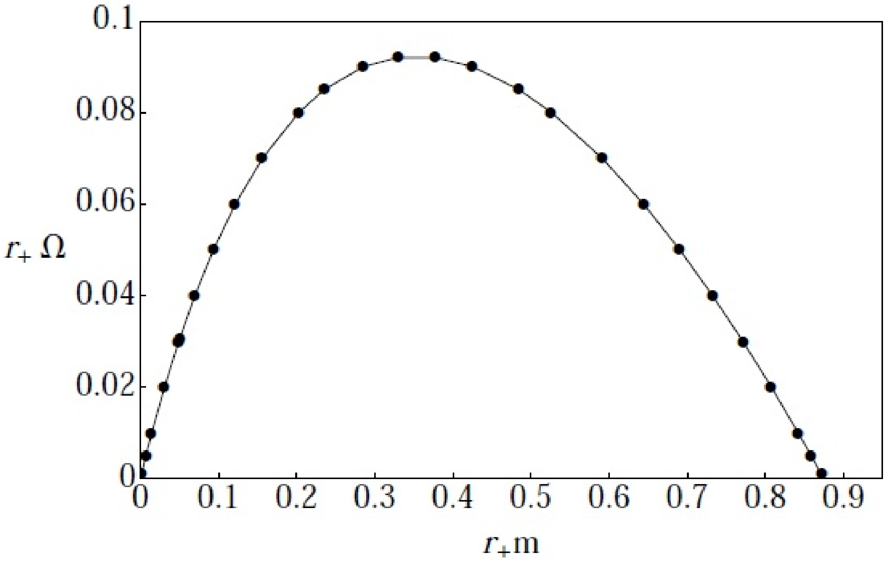

Figure 6.2: μ (m in the graphic) and Ω have been rescaled. Static Black String in $D = 5$ with $n = 1$. Taken from [43].

Since the perturbation h_{zz} and the components $R_{z\mu\nu\sigma}$ of the Riemann tensor vanish, equation (6.8) reduces to the 4-dimensional Lichnerowicz operator plus a mass term:

$$\Delta_L h_{\mu\nu} = \left\{ \Delta_L^{(4)} + \frac{\partial^2}{\partial z^2} \right\} h_{\mu\nu} = \left\{ \Delta_L^{(4)} + \mu^2 \right\} h_{\mu\nu} = 0 \tag{6.27}$$

If a mode is pure gauge it can only correspond to a 4-dimensional change of coordinates and therefore will not have any z dependence so the mass term will drop. Thus, we conclude that any solution of the *massive* 4-dimensional Lichnerowicz operator must be a physical mode.

This threshold has the important consequence of fixing a limit for the dimensions of the BH or Black String to develop a GL instability. The size of the horizon r_+ must be large enough to accomodate the modes of λ^{GL}. It is interesting to look closer at the case when the extra dimension is periodically identified. In this case, the values of μ will be quantized as $\mu = \frac{2\pi n}{L}$ with $n = 1, 2, ...$ One can readily see that, if L is large enough, the first value of μ will be larger than the threshold μ^{GL} and the system will not develop any instability whatsoever. In the context of String Theory, this has the consequence of breaking the duality that is obtained by replcing the radius of the closed extra dimensions L by $1/L$. For appropriate values of L we may have an instability while, when going to the spacetime where, instead, we take $1/L$, we may have none. Winding modes could be developed, but we will not be concerned with more complicated scenarios here ([43]).

To close this section, we will try to give an intuitive idea of what is happening by working out the approximate position of the apparent horizon. We can anticipate that we will have an ocillatory behaviour somehwere because of the exponentials in the solution but let us proceed in more detail. We will work in Eddington-Finkelstein coordinates:

$$ds^2 = -\frac{(r - r_+)}{r} du^2 + 2du dr + r^2 d\Omega_2^2 + dz^2 \tag{6.28}$$

where $u = t + r + r_+ \log(r - r_+)$. The radial lightrays for this metric are given by:

$$\begin{cases} u = \text{const.} & \text{incoming} \\ \frac{dr}{du} = \frac{r - r_+}{2r} & \text{outgoing} \end{cases} \tag{6.29}$$

When the metric is not perturbed, the apparent horizon is at $r = r_+$. Once we perturb the metric, the radial geodesics equation for the outgoing lightrays becomes:

$$h_{uu} - \frac{r - r_+}{r} + 2\frac{dr}{du}(1 + h_{ur}) + h_{rr}\left(\frac{dr}{du}\right)^2 = 0 \tag{6.30}$$

from which we can solve for $\frac{dr}{du}$ to first order and get:

$$\frac{dr}{du} \approx \frac{r-r_+}{2r} - \frac{r-r_+}{2r}h_{ur} - \frac{1}{2}h_{uu} - \frac{(r-r_+)^2}{8r^2}h_{rr} \tag{6.31}$$

Finding the zero for the above equation will give us the approximate position of the apparent horizon:

$$r \approx r_+(1+h_{uu}) + \lim_{r\to r_+}\left[(r-r_+)h_{ur} + \frac{(r-r_+)^2}{4r_+}h_{rr}\right] \tag{6.32}$$

Thus, taking into account the form of the perturbation (6.21), taking the limit and the real part, we find the approximate position of the apparent horizon:

$$r \approx r_+ + const.\,\cos(\mu z) \tag{6.33}$$

when we include one of the angular variables we obtain a surface generated by rotating about the z axis the cosine function shifted by r_+. It looks like a cylinder with its surface distorted in a way given by the cosine function. See [42] or [43] for graphics. The details of the above calculations may be found in the corresponding appendix at the end of the dissertation.

A final comment on more general settings for the instability such as charged solutions: these cases can also develop a GL type of instability though the equations are much more cumbersome to deal with and we must take into account the perturbation of the extra fields: the dilaton and the "magnetic" field strength. The *low charge* and the *fully charged* cases have been studied in detail in [43] but the main features of the GL are equally illustrated by the neutral case which can be worked out in detail much more easily.

6.3 Final state of the Gregory-Laflamme Instability

The final state of the GL instability has remained a mistery for years since the only way to deal with it was by resorting to numerical methods, it was not before a few years after the publication of the first papers by Gregory and Laflamme that some simulations were carried out and we were able to gain some insight into the evolution of the instability. Happily, it turned out to be quite what Gregory and Laflamme had expected: the Black String fragmented into an infinite collection of isolated BHs. However, quantum gravity is neccesary to account for and deal with the naked singularites that arise when the pinch-off of the Black String takes place and the BHs break "loose".

We will present a summary of the most striking features of the evolution of a 5-dimensional Black String as given in [44]. We will not linger in the details of the implementation of the program but refer the reader to the relevant paper for a more complete account of it.

The results we are going to present here correspond to a 5-dimensional Black String in vacuum perturbed in the direction of the string. For calculation purposes, it is convenient to use *harmonic coordinates* which, by imposing certain gauge conditions, satisfy:

$$\nabla^{\nu}\nabla_{\nu}x^{\mu} = 0 \tag{6.34}$$

Besides, they are particularly well adapted to Black Strings since they are regular at the horizon. In order to see what the final state is, we will look at the apparent horizon (AH) of the Black String which is almost indistingushible from the event horizon since the latter cannot be defined properly due to the existence of naked singularities as the Black String radius shrinks to zero size.

To study the dynamics of the AH several variables, which have intrinsic meaning for the Black String, are monitored, these are:

- The apparent horizon radius $R_{AH}(t, z)$.
- The total horizon area $A(t)$.
- The two curvature invariants defined by $I \equiv R_{\nu\tau\sigma\mu}R^{\nu\tau\sigma\mu}$ and $J \equiv R_{\nu\tau\alpha\beta}R^{\alpha\beta\sigma\mu}R_{\sigma\mu}{}^{\nu\tau}$.

For computational and graphical purposes we rescale the last two as:

$$K \equiv \frac{IR_{AH}^4}{12}, \quad S \equiv 27(12J^2I^{-3} - 1) + 1 \tag{6.35}$$

Figure 6.3 shows a graphic of the $A(t)$ rescaled to the original total horizon area A_0 in terms of the time. We can see at once that it never decreases as it was expected from thermodynamics. The graphic shows the result for three different resolutions of the programm that was used and we refer the reader to [44] for a description of them. Following this, we have three different figures depicting slices of the AH where we can appreciate the evolution of the radius $R_{AH}(t, w)$. In the Figure 6.4, R stands for areal radius and the coordinate Z has been chosen in such a way that the proper length of the horizon in the spacetime direction z (for fixed t, θ and φ) is equal to the Euclidean length of $R(Z)$ of the diagram.

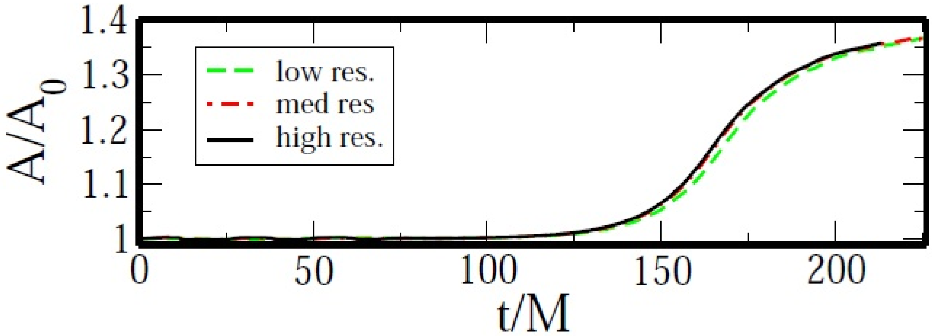

Figure 6.3: $A(t)$ rescaled to the original total horizon area A_0 *vs.* time t rescaled to the initial masss per unit length M. Figure taken from [44].

One can readily see how the fragmentation of the Black Strng is taking place in a self-similar cascade which will develop into a fractal structure. We have an infinite collection of BHs linked by increasingly thin pieces of Black String whose radius will eventually shrink to zero size thus unveiling a naked singularity. At each stage,each Black String segment develops a protrusion which eventually evolves into a BH fuelling the self-similar cascade. Finally, Figure 6.5 shows the evolution of the curvature invariants superposed to the plot of the AH radius. It is also clear that whenever we find a quasi-spherical protrusion in the string, whatever the size of it, we find the variables S and K to be ~ 6 whereas, they are ~ 1 for the Black String segments that link them.

Let us now proceed to the extrapolation of the above results to the final state. In order to do this, we need to know the time at which the Black String segments have not yet reached zero size so that we can still rely on the classical picture we have been using. At this time, (see the ubiquitous [44] for an estimate) we look at the various parameters we have been considering. The curvature invariants go like $\sim r^{-4}$ just outside the Black String segments and recalling that in harmonic coordinates the time is regular everywhere from some distance inside the AH outwards. Since this is precisely the time measured by an observer at infinity we see that as $r \to 0$ the curvature invariants diverge unveiling a *naked singularity* which is a violation of the *cosmic censorship hypothesis.*

To close this section, let us point out some striking resemblances of this problem. The way in which the Black String pinches-off and the self-

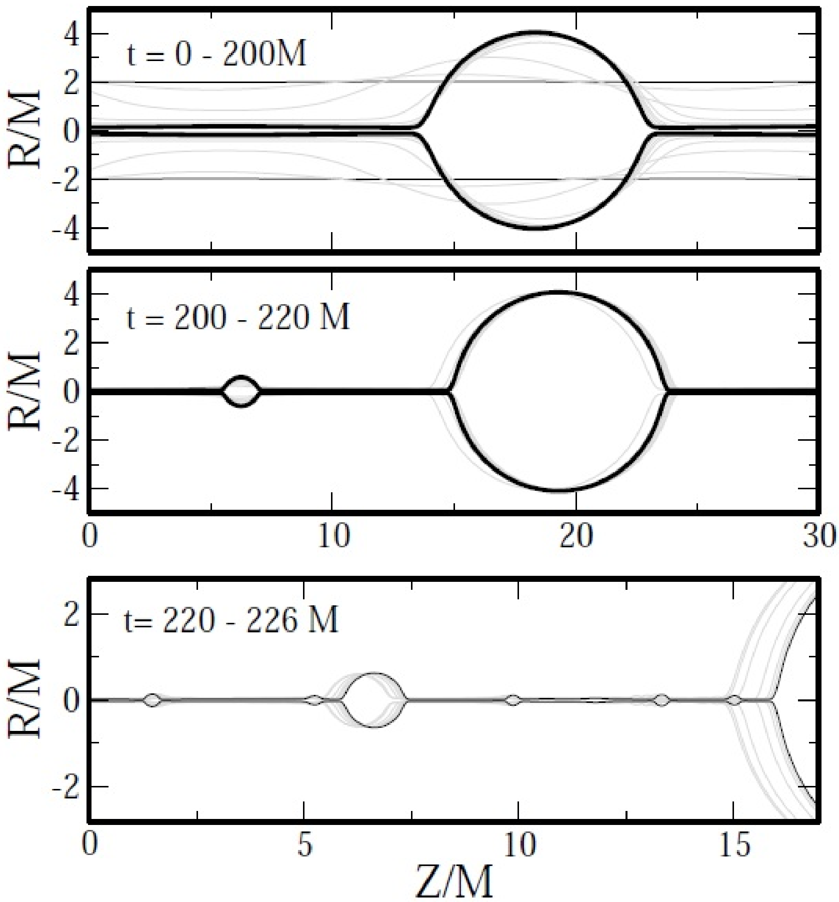

Figure 6.4: Areal radius R *vs.* embedding coordinate Z for various times. All of them are rescaled with the initial masss per unit length M. Figure taken from [44].

similar cascade is qualitatively the same as the one which takes place for the Rayleigh-Plateau problem (the following video [48] reflects perfectly the instability and as we predicted for the AH of the BH, before the fragmentation occurs its shape is that of a surface of revolution with its contour given by the cosine fucntion) and the radius of the fluid column goes like:

$$R \propto (t_c - t) \tag{6.36}$$

Using this fact and the data obtained in the calculation, Lehner and Pretorius have shown in their paper that the proper length of the AH, prior to pinch-off, is given by:

$$L_p(t) \propto (t_c - t)^{1-d} \tag{6.37}$$

where d stands for the Hausdorff dimension of the end state AH's shape.

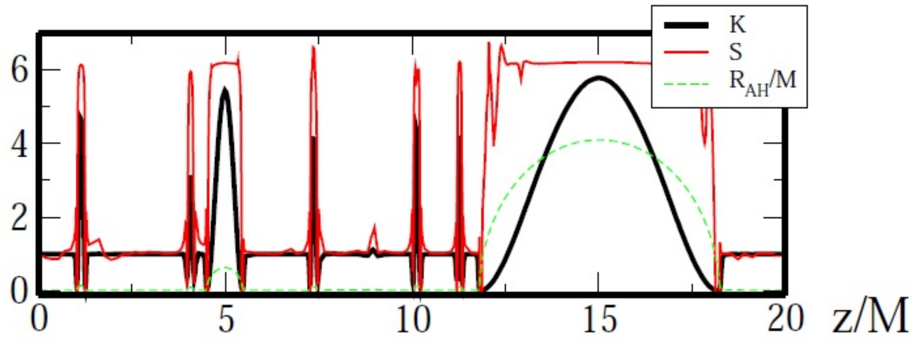

Figure 6.5: Values for the different curvature invariants K and S as well as for the areal radius R on the horizon at *vs.* the rescaled cartesian coordinate z. Figure taken from [44].

From the slope of L_p in Figure 6.5 we conclude that $d \simeq 1.05$ which corresponds to a fractal curve ($d = 1$ corresponds to a non-fractal curve) as we have been anticipating. This fractal structure is obtained by succesively replacing a string segment by a similar length segment with a small semi-circular protrusion.

6.4 Gregory-Laflamme Instability in Static Blackfolds

In this section we will illustrate how the Blackfold approach can also predict, to the lowest orders, the instability discovered by Gregory and Laflamme. We will begin by looking at the simplest case, namely, that of a simple fluid without any dispitavie effects and then we will turn to viscous fluids to go one order further in the μ expanssion of the disperssion relation.

We recall that we are working with perturbation wavelengths which are long compared to the chracteristic length of the horizon and still, small enough to make the worldvolume, parametrized by (t, z^i) with $i = 1, ..., p$, look flat $K^{\sigma}_{\mu\nu} \approx 0$ so:

$$r_o \lll \lambda \lll R \tag{6.38}$$

Working with an static fluid we introduce the following perturbations:

$$\delta\varepsilon, \quad \delta P = \frac{dP}{d\varepsilon}\delta\varepsilon, \quad \delta u^a = (0, v^i), \quad \delta X^m = \xi^m \tag{6.39}$$

where X^m are the transverse coordinates to the worldvolume which are held constant prior to the perturbation. The remaining transverse coordinates are held at constant value. Working at linear order in the perturbation, the stress-energy tensor of the Blackfold (3.10) gets the following contributions:

$$T^{tt} = \varepsilon + \delta\varepsilon, \quad T^{ti} = (\varepsilon + P)v^i, \quad T^{ij} = \delta^{ij}\left(P + \frac{dP}{d\varepsilon}\delta\varepsilon\right) \tag{6.40}$$

The extrinsic curvature is given, to first order, by:

$$\delta K_{ab}{}^{m} = \partial_a \partial_b \xi^m \tag{6.41}$$

On the one hand, the extrinsic eqeuations (3.30), yield:

$$(\varepsilon\partial_t^2 + P\sum_{i=1}^{p}\partial_i^2)\xi^\rho = 0 \tag{6.42}$$

See (D.17). Comparison with the wave equation:

$$\left(\Delta - \frac{1}{c_T^2}\partial_t^2\right)\xi^\rho = 0 \tag{6.43}$$

yields the transverse elastic oscillations:

$$c_T^2 = -\frac{P}{\varepsilon} \tag{6.44}$$

On the other hand, the intrinsic equations (3.31), can be recast as:

$$\partial_t^2 T^{tt} - \partial_i \partial_j T^{ij} = 0 \tag{6.45}$$

See (D.19). Which, according to (6.40), give:

$$\left(\partial_t^2 - \frac{dP}{d\varepsilon}\sum_{i=1}^{p}\partial_i^2\right)\delta\varepsilon = 0 \tag{6.46}$$

See (D.21). Again, comparison with the wave equation gives sound-mode oscillations:

$$c_L^2 = \frac{dP}{d\varepsilon} \tag{6.47}$$

From these expressions we can conclude that, for a Blackfold with equation of state such that:

$$\frac{P}{\varepsilon}\frac{dP}{d\varepsilon} > 0 \tag{6.48}$$

they imply that:

$$c_L^2 \, c_T^2 < 0 \tag{6.49}$$

which is the condition for having an instability, either in the tranverse or longitudinal direction. Now, we can substitute the values of the stress-energy tensor of a neutral Balckfold (3.11) into the expressions above and we see that:

$$c_T^2 = -c_L^2 = \frac{1}{n+1} \qquad \text{(Neurtal Blackfold)} \tag{6.50}$$

As we studied previously, Blackbranes develop GL instability in such a way that the horizon radius varies as:

$$\delta r_o \sim e^{\Omega t} e^{i\mu_j z^j} \tag{6.51}$$

Recall that there is a threshold for the instability to occur beyond which it disappears: λ^{GL}. The threshold makes $\Omega = 0$ and this zero-mode has the smallest (highest μ) of the wavelengths allowed for the GL instability. However, we saw that we can go to arbitrarily low values of μ, that is, to arbitrarily long wavelengths (see last pragraph of page 54) and these latter modes should arise within the Blackfold approach since we are working in this limit. The sound-mode instability found above corresponds precisely to this limit $\Omega, \mu \to 0$ of the GL instability. This can be seen if we recall that variations of the pressure and energy density of the Blackfold produce variations of the horizon radius: $\delta\varepsilon \sim \delta P \sim \delta r_o$. Using (6.46), (6.50) and (6.51) it can be shown that the disperssion relation is:

$$\Omega = \frac{1}{\sqrt{n+1}}\mu \tag{6.52}$$

See (D.23). This result is in good agreement with the slopes of the graphics obtained by Gregory [42] and Laflamme[43], one of them was included above and labelled as Figure 6.2 . Note that the Gibbs-Duhem relation $dP = sd\mathcal{T}$, which the Blackfold satisfies, implies in general that:

$$\frac{dP}{d\varepsilon} = s\frac{d\mathcal{T}}{d\varepsilon} = \frac{s}{C_v} \tag{6.53}$$

where C_v is the specific heat at constant volume. Now, by using equations (3.12) and (3.15), a little computation gives:

$$\Omega = \sqrt{\frac{s}{|C_v|}}\mu \tag{6.54}$$

See (D.27). Thus, we have shown that we can recover the GL instability in the large wavelength regime by simply using the Blackfold approach which is far simpler than solving the perturbation equations solved by Gregory and Laflamme.

We are now going to deal with the case of a viscuous fluid which will be a key point when we relate the classical instability to the local theromodynamics of the Blackfold. The details of the setup can be found in [49]. The main idea is to introduce perturbations in the velocity coordinates of the Blackbrane and thus get a perturbed metric with which we compute the stress-energy tensor of the viscous fluid. The stress-energy tensor of a viscous fluid is:

$$T_{ab} = \varepsilon u_a u_b + P\,\Pi_{ab} - \zeta\theta\Pi_{ab} - 2\eta\sigma_{ab} + \mathcal{O}(\partial^2) \tag{6.55}$$

where:

$$\Pi_{ab} = \eta_{ab} + u_a u_b, \quad \theta = \partial_a u^a, \quad \sigma_{ab} = \Pi_a^c \Pi_b^d \partial_{(c} u_{d)} - \frac{\theta}{p}\Pi_{ab} \tag{6.56}$$

and for the perturbed neutral Blackbrane we have:

$$\eta = \frac{\mathcal{A}_{(n+1)}}{16\pi G} r_o^{n+1}, \quad \zeta = \frac{\mathcal{A}_{(n+1)}}{8\pi G} r_o^{n+1}\left(\frac{1}{p} + \frac{1}{n+1}\right) \tag{6.57}$$

For this case, we consider again perturbations of the following form:

$$\varepsilon \to \varepsilon + \delta\varepsilon, \quad P \to P + c_L^2 \delta\varepsilon, \quad u^a = (1, 0, ...) \to (1, \delta v^i) \tag{6.58}$$

with:

$$\delta\varepsilon(t, z^i) = \delta\varepsilon\; e^{i\omega t + i\mu_j z^j}, \quad \delta v^i(t, z^i) = \delta v^i\; e^{i\omega t + i\mu_j z^j} \tag{6.59}$$

The equations of motion are found by simply requiring the conservation of the stress-energy tensor $\partial_a T^{ab} = 0$. If we work to linear order in the perturbations δv^i and $\delta\varepsilon$, we get:

$$\begin{aligned} \omega\delta\varepsilon + (\varepsilon + P)\mu_i \delta v^i + \mathcal{O}(\mu^3) &= 0 \\ i\omega(\varepsilon + P)\delta v^j + ic_L^2 \mu^j \delta\varepsilon + \eta\mu^2 \delta v^j + \mu^j \left(\left(1 - \tfrac{2}{p}\right)\eta + \zeta\right)\mu_i \delta v^i + \mathcal{O}(\mu^3) &= 0 \end{aligned} \tag{6.60}$$

See (D.41) and (D.45). Solving for $\delta\varepsilon$ in the first equation and plugging this result into the second we get the disperssion relation:

$$\omega - c_L^2 \frac{\mu^2}{\omega} - i\frac{\mu^2}{sT}\left(2\left(1 - \frac{1}{p}\right)\eta + \zeta\right) + \mathcal{O}(\mu^3) = 0 \tag{6.61}$$

where $\mu = \sqrt{\mu_i \mu^i}$ and we have used the Gibbs-Duhem relation $\varepsilon + P = sT$. When $c_L^2 > 0$ the fluid is stable and the viscosity term only adds a complex modification for the ω which results into a damped oscillation. However, we have $c_L^2 < 0$ ((6.50)) so the ω is purely imaginary and the waves are unstable. Setting, as before, $\omega = -i\Omega$ and requiring that $\Omega > 0$, we find, to order μ^2:

$$\Omega = \sqrt{-c_L^2}\mu - \left(\left(1 - \frac{1}{p}\right)\eta + \frac{\zeta}{2}\right)\frac{\mu^2}{sT} + \mathcal{O}(\mu^3) \tag{6.62}$$

See (D.51). For the Blackbrane, using (6.50), we find:

$$\Omega = \frac{1}{\sqrt{n+1}}\mu + \mathcal{O}(\mu^2) \tag{6.63}$$

We can see at once that, to first order, we have got equation (6.52) which we obtained in the perfect fluid approximation. It has been argued that this dispersion relation up to order μ^2 reproduces with great accuracy the numerical results obtained for the limit $n \to \infty$. However, we do not know of the existence of a rigurous proof of this fact, the interested reader might find an heuristic argument in [49].

6.5 Gregory-Laflamme Instability in Boosted Black Strings

This section contains, as far as we know, new material on the subject. In the previous one, we followed Emparan's approach to show that the GL instability arises in static Blackfolds just as it was predicted by Gregory and Laflamme. Here, we will rely on that approach to show how the same GL instability develops in a boosted Black String. The fact that we have chosen the boosted Black String to make such a fruitful attempt is that we can use the numerical results provided by [60] to confirm our analytical results. The question as to wether this approach would help to derive the GL instability for more general Blackfolds, apart from Black Strings, or not is left unaswered for the complexity of the equations obtained, but it could be investigated in the future with the aid of numerical simulations.

We begin with the set up which is rather similar to that of the previous section. Note, however, that we will proceed first with the longitudinal modes which are, strictly speaking, the ones that Gregory and Laflamme considered. The preturbation of the transverse coordinates will be considered at the end of this section. The fact that we are now working with a boosted Black String is reflected in the velocity u^a having a non-zero contribution at zeroth order in the perturbation. So, working with the same assumptions as before ((6.38) and above), we have the following perturbations:

$$\delta\varepsilon, \quad \delta P = \frac{dP}{d\varepsilon}\delta\varepsilon, \quad \delta X^m = \xi^m \tag{6.64}$$

Note now that, unlike the previous section, we have:

$$u^a = (1, u^i) \Longrightarrow u^a + \delta u^a = (1, u^i + \delta u^i) \tag{6.65}$$

This gives the stress-energy tensor of the boosted Blackfold (3.10) the following contributions up to second order, later on we will only retain first order terms and drop the rest:

$$\begin{aligned} T^{tt} &= \varepsilon + \delta\varepsilon \\ T^{ti} &= (\varepsilon + \delta\varepsilon + P + \delta P)(u^i + \delta u^i) \\ T^{ij} &= (\varepsilon + \delta\varepsilon + P + \delta P)(u^i u^j + \delta u^i u^j + u^i \delta u^j) + (P + \delta P)\delta^{ij} \end{aligned} \tag{6.66}$$

Let us now proceed with the intrinsic equations (D.18) as we announced, for the above stress-energy tensor, to first order in the perturbations, they read:

$$\begin{aligned}
&\partial_t\delta\varepsilon + (\varepsilon+P)\partial_i\delta u^i + u^i\left(1+\tfrac{dP}{d\varepsilon}\right)\partial_i\delta\varepsilon = 0\\
&(\varepsilon+P)\partial_t\delta u^j + u^j\left(1+\tfrac{dP}{d\varepsilon}\right)\partial_t\delta\varepsilon + \tfrac{dP}{d\varepsilon}\partial^j\delta\varepsilon + u^j u^i\left(1+\tfrac{dP}{d\varepsilon}\right)\partial_i\delta\varepsilon\\
&+(\varepsilon+P)(u^i\partial_i\delta u^j + u^j\partial_i\delta u^i) = 0
\end{aligned} \tag{6.67}$$

In the case of a Black String, we have $i = z$, $n = 1$, $u^z = \tanh\sigma$ and $\frac{dP}{d\varepsilon} = -\frac{1}{2}$. Plugging in this in the above equations we get:

$$\begin{aligned}
&\partial_t\delta\varepsilon + (\varepsilon+P)\partial_z\delta u^z + \tanh\sigma\tfrac{1}{2}\partial_z\delta\varepsilon = 0\\
&(\varepsilon+P)\partial_t\delta u^z + \tfrac{\tanh\sigma}{2}\partial_t\delta\varepsilon - \tfrac{1}{2}\partial_z\delta\varepsilon + \tfrac{\tanh^2\sigma}{2}\partial_z\delta\varepsilon\\
&+2(\varepsilon+P)\tanh\sigma\ \partial_z\delta u^z = 0
\end{aligned} \tag{6.68}$$

The generic form for the perturbation is as in (6.59):

$$\delta\varepsilon(t,z) = \delta\hat{\varepsilon}\ e^{i\omega t+ikz}, \quad \delta u^z(t,z) = \delta\hat{u}^z\ e^{i\omega t+ikz} \tag{6.69}$$

where $\delta\hat{\varepsilon}$ and $\delta\hat{u}^z$ are constants. The equations (6.68) and the above form of the perturbations yield a system of two equations for $\delta\hat{\varepsilon}$ and $\delta\hat{u}^z$. In order to have a non trivial solution we need the determinant of its coefficients to vanish (See (D.52)) and we get the dispersion relation at once:

$$\Omega = \left(\pm\frac{1}{\cosh\sigma}\frac{1}{\sqrt{2}} - i\ \tanh\sigma\right)k \tag{6.70}$$

We are interested in the real part of the frequency which is the one that will determine the stability of the perturbation. Before examining the behaviour at limiting cases we note the following relation, up to a sign, stated also in [60]:

$$\mathrm{Re}\ \Omega = \pm\frac{\tilde{\Omega}}{\cosh\sigma} \tag{6.71}$$

Where $\tilde{\Omega}$ is the frequency of the static case (6.52) with $n = 1$. It is straightforward to see from the above equation that when we have no boost $\cosh\sigma = 1$ and we recover the static case:

$$\mathrm{Re}\ \Omega = \pm\tilde{\Omega} \tag{6.72}$$

As the speed of the boost increases towards that of light $\cosh\sigma$ increases too, making the slope of the dispersion relation to decrease with respect to that of the static case. In the limit of luminal boost velocity we have $\cosh\sigma \to \infty$ and therefore:

$$\mathrm{Re}\ \Omega \to 0 \tag{6.73}$$

The ultimate check for this result are the numerical results obtained in [60]. They computed numerically the dispersion relations for various values of the boost velocity and plotted them in a graphic which we show below in Figure 6.6. It is apparent how the slope decreases towards zero as we increase the velocity of the boost. Note that, for the static case, $\cosh\sigma = 1$, we have the same curve as Gregory and Laflamme obtained and we showed in Figure 6.2. This latter check can be viewed better in Figure 6.7 which is the same computation but in the proper frame where, as expected, the dispersion relation look nearly like the static case. If we look at the red lines, we see at once that, upon rescaling, they are exactly the same. Thus confirming that when the boost velocity is zero we, indeed, recover the results from the static case.

We compute the extrinsic equations as we did in the previous section, only now we have to use the new stress-energy tensor (6.66). The extrinsic curvature tensor is still given, to first order, by (6.41) so the extrinsic equations (3.30) read:

$$[\varepsilon\partial_t^2 + (\varepsilon + P)u^i\partial_i\partial_t + (\varepsilon + P)u^iu^j\partial_i\partial_j + P\partial_i\partial^i]\xi^\rho = 0 \tag{6.74}$$

See (D.54). For the case of a boosted Black String at hand it simplifies to (we will simplify it further later):

$$[\varepsilon\partial_t^2 + (\varepsilon + P)\tanh\sigma\partial_z\partial_t + (\varepsilon + P)\tanh^2\sigma\partial_z\partial_z + P\partial_z\partial^z]\xi^\rho = 0 \tag{6.75}$$

In order to get a dispersion relation we can Fourier transform $\xi^\rho(t, z)$:

$$\xi^\rho(t, z) = \int \frac{d\omega\, dk}{(2\pi)^2} e^{it\omega} e^{-ikz}\hat{\xi}^\rho(\omega, k) \tag{6.76}$$

which, after dividing by ε, yields for (6.75):

$$\omega^2 - \omega\, k\left(1 + \frac{P}{\varepsilon}\right)\tanh\sigma + k^2\left\{\left(1 + \frac{P}{\varepsilon}\right)\tanh^2\sigma + \frac{P}{\varepsilon}\right\} = 0 \tag{6.77}$$

In our case we have $n = 1$ so that $\frac{P}{\varepsilon} = -\frac{1}{n+1} = -\frac{1}{2}$ and the disperssion relation, finally, looks like:

$$\omega^2 - \omega\, k\, \frac{1}{2}\tanh\sigma + k^2\, \frac{1}{2}(\tanh^2\sigma - 1) = 0 \tag{6.78}$$

We can solve for ω:

$$\omega = \frac{k}{4}\left(\tanh\sigma \pm \sqrt{8 - 7\tanh^2\sigma}\right) \tag{6.79}$$

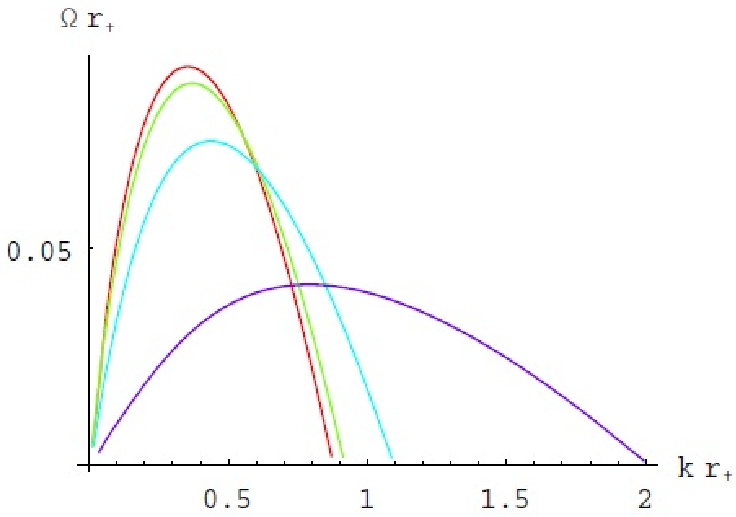

Figure 6.6: Dispersion relations for various values of the boost velocity in the LAB frame: $u^z = 0$ in red, $u^z = 0.3$ in green, $u^z = 0.6$ in blue and $u^z = 0.9$ in purple. Figure taken from [60].

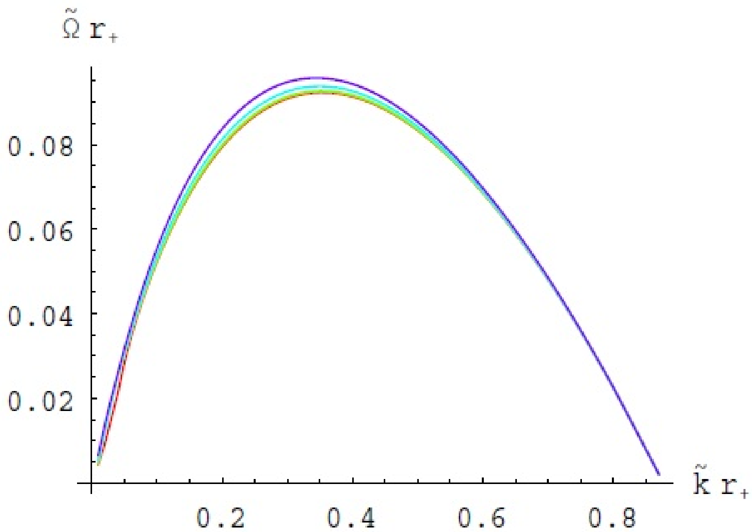

Figure 6.7: Dispersion relations for various values of the boost velocity in the proper frame, again we have: $u^z = 0$ in red, $u^z = 0.3$ in green, $u^z = 0.6$ in blue and $u^z = 0.9$ in purple. Note that the red curve is the same as that of the previous figure, the static case. Figure taken from [60].

Had we done this computation, using (6.76), for the static case, we would have come to the conclusion that, for $n = 1$, the dispersion realtion obtained by using (6.42), (6.44) and (6.50) was:

$$\omega = \frac{k}{\sqrt{2}} \tag{6.80}$$

which is precisely what we obtain from (6.79) when we take the static limit making $\tanh\sigma = 0$.

It is worth noting that, unlike the static case, longitudinal and transverse perturbations do not propagate with the same absolute value of the velocity as can be seen by comparing (6.70) and (6.79). We know that our dispersion relation is of the form $\omega^2 = c_T^2 k^2$ (cf. (6.79)), from the above we see that, according to wether we pick up the minus or plus sign we have a positive or negative velocity c_T but we always have $c_T \in \mathbb{R}$ so the exponential in (6.76) is oscillatory and bounded. No instability arises in the transversal modes.

To close this section, we point out that we could derive from the above dispersion relations the ones which we would expect to hold, at least for small k, in the Black Ring case by simply using the condition of equilibrium for the Black Ring (5.39). For the Black Ring in 5 spacetime dimensions we have $n = 1$, so the equilibrium condition gives a boost velocity $u^z = \tanh\sigma = \frac{1}{\sqrt{2}}$.

6.6 Gregory-Laflamme Instability, Thermodynamics and caged Blackbranes

We devote this last section to present and derive some results of a rather new approach of studying the stability of Blackbranes by putting them inside a box and varying the latter's size. In the present case, we will study how the Blackbrane reacts to perturbation when placed inside a cylindrical cavity with fixed R. By doing this, we will be able to study several aspects of the Blackbrane dynamics and its stability:

Correlated Instabilities: By this we mean the relationship between the local thermodynamics of the membrane and its dynamical stability. This was studied by Gubser and Mitra [50] and led to the Correlated Stability Conjecture (CSC) which can be succitly stated as follows [52]: *translationally invariant horizons have a tachyonic perturbation mode if and only if they are locally thermodynamically unstable.* A tachyon is a zero-mode with finite wavelenght (or finite wavenumber μ, we will

give a proper definition of it in due course) which marks the threshold for the GL instability. We will see that, inside a cavity, the Blackbrane's specific heat changes sign when the radius of the cavity reaches a critical value R_c. For $R > R_c$, the specific heat is negative and, locally, the brane is thermodynamically unstable, whereas for $R < R_c$, the specific heat is positive and the brane is thermodynamically stable. We will see that both cases lead, respectively, to dynamical instability or to stability. However, the aforementioned formulation of the CSC is incorrect, there are counterexamples, given by Gubser and Mitra themselves, that show Blackbranes which are thermodynamically stable and nonetheless, have tachyonic modes [51]. We will see that local thermodynamical stability is linked to a kind of modes called *ghosts*. Before continuing, let us give a proper definition of these two modes, ghosts and tachyons.
We assume we have a dispersion relation of the form:

$$\omega^2 = c_L^2 \mu^2 + m^2 \tag{6.81}$$

- We say we have a *Tachyon* when $m^2 < 0$. Further, a static zero-mode ($\omega = 0$) is achieved when also $\mu = \sqrt{-m^2/c_L^2}$.
- We say we have a *Ghost* when $c_L^2 < 0$. It is called a *Massless Ghost* if we also have $\mathrm{Re}\,\omega = 0$ and $\Omega \equiv -\mathrm{Im}\,\omega = \mu\sqrt{-c_L^2}$.

Thermodynamical instabilities of a translation invariant horizon are connected to massless ghosts: a horizon which is translation-invariant can support perturbations of arbitrarily long wavelength. Modes with $\Omega, \mu \to 0$, or equivalently $\Omega \to 0, \lambda \to \infty$, are hydrodinamic modes corresponding to fluctuations of either conserved quantities, or massless modes, we will concern ourselves with the first.

As we have seen in a previous section, when we consider fluctuation of a conserved quantity, such as the energy of a neutral Blackbrane, we arrive at the result that:

$$c_L^2 = \frac{s}{C_v} \tag{6.82}$$

This relation illustrates our previous claim on thermodynamic instability and unstable perturbation or massless ghosts, namely: whenever we have a local thermodynamical instability and $C_v < 0$, since the entropy is always positive, we must have $c_L^2 < 0$. This condition, as we saw when deriving the dispersion relation (6.52), leads to the appearance of an instability, which means ω is purely imaginary and hence we have a massless ghost. Thus, we have shown that thermodynamical instabilities are associated with massless ghosts that produce unstable oscillations. This arguments suggest that we recast the conjecture as follows [52]: *translationally invariant horizons have massless ghost excitations if and only if they are locally thermodynamically unstable. The ghost is a longwavelength, low imaginary frequency, hydrodynamic instability of the horizon.* This statement has been coined as *Correlated Hydrodynamic Stability* (CHS).

The ghost is usually accompanied by a tachyonic mode: the horizons are stable to very short wavelengths so the ghost instability must disappear $\mathrm{Im}\,\omega = 0$ at some finite $\mu^{GL} > 0$. If also $\mathrm{Re}\,\omega = 0$, we have a zero-mode which, according to our previous definitions, is a tachyon. So a ghost perturbation will be typically accompanied by a tachyonic mode at finite μ. The converse is, however, not true: a tachyonic mode need not evolve into a ghost instability at large wavelenghts and hence, does not have to be related to any local thermodynamical instability.

Membrane Rigidity: Analogous to the case of an oscillating membrane in classical mechanics, the more rigid the membrane is the more stable it is to perturbations, this is reflected in a high speed of sound in the media, roughly, the speed of sound goes like $c_L^2 \sim \frac{1}{R}$. This is because fixing the wall metric at a finite distance R constraints the membrane connected to it and makes it harder for the membrane to fluctuate freely.

Viscosities do not run with R: We work to first order in derivatives and the stress-energy tensor of the fluctuating Blackbrane allows us to compute the shear and bulk viscosities which will be found to be independent of R.

Spectrum: As we have already seen, the inclusion of dissipative effects will lead to an improved dispersion relation including higher order terms. We will study how the frequency of the oscillations changes with R.

6.6.1 Static Blackbrane in cylindrical cavity

We have a black p-brane in $D = 3 + p + n$ spacetime dimensions with the metric given by (3.4). We put it inside a cylindrical cavity bounded by a wall along the Blackbrane coordinates $\{\sigma^a\}$ whose transverse section are $S^{(n+1)}$ spheres with radius $r = R$.

This induces a metric on the walls given by:

$$\hat{h}_{\mu\nu}dx^\mu dx^\nu = \hat{h}_{ab}d\sigma^a d\sigma^b + R^2 d\Omega^2_{(n+1)} \tag{6.83}$$

where

$$\hat{h}_{ab} = -\hat{u}_a\hat{u}_b + \hat{P}_{ab}, \quad \hat{u}_a = u_a\sqrt{f(R)}, \quad P_{ab} = \hat{P}_{ab} \tag{6.84}$$

and $f(r) = 1 - \frac{r_o^n}{r^n}$. From now on, all the quantities measured on the wall will be denoted by letters with a hat. The extrinsic curvature tensor is:

$$\Theta_{\mu\nu} = -\frac{1}{2}\sqrt{f(R)}\partial_R \hat{h}_{\mu\nu} \tag{6.85}$$

with this we can obtain the Brown-York [53] quasilocal stress-energy tensor measured on the wall by using the following formula:

$$\hat{T}_{ab} = \frac{\mathcal{A}_{(n+1)}}{8\pi G} R^{n+1}(\Theta_{ab} - \hat{h}_{ab}\Theta) \tag{6.86}$$

which turns out [52] to have the form of a perfect fluid stress-energy tensor with the following parameters:

$$\hat{\varepsilon} = -\frac{\mathcal{A}_{(n+1)}}{8\pi G}(n+1)R^n\sqrt{f(R)} \tag{6.87}$$

$$\hat{P} = -\hat{\varepsilon} + \frac{\mathcal{A}_{(n+1)}}{8\pi G} R^{n+1}\partial_R\sqrt{f(R)} = -\hat{\varepsilon} + \frac{\mathcal{A}_{(n+1)}}{16\pi G}\frac{n r_o^n}{\sqrt{f(R)}} \tag{6.88}$$

and the entropy density and temperature are:

$$s = \frac{\mathcal{A}_{(n+1)}}{4\pi G} r_o^{n+1} \tag{6.89}$$

$$\hat{\mathcal{T}} = \frac{n}{4\pi r_o\sqrt{f(R)}} \tag{6.90}$$

Also, the following two thermodynamical relations hold:

$$\hat{\varepsilon} + \hat{P} = s\hat{\mathcal{T}} \qquad \text{Euler relation} \tag{6.91}$$

$$d\hat{\varepsilon} = \hat{\mathcal{T}} ds \qquad \text{First law} \tag{6.92}$$

Both $\hat{\varepsilon}$ and $\hat{P}$ diverge as $R \to \infty$ but since we are working with finite R we can simply neglect this behaviour. See [52] for an argument on how to cure this divergence.

6.6.2 Perturbing the Blackbrane

We will not linger in the details of the calculation but simply give an outline of how it goes, the details may be found again in [52]. We promote the Blackbrane parameters to collective variables depending on the worldvolume coordinates $\{\sigma^a\}$: $\{u_a, r_o\} \to \{u_a(\sigma^b), r_o(\sigma^b)\}$. Keeping R fixed, we add correcting functions $f_{\mu\nu}$ to the metric. The EE $R^r{}_a = 0$ do not involve any $f_{\mu\nu}$ so they are simply constraints:

$$\nabla_a \ln r_o^{n+1} = \theta u_a + (n+1)a_a \tag{6.93}$$

For a detailed account of how to set the appropriate boundary conditions and thus compute the correcting functions we refer the reader to the apendix A of [52]. We will skip technical details and go straight to the key results which will illustrate the physical behaviour of the Blackbrane-cavity system. The appendix B of the above paper also computes the relation between some important magnitudes of the Blackbrane and its values measured on the wall:

$$\hat{\Xi} = \frac{\Xi}{\sqrt{f(R)}}, \qquad \Xi = u^a, \theta, \sigma^{ab} \tag{6.94}$$

$$\hat{a}_a = a_a + \frac{1}{\sqrt{f(R)}} \Pi_a{}^b \partial_b \sqrt{f(R)} \tag{6.95}$$

the expression of the acceleration will be very important when studying the stability. To make things clearer, let us introduce the Newtonian potential ϕ which depends on $\{\sigma^a\}$ through r_o:

$$f(R) = e^{2\phi} \tag{6.96}$$

we define its spatial gradiaent as:

$$\nabla_a \phi \equiv \Pi_a{}^b \partial_b \phi \tag{6.97}$$

and we can rewrite (6.95) in a more compact form:

$$\hat{a}_a = a_a + \nabla_a \phi \tag{6.98}$$

Having no cavity is equivalent to taking the limit $R \to \infty$ so that $\hat{h}_{ab} \to \eta_{ab}$ and $\hat{u}_a \to u_a$. In this limit, we recover all the results of section 6.4 and therefore we refer to that section for details. However, in the case of the finite cavity we want to deal with, redshift gradients will be present and control the dynamics of the system. Let us project the constraint derived from the EE (6.93):

$$\Pi_a{}^b \nabla_b \ln r_o = a_a \tag{6.99}$$

which we can use to derive (see (D.56)) an important result:

$$\nabla_a \phi = -\frac{n}{2}\left(\frac{1}{f(R)} - 1\right) a_a \tag{6.100}$$

For the whole range of R we have that $f(R) < 1$ so we see that $\nabla_a \phi$ is directed in the opposite direction to that of a_a and hence, opposes the growth of unstable modes. Further, by decreasing R we can reach a value:

$$R_c = r_o \left(\frac{n+2}{2}\right)^{\frac{1}{n}} \tag{6.101}$$

such that:

$$\nabla_a \phi = -a_a \Longrightarrow \hat{a}_a = 0 \tag{6.102}$$

so we can summarize the behaviour against instabilities of the Blackbrane as follows:

$R > R_c$: Even though the redshift gradient opposes the growth of inbstabilities, it is not sufficient to cancel them so we have unstable growth of the perturbations.

$R = R_c$: Instability threshold at which the Blackbrane, simply, does not react to perturbations and hence the instability disappears.

$R < R_c$: The acceleration measured on the wall $\hat{a}_a$ is now directed opposite to the growth of instabilities overtaking them and so rendering the Blackbrane stable.

Let us now see how other dynamical and thermodynamical quantities support this behaviour. The following relation holds amongst the accelerations and speeds of sound in the Blackbrane and on the wall, see (D.58):

$$\frac{\hat{a}_a}{a_a} = \frac{\hat{c}_L^2}{c_L^2} \tag{6.103}$$

little manipulation leads to:

$$\hat{c}_L^2 = \frac{-1}{n+1}\left(\frac{1-(R_c/R)^n}{f(R)}\right) \tag{6.104}$$

which can account for the change in the behaviour of the effective fluid with respect to instabilities in the following way, supporting our previous description:

$R > R_c$: $\hat{c}_L \in \mathbb{C}$, the Blackbrane is unstable.

$R = R_c$: $\hat{c}_L = 0$, threshold of the instability.

$R < R_c$: $\hat{c}_L \in \mathbb{R}$, the Blackbrane is stable.

If we think in terms of classical mechanics, the speed of sound is a measure of the rigidity of the brane, the higher it is, the more rigid the brane and thus the more stable it is. In the light of the above results, as we proceed from high values of R towards lower ones, we increase the rigidity of the Blackbrane thus making it more stable.

It is illustrative to look at the form of the constraint equations (6.93), see (D.60) for details:

$$\hat{\nabla}_a \ln r_o^{n+1} = \hat{\theta}\hat{u}_a - \frac{1}{\hat{c}_L^2}\hat{a}_a \tag{6.105}$$

r_o^{n+1} is proportional to the entropy density s which, as we saw, does not run with R; so all the R dependence of the Blackbrane effective fluid dynamics is encoded within the modified acceleration term in $\hat{c}_L^2$ in the manner explained above.

Let us now provide proof of the link between thermodynamics and stability announced at the beginning of this section. By differentiating the Euler relation (6.91) above and using the First law (6.92) we obtain:

$$\hat{c}_L^2 = \left(\frac{d\hat{P}}{d\hat{\varepsilon}}\right)_R = s\frac{d\hat{\mathcal{T}}}{d\hat{\varepsilon}} = \frac{s}{\hat{C}_v} \tag{6.106}$$

so, again, since s does not run with R and it is always positive, the thermodynamical stability will be determined by the sign of $\hat{C}_v$ which is linked to the dynamical stability of the Blackbrane by the above equation and could be summarized, with the aid of (6.104), as:

$R > R_c$**:** $\hat{C}_v < 0 \Leftrightarrow \hat{c}_L \in \mathbb{C}$

$R = R_c$**:** $\hat{C}_v \to \infty \Leftrightarrow \hat{c}_L = 0$ (Critical Point: 2^{nd} Order Phase Transition)

$R < R_c$**:** $\hat{C}_v > 0 \Leftrightarrow \hat{c}_L \in \mathbb{R}$

This confirms what we aticipated at the begining of the section in the point "Correlated Instabilities".

6.6.3 Viscous Blackbrane Dynamics

Most of the results have already been presented in section 6.4 so we are just going to point out what changes when we put the Blackbrane inside the box. In the case of an uncaged Blackbrane, we can rewrite the shear and bulk viscosity (6.57) in terms of the entropy density (3.12):

$$\eta = \frac{s}{4\pi}, \quad \zeta = 2\eta\left(\frac{1}{p} - c_L^2\right) \tag{6.107}$$

The values that are obtained in the case of a caged Blackbrane are computed from the stress-energy tensor with viscous terms in [52]:

$$\hat{\eta} = \frac{s}{4\pi}, \quad \hat{\zeta} = \frac{s}{2\pi}\left(\frac{1}{p} + \frac{1}{n+1}\right) \tag{6.108}$$

plugging (6.57) into the bulk viscosity expression we find:

$$\hat{\zeta} = \frac{s}{2\pi}\left(\frac{1}{p} - c_L^2\right) \tag{6.109}$$

This is a rather unexpected result since we could have naively guessed that it would go with $\hat{c}_L^2$ instead of simply c_L^2. However, this cannot be the case if we want both viscosities to be R independent as we anticipated at the beginning of the section. Therefore we conclude that both $\frac{\hat{\zeta}}{s}$ and $\frac{\hat{\eta}}{s}$ do not depend on R.

Also, we have a modified dispersion relation for the unstable modes:

$$\Omega = \sqrt{-\hat{c}_L^2}\mu - \left(\left(1 - \frac{1}{p}\right)\hat{\eta} + \frac{\hat{\zeta}}{2}\right)\frac{\mu^2}{s\hat{T}} + \mathcal{O}(\mu^3) \tag{6.110}$$

For the $n = 1$ case, the plot of this dispersion realtion is qualitatively similar to that one shown in Figure 6.2 (Figure 1 of [52]) confirming the existence or absence of GL type instabilities in a range of μ depending on the value of R as we have summarized above. We note that, the higher the R, the wider the range of μ in which we can find unstable modes. As the instability gets weaker, this range shrinks to zero, presumably, at $R = R_c$ and for $r_o < R < R_c$ the frequency Ω has, to μ^2 order, negative real and imaginary pieces which are damped oscillations and therefore stable. See equation (5.7) of [52].

6.6.4 Critical Behaviour

When we approach the Critical Point $\hat{C}_v \to \infty$ and $\hat{c}_L = 0$, we are at the threshold of the instability and the hydrodinamic mode is about to become a ghost. On the one hand, the fact that $\hat{c}_L = \left(\frac{d\hat{P}}{d\hat{\varepsilon}}\right)_R = 0$ means, physically, that the fluid does not produce pressure gradients when its energy density is locally perturbed. On the other hand, $\frac{d\hat{T}}{d\hat{\varepsilon}} = \frac{1}{\hat{C}_v} = 0$ means that, the same perturbation of the energy density does not produce any temperature gradients in the fluid that restore thermodynamical equilibrium. The two phenomena are related by the relation $d\hat{P} = sd\hat{T}$ which allows us to link thermodynamics with dynamical stability of the Blackbrane. At the threshold, the tachyonic zero-mode becomes massless, which translates into an infinite wavelength, however, this mode cannot be accounted for by a simple hydrodinamical model such as ours and hence the calculations need not be accurate near this Critical Point.

6.6.5 1^{st} *vs.* 2^{nd} Order Phase Transitions

If this setting were physically realizable we would move from stability to instability, that is, from low values of R to higher ones up until we reached the Critical Point (CP) at R_c. However, this point is never reached because there exists a 1^{st} Order Phase Transition before we get to the 1^{st} Order Phase Transition (OPT) which prevents the system from reaching the CP. This happens at:

$$R_1 = \left(\frac{n+2}{2(n+1)}\right)^{\frac{1}{n}} R_c < R_c \tag{6.111}$$

where the pressure of the caged Blackbrane equals that of Minkowski space in the same cavity (See (D.62)-(D.73)). From here on until the CP, the pressure of the Minkowski background is larger than that of the Blackbrane. Therefore, when the Blackbrane is taken to this state it will undergo nucleation of Minkowski (vacuum) bubbles within its worldvolume and these will begin to expand as we increase R towards the CP. This phase transition to vacuum is what prevents the Blackbrane from reaching the critical state.

Chapter 7

Black Holes and Black Rings in Anti-de Sitter Background

We devote this chapter to a brief review of the results that have been obtained throughout the last years for the same kind of objects we have studied to this point, namely: Black Holes and Black Branes. These are more complicated solutions to deal with than those for a Minkowski background but, this notwithstanding, some results have already been derived which remind greatly of many features we have been studying in detail in the previous chapters: brane limits, GL instability, thermodynamics and correlated instabilities. We will, in most cases, give the most important results without any attempt to derive them but proper references will be given where appropriate. This chapter is mainly included for completeness since, although higher dimensional gravity in vacuum might be very interesting in itself as a low energy limit of string theory (as we pointed out in the introduction) it is the first step towards the understanding of gravity in non-empty universes useful for AdS/CFT.

7.1 Rotating Black Holes and Black Ring in Anti-de Sitter

Rotating Black Hole solutions were first found by Kerr. They were solution to the Einstein equations in vacuum. Its generalization for spacetimes with cosmological constant Λ soon followed [54]. Afterwards, Myers and Perry found a generalization of vacuum solutions for higher dimensional spacetimes, both non-spinning and spinning, we have been concerned with them in chapter 4. The analogous to this last family of solutions are the so called *Kerr-de Siter* metrics. They are higher dimensional spinning generalizations of the solutions found by Carter, Hawking, Hunter and Taylor-

Robinson [55]. The general form of this family of metrics has been studied in full by Gibbons, Lu, Page and Pope [56] and its physical properties by Gibbons, Perry and Pope [57].

The situation for the Black Ring is such that it will allow us to use certain results that have already been presented in previous chapters. As usual, we are looking for an object whose horizon topology is $S^1 \times S^{d-3}$ (when working in d spacetime dimensions). The ring will be again chracterized by two radii: the horizon radius r_o and the radius of the ring itself R. However, we now need to consider one more length scale, the characteristic radius of curvature of the Anti-de Sitter (AdS) background, denoted here by L. See section 2.2 of [58] for a succint but very thorough summary of AdS geometry. It is to be expected that for $R \lll L$ the solution will be a slight modification of the vacuum solutions but here, we will present results based on assumptions which render them valid whenever $r_o \lll \min(R, L)$ without any hierarchy between R and L. See [59] and its apendices for details of the calculation which relies heavily on the method of matched asymtotic expansion that we described at the begining.

Consider the metric of a general AdS$_d$ spacetime:

$$ds^2 = -V(\rho)d\tau^2 + \frac{d\rho^2}{V(\rho)} + \rho^2[d\Theta^2 + \sin^2\Theta \, d\Omega^2_{d-4} + \cos^2\Theta \, d\psi^2] \tag{7.1}$$

where

$$V(\rho) = 1 + \frac{\rho^2}{L^2} \tag{7.2}$$

and we shall place the ring at $R = \rho$ and $\Theta = 0$. We also define proper time and length coordinates along the worldsheet of the ring t, z:

$$t = \tau\sqrt{V(R)}, \qquad z = R\psi \tag{7.3}$$

where z must be regarded as periodic: $z \sim z + 2\pi R$ and R is the ring radius. Since we are far away enough from the ring, the stress-energy tensor will be that of a circular source of mass we used in (5.28) for the case of $d = 5$. The conservation of the tress-energy tensor gives the equilibrium condition for the Black Ring, in our case, for the thin ring, this is equivalent to requiring the extrinsic equations to hold. It reads:

$$\frac{(R/L)^2}{1 + (R/L)^2} T^\tau{}_\tau + T^\psi{}_\psi = 0 \tag{7.4}$$

since $T^\psi{}_\psi = T^z{}_z = T_{zz}$ we see at once that, unlike the vacuum case where $T_{zz} = 0$, we need a fluid with pressure to balance the AdS attractive potential. Using the stress-tensor we have mentioned above (its general form might be found in [59]) and the fact that $T^{\tau}{}_{\tau} = T^{t}{}_{t} = -T_{tt}$ we arrive at the equilibrium condition for the AdS Black Ring:

$$\sinh^2\alpha = \frac{1 + (d-2)\left(\frac{R}{L}\right)^2}{d-4} \tag{7.5}$$

Note that as we take the limit $L \to \infty$ leaving R fixed, we obtain the expression we obtained for the vacuum Black Ring. The cosmological constant has the effect of making the equilibrium condition for the Black Ring dependent on the radius: the longer the ring, the stronger the AdS potential attraction and the higher the velocity of the boost needed to balance the tension and potential attraction.

We also expect our Black Ring, when it is sufficiently thin, to develop GL type instabilities, but we will study them in section 7.3. For an account of the influence of boosts on the GL instability parameters see section 3 of [60]. It has been shown [59] that the Black Ring satisfies the First Law of Black Hole Thermodynamics but does not satisfy the Smarr relation, this failure is due to the presence of the extra length scale L of AdS. In [61], they have computed the upper bounds for the AdS spinning solutions, and for the case of one angular momentum, we have that:

$$J \leq ML \tag{7.6}$$

this equation is saturated in the limit of very long Black Rings $R \to \infty$ keeping L fixed since the difference between both of the above quantities decreases to zero as $R \to \infty$.

7.2 Black Rings *vs.* Rotating Black Holes in AdS

The situation now is rather similar to that encountered when we compared MPBH with BR, birefly, our limit $a \to L$ and the saturated bound (7.6) is the equivalent to the limit $a \to \infty$ for fixed mass of the MPBH . In the case $d = 5$, solutions for the spinning BH exist provided $a^2 < 2m$ and the imposibility of taking the limit $a \to L$, keeping the value of the mass finite, makes it imposible for the bound (7.6) to saturate. The limit $a^2 \to 2m$ corresponds to a naked singularity with zero horizon surface area. Nevertheless, when we have $d \geq 6$, using arguments similar to those used by Myers and Perry in [14], one can see that, for any value of $a \in [0, L)$ and finite mass, there is always a solution for the even horizon of the spinning AdS BH. In this case, when we take the limit $a \to L$ and keep the mass fixed at a finite value, the bound (7.6) is saturated. In Figure 7.1 we can see the case $d = 5$, the thin lines correspond to rotating BH and the solid ones to the thin Black Ring approximation. We can see how the spin of the BR reach the bound but the spining BH doesn't. The dashed line corresponds to the area where the thin ring approximation breaks down. The resemblance with Figure 5.1 depicting the behaviour of the vacuum analogues is apparent. Figure 7.2 depicts the $d = 7$ which, as we have already said, is qualitatively very different from the previous case. In particular, we can see at once how both solutions, the BH and the BR, reach the bound $J = ML$ and how the latter have higher area than the former near the bound. This can also be seen by comparing the analytical expressions for the area of both solutions, see section 3.1 of [59].

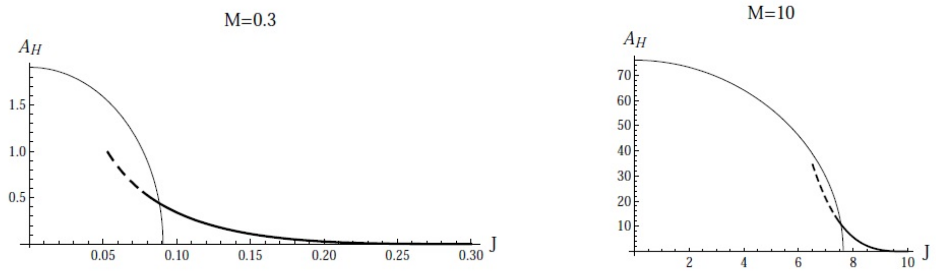

Figure 7.1: Plots of the horizon area A_H *vs.* the angular momentum J for two values of the mass M. $d = 5$. Thin lines correspond to AdS rotating BH and the solid ones to AdS thin Black Rings. The dasehd line shows the limit where the thin ring approximation breaks down. Taken from [59].

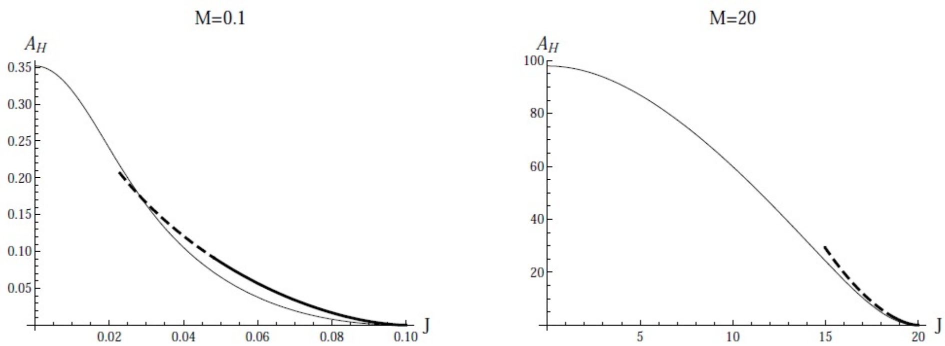

Figure 7.2: Plots of the horizon area A_H *vs.* the angular momentum J for two values of the mass M. $d = 7$. Thin lines correspond to AdS rotating BH and the solid ones to AdS thin Black Rings. The dasehd line shows the limit where the thin ring approximation breaks down. Taken from [59].

Following [36], when we take the limit $a \to L$ in the case $d \geq 6$, we can identify the chracteristic lengths in both, transverse and parallel, directions of the horizon $\ell_{\perp}$ and $\ell_{\parallel}$. Using the expressions in [59], it can be shown that the quotient $\frac{\ell_{\parallel}}{\ell_{\perp}} \sim \Xi^{-\frac{d-1}{2(d-5)}}$ diverges as $a \to L$ and $\Xi \to 0$. This means that the parallel dimension is much larger than the transverse, that is, the BH pancakes along the plane of rotation and we have a membran- like object. This can be justified analytically also, [59]: the limiting metric of the BH in AdS is, up to a conformal factor, that of a Black Membrane (4.23). There is also a more exotic limiting case which has no vacuum analogue, it has the pecularity of possesing a horizon topology $\mathbb{H}^2 \times S^{d-4}$, details can be found at the end of section 3 in [59] and references therein.

Let us finish this section by providing a classification of the Black Rings in AdS accoording to the hierarchy of the different length scales involved. There are more possibilities than just thin and fat Black Rings as in the vacum case since now we do not have just two scales but three:

- *Thin* rings have $r_o \lll R$ and *fat* rings $r_o \sim R$
- *Small* rings have $r_o < L$ and *large* rings $r_o > L$
- *Short* rings have $R < L$ and *long* rings $R > L$

All the results above apply to *small* rings, wether *short* or *long*, as we pointed out in the paragraph above (7.1).

7.3 Gregory Laflamme Instability in AdS

The first account on the stability of BH in AdS was given for the $d = 4$ case in [62]. However, we are interested in higher dimensional cases and so we will review some of the most important facts found by Delsate *et al.*. In particular, we would like to know wether or not the Gubsner-Mitra conjecture still applies in AdS as it did in asymptotically flat backgrounds. The procedure to study the GL instability of AdS Black Strings followed by them in [63] is very similar to that described above, used by Gregory and Laflamme in their original papers. Note that, when we describe AdS Black Rings or BHs within the Blackfold formalism, the study of the GL instabilities is analogous to that carried out in the previous chapters, there are some differences, however, when we do not resort to the aid of the Blackfold approach. We will not make use of Blackfold here. The metric of a non-uniform BR is used and a perturbation is introduced and parametrized by some set of functions. The EE give a pair of ODE's involving these functions and they are numerically integrated with the appropriate boundary conditions.

For the case of BR, with horizon topology $S^1 \times S^{d-3}$, it was found that, for fixed r_o, the stability depended on the value of the cosmological constant Λ in the following way:

- $\Lambda \in (\Lambda_c, 0] \Longrightarrow$ Classically Unstable
- $\Lambda \in (\infty, \Lambda_c) \Longrightarrow$ Classically Stable

where Λ_c is the critical value of the cosmological constant. Instead, if we fix the cosmological constant and let r_o vary, we otain the following result:

- $r_o > r_o^{crt} \Longrightarrow k^2 < 0 \Longrightarrow k \in \mathbb{C}$ Classically Stable
- $r_o < r_o^{crt} \Longrightarrow k^2 > 0 \Longrightarrow k \in \mathbb{R}$ Classically Unstable

Figure 7.3 plots k^2 *vs.* $r_h = r_o$.

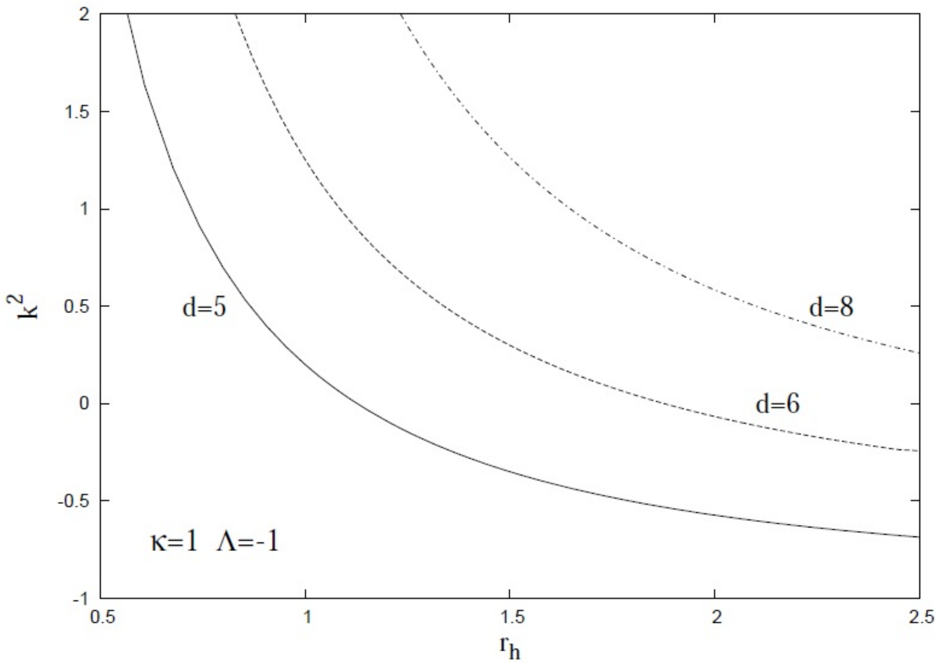

Figure 7.3: k^2 *vs.* r_o for various values of d. $k^2 = 0$ gives the critical value r_o^{crt}. For r_o above it we have stability, instability ortherwise. Taken from [63].

We can also draw some conclusions about the Gubsner-Mitra conjecture by looking at the plot of the temperature against the entropy. The results are seen to confirm it, that is, thermodynamical stability corresponds to a dynamically stable configuration whereas, thermodynamical instability corresponds to an unstable configuration. This is depicted in Figure 7.4 below.

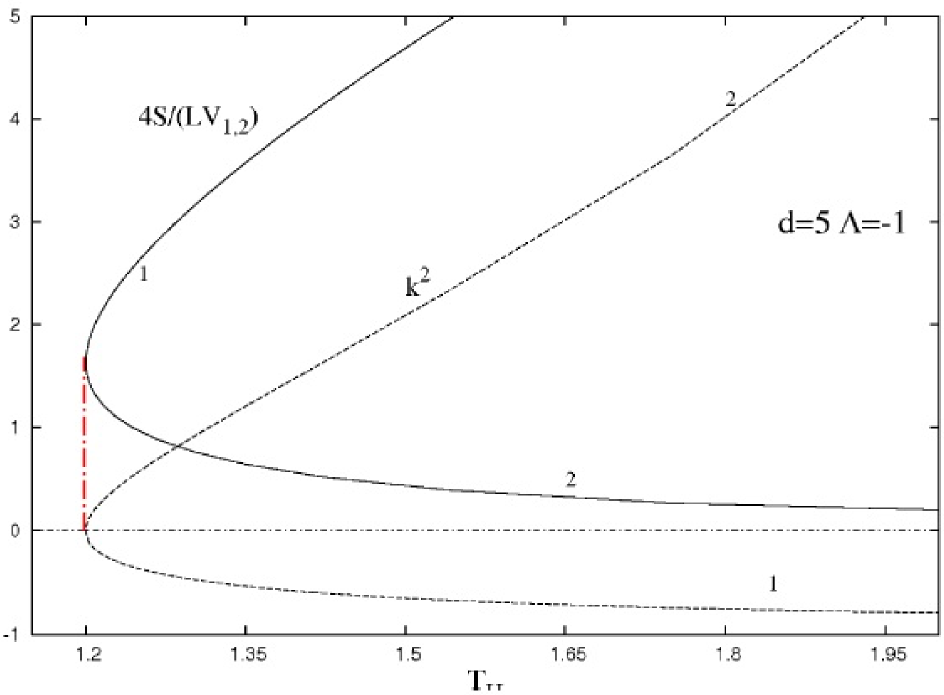

Figure 7.4: The entropy as well as k^2 are plotted in the graphic and we can see at once the correspondenc between thermodynamics and dinamical stability. The thermodynamically stable brach joins the unstable one at precisely the same temprature (marked in red) at which the dynamically stable branch $k^2 < 0$ joins the unstable one $k^2 > 0$. Taken from [64].

Chapter 8

Closing Remarks

One of the lessons learnt in this dissertation is that the Blackfold approach can reproduce, in the appropriate limits we pointed out, the results obtained by using the exact solutions. This can be done analytically and here lies the power of this method since, for instance, no analytical solution is known for Black Rings in 6 spacetime dimensions and one has to resort to numerical methods to get results. Although the Blackfold approach does not provide us with exact solutions it can give us a very complete picture of the physics of the problem.

Also, it is possible to use methods typically used in fluid dynamics which permit gaining insight and a more intuitive picture of the higher dimensional Black Hole dynamics, specially of its event horizons' evolution. This fluid analogy, makes it possible for us to study the thermodynamics of Blackfolds by simply applying the thermodynamics of a fluid to the collective variables of the Blackfold and deriving the consequences.

As far as we are concerned, there is still much work to do. In the Black Rings domain for instance, no analytical solution for spacetimes of dimensionality higher than 5 have been discovered. This would also open the door to the inclusion of more angular momenta, as in the case of the MPBH, for a given solution of the Black Ring (a 5-dimensional Black Ring with two angular momenta was found in [65]). There is also a lot of uncertainity as to which the final state of the Gregory Laflamme instability is, some numerical calculations have been carried out but, as we have seen, they led to naked singularities and to regimes where quantum gravity plays a crucial role.

To close this dissertation, we would like to point out a possible path of research concerning AdS which we are not sure wether it has alredy been explored or not. In his PhD thesis [64], Delsate suggests (page 105), following Hawking and Page [66], as a way of thinking of the AdS background characteristic length L and of its influence on the stability of the Black Ring, regarding it as box in which the Black Ring is placed. Two years later, Emparan and Martinez studied the effects on dynamical stability [52] of placing a Blackbrane inside a box of finite size. However, they do not make any reference to a possible application to AdS. With this ideas, we are tempted to think that it might be interesting to attempt to reproduce certain AdS solutions in the mebrane limit by simply placing a Blackbrane into the appropriate Box with the correct boundary conditions whose effect on the brane is the same as that of the curvature of the AdS background. We have seen how much simpler the Blackfold approach can make things. It has provided analytical approximations to problems for which the only way to be dealt with were numerical methods. Therefore, having a model which, although in certain limits and only to leading order, could give analytical results that matched the already existing numerical simulations, would be very helpful and would throw some light into the physics of the AdS solutions. This suggestion for future work together with the contents of section 6.5 are the *contributions* of this dissertation on the topic of Blackfolds.

Acknowledgements

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Appendix A

Black Hole Motion

In the appendices we present the derivation of some of the formulae encountered throughout the dissertation. They have been named exactly as the corresponding chapters and, althought the labelling of the equations does not match, each derivation has been appropriately referred in the text.

We are interested in finding the mass of a higher dimensional Schwarzschild-Tangherlini BH. The metric was given in (2.7) and (2.8). In order to use the ADM method we need to express it in cartesion coordinates, this can be achieved by going to isotropic coordinates so that the metric finally looks like:

$$ds^2 = -\kappa^2(\rho)dt^2 + \lambda^2(\rho)[d\rho^2 + \rho^2 d\Omega^2_{D-2}] \tag{A.1}$$

Confronting this equation with (2.7) we get the following conditions upon the coordinates:

$$r^2 = \lambda^2(\rho)\rho^2 \tag{A.2}$$

$$\lambda(\rho)d\rho = \pm\left(1 - \frac{C_D}{r^{D-3}}\right)^{-\frac{1}{2}} dr \tag{A.3}$$

Combining both and requiring that $\rho \to \infty$ when $r \to \infty$ we pick up the plus sign and get the following differential equation:

$$\frac{d\rho}{\rho} = \frac{dr}{\sqrt{r^2 - C_D r^{5-D}}} \tag{A.4}$$

Which can be integrated to give [24]:

$$r = \rho\left[2^{D-4}\left(1 + \frac{C_D}{2^{D-2}\rho^{D-3}}\right)^2\right]^{\frac{1}{D-3}} \tag{A.5}$$

We are only interested in the spatial part of the metric which now will look like:

$$ds^2 = -\kappa^2(\rho)dt^2 + \left[2^{D-4}\left(1 + \frac{C_D}{2^{D-2}\rho^{D-3}}\right)^2\right]^{\frac{2}{D-3}}[d\rho^2 + \rho^2 d\Omega^2_{D-2}] \tag{A.6}$$

Furthermore, we perform the following rescaling of the radial coordinate in order to decompose the metric as shown in (2.9):

$$R \equiv 2^{\frac{D-4}{D-3}}\rho \tag{A.7}$$

So, we finally have the metric in the form:

$$ds^2 = -\hat{\kappa}^2(R)dt^2 + \left(1 + \frac{C_D}{4R^{D-3}}\right)^{\frac{4}{D-3}}[dR^2 + R^2 d\Omega^2_{D-2}] \tag{A.8}$$

Now, let us expand the scale factor when $r \to \infty$, that is when $R \to \infty$, in terms of $\epsilon \equiv \frac{C_D}{R^{D-3}} \lll 1$, a similar expansion should be carried out in the temporal part in order to get the desired "Minkowski+terms" metric structure:

$$\left(1 + \frac{x}{4}\right)^{\frac{4}{D-3}} = 1 + \frac{x}{D-3} + \mathcal{O}(\epsilon^2) \tag{A.9}$$

The expanded metric is:

$$ds^2 = -\hat{\kappa}^2(R)dt^2 + \left[1 + \frac{1}{D-3}\frac{C_D}{R^{D-3}} + \mathcal{O}(\epsilon^2)\right][dR^2 + R^2 d\Omega^2_{D-2}] \tag{A.10}$$

According to R.Wald [25] the formula for the ADM mass is given in the appropriate units by:

$$E = M_{ADM} = \frac{1}{16\pi G}\lim_{R\to\infty}\sum_{\mu,\nu=1}^{D-1}\oint_{S_R^{D-2}}\left(\frac{\partial h_{\mu\nu}}{\partial x^\mu} - \frac{\partial h_{\mu\mu}}{\partial x^\nu}\right)N^\nu R^{D-2} d\Omega_{D-2} \tag{A.11}$$

Where $R^2 \equiv \sum_{\mu=1}^{D-1}(x^\mu)^2$, N^ν is the vectro normal to the surface of the $(D-2)$-dimensional sphere with radius R upon which we perform the integration. The spatial part of the metric in cartesian coordinate is:

$$dl^2 = \left[1 + \frac{1}{D-3}\frac{C_D}{R^{D-3}} + \mathcal{O}(\epsilon^2)\right][\sum_{\mu=1}^{D-1}(dx^\mu)^2] \tag{A.12}$$

Therefore, the pertirbation to the Minkowskian metric takes the form:

$$h_{\mu\nu} = \delta_{\mu\nu}h = \delta_{\mu\nu}\frac{C_D}{D-3}R^{-(D-3)} \tag{A.13}$$

The partial derivatives of the elements of the metric are:

$$\frac{\partial h_{\mu\nu}}{\partial x^\sigma} = -\delta_{\mu\nu}x_\sigma C_D R^{-(D-1)} \tag{A.14}$$

Substituting into (A.11) and rearranging temrs we get the following expression:

$$\begin{aligned}
M_{ADM} = \frac{1}{16\pi G}\lim_{R\to\infty}\int_{S_R^{D-2}}(D-2)C_D R^{-(D-1)}(\sum_{\sigma=1}^{D-1}x_\sigma N^\sigma)R^{D-2}d\Omega_{D-2} = \\
\frac{(D-2)C_D}{16\pi G}\lim_{R\to\infty}\int_{S_R^{D-2}}R^{-(D-1)}RR^{D-2}d\Omega_{D-2} = \\
\frac{(D-2)C_D}{16\pi G}\lim_{R\to\infty}\int_{S_R^{D-2}}R^{-(D-2)}R^{D-2}d\Omega_{D-2} = \\
\frac{(D-2)C_D}{16\pi G}\int_{S_R^{D-2}}d\Omega_{D-2} \Longrightarrow \\
M_{ADM} = \frac{C_D(D-2)\mathcal{A}_{(D-2)}}{16\pi G} \qquad \text{(A.15)}
\end{aligned}$$

Let us now derive both equations of motion for the BH, the first one is:

$$\begin{aligned}
u_\nu\bar{\nabla}_\mu T^{\mu\nu} = -u_\nu u_\mu u^\sigma\nabla_\sigma\left(m(\tau)u^\mu u^\nu\right) = \\
-u_\nu u_\mu u^\tau u^\mu u^\nu\partial_\tau m(\tau) - 2m(\tau)u_\mu u^\mu u^\sigma\left(u_\nu\nabla_\sigma u^\nu\right) = \\
-u^\tau\partial_\tau m(\tau) = 0 \Longrightarrow \\
\partial_\tau m(\tau) = 0 \qquad \text{(A.16)}
\end{aligned}$$

Taking into account (2.19) and the normalization condition of the velocity four-vector we get, after applying Leibniz rule a few times, the second one:

$$\begin{aligned}
(g_{\sigma\nu} + u_\sigma u_\nu)\bar{\nabla}_\mu T^{\mu\nu} = -(g_{\sigma\nu} + u_\sigma u_\nu)u_\mu u^\rho\nabla_\rho T^{\mu\nu} = \\
-g_{\sigma\nu}u_\mu u^\rho\nabla_\rho(mu^\mu u^\nu) - u_\sigma u_\nu u_\mu u^\rho\nabla_\rho(mu^\mu u^\nu) = \\
m\{u_\sigma u_\mu a^\mu + a_\sigma\} = m(\tau)a_\sigma = 0 \Longrightarrow \\
m(\tau)a_\sigma = 0 \qquad \text{(A.17)}
\end{aligned}$$

Where the acceleration is defined by $a^\sigma = u^\mu\nabla_\mu u^\sigma$

Appendix B

Blackfold Dynamics

In order to find the expression for the stress-energy tensor, we must first write the metric (3.4) in isotropic coordinates and go to its asymptotic limit so that we can decompose it as in (2.10). An analogous procedure to the one followed in the previous section with the replacement $D \to (D-p)$ gives:

$$ds^2 = \left\{ \eta_{ab} + \frac{C_{D-p}}{R^{D-p-3}} \left(1 + \frac{C_{D-p}}{4R^{D-p-3}} \right)^{-2} u_a u_b \right\} d\sigma^a d\sigma^b + \left(1 - \frac{1}{(D-p-3)} \frac{C_{D-p}}{R^{D-p-3}} + \mathcal{O}(\epsilon^2) \right) \sum_{i=1}^{D-p-1} (dx^i)^2 \tag{B.1}$$

where $R^2 = \sqrt{\sum_{i=1}^{n+2} (x^i)^2}$, $n = D-p-3$ and x^i are the usual spherical coordinates in $(n+1)$ dimensions and now $\epsilon \equiv \frac{C_{D-p}}{R^{D-p-3}}$. We can easily read off the elements of the perturbed metric, there are two kinds, worldvolume components with indices $a, b, c = 0, 1, ...p$:

$$h_{ab} = \frac{C_{D-p}}{R^{D-p-3}} \left(1 + \frac{C_{D-p}}{4R^{D-p-3}} \right)^{-2} u_a u_b \tag{B.2}$$

and transverse coordinates components with indices $i, j = 1, ..., D-p-1$:

$$h_{ij} = \delta_{ij} h = \delta_{ij} \frac{1}{(D-p-3)} \frac{C_{D-p}}{R^{D-p-3}} \tag{B.3}$$

The stress-energy tensor in the ADM formalism is given by the expression:

$$T_{ab} = \frac{\lim_{R\to\infty} \sum_{i,j=1}^{n+2}}{16\pi G} \oint_{S_R^{n+1}} [\eta_{ab} \left(\left(\sum_{c=0}^{p} \partial_i h_{cc} \right) + \partial_i h_{jj} - \partial_j h_{ji} \right) - \partial_i h_{ab}] d\mathcal{A}_{n+1}^i \tag{B.4}$$

where $d\mathcal{A}_{n+1}^i \equiv N^i R^{n+1} d\Omega_{n+1}$ and N^i is a radial unitary vector pointing outwards and normal to the surface of integration. Since the latter is the

$(n+1)$-dimensional sphere we have $x^i = RN^i$ so:

$$\sum_{i=1}^{n+2} x_i N^i = R \sum_{i=1}^{n+2} N_i N^i = R \tag{B.5}$$

Let us now collect the various partial derivatives that we need in order to compute the stress-energy tensor. Thus, making use of (B.2), (B.3) and (B.4) one finds:

$$\partial_j h_{ab} = x_j u_a u_b \frac{C_{D-p}(D-p-3)}{R^{D-p-1}} \left(1+\frac{\epsilon}{4}\right)^{-2} \left[\frac{\epsilon}{2}\left(1+\frac{\epsilon}{4}\right)^{-1} - 1\right] \tag{B.6}$$

It is straightforward to see that:

$$\lim_{R\to\infty} \partial_j h_{ab} = -x_j u_a u_b \frac{C_{D-p}(D-p-3)}{R^{D-p-1}} \tag{B.7}$$

we have not removed the power of R in the denominator on purpose because, as we will see, it cancels out with the one coming from the area element of the integral. The other derivatives we need are:

$$\partial_k h_{ij} = -x_k \frac{C_{D-p}}{R^{D-p-1}} \delta_{ij} \tag{B.8}$$

Now let us compute the first term in the integral, the cofactor of the Minkowskian metric. For simplicity we will take the limit where appropriate and keep the R's elsewhere in order to make the cancellation more explicitly:

$$\begin{aligned}\sum_{i,j=1}^{D-p-1} \left(\left(\sum_{c=0}^{p} \partial_i h_{cc}\right) + \partial_i h_{jj} - \partial_j h_{ji}\right) N^i = \sum_{i,j=1}^{D-p-1} \left(-x_i N^i \left(\sum_{c=0}^{p} u_c u^c\right) \frac{C_{D-p}}{R^{D-p-1}}\right. \\ \left.(D-p-3) - x_i N^i \frac{C_{D-p}}{R^{D-p-1}} \delta_{jj} + x_j N^i \frac{C_{D-p}}{R^{D-p-1}} \delta_{ij}\right) = \frac{C_{D-p}(D-p-3)}{R^{D-p-2}} \\ -\frac{C_{D-p}}{R^{D-p-2}}(D-p-1) + \frac{C_{D-p}}{R^{D-p-2}} = \frac{C_{D-p}}{R^{D-p-2}}(D-p-3-D+p+2) = \\ -\frac{C_{D-p}}{R^{D-p-2}}\end{aligned} \tag{B.9}$$

The second term yields:

$$\begin{aligned}\sum_{i=1}^{D-p-1} \partial_i h_{ab} N^i = \sum_{i=1}^{D-p-1} -N^i x_i u_a u_b \frac{C_{D-p}(D-p-3)}{R^{D-p-1}} = \\ -u_a u_b \frac{C_{D-p}(D-p-3)}{R^{D-p-2}}\end{aligned} \tag{B.10}$$

Collecting all the terms we have:

$$\begin{aligned}T_{ab} = \frac{1}{16\pi G}\oint_{S_R^{n+1}}\left[\eta_{ab}\left(-\frac{C_{D-p}}{R^{D-p-2}}\right) + u_a u_b\frac{C_{D-p}(D-p-3)}{R^{D-p-2}}\right]R^{D-p-2}d\Omega_{n+1} &=\\ \frac{C_{D-p}}{16\pi G}\oint_{S_R^{n+1}}\left[u_a u_b(D-p-3) - \eta_{ab}\right]\frac{1}{R^{D-p-2}}R^{D-p-2}d\Omega_{n+1} &=\\ \frac{C_{D-p}}{16\pi G}\left[u_a u_b(D-p-3) - \eta_{ab}\right]\oint_{S_R^{n+1}}d\Omega_{n+1} &=\\ \frac{C_{D-p}}{16\pi G}\left[u_a u_b(D-p-3) - \eta_{ab}\right]\mathcal{A}_{(D-p-2)} &\Longrightarrow\\ T_{ab} = \frac{r_o^{D-p-3}\mathcal{A}_{(D-p-2)}}{16\pi G}(u_a u_b(D-p-3) - \eta_{ab})\end{aligned} \tag{B.11}$$

This is our formula (3.7).

Let us derive the identity (3.21):

$$h^{\mu}{}_{\nu}\partial_a x^{\nu} = h^{\mu\sigma}g_{\sigma\nu}\partial_a x^{\nu} = \gamma^{cb}\partial_c x^{\mu}(\partial_b x^{\sigma}g_{\sigma\nu}\partial_a x^{\nu}) = \gamma^{cb}\partial_c x^{\mu}\gamma_{ba} = \delta^c_a\partial_c x^{\mu} = \partial_a x^{\mu} \tag{B.12}$$

We show how to derive the Blackfold equations from (3.32) by projection.
We first project in along u_μ:

$$\begin{aligned}u_\mu u^\mu u^\nu\bar\nabla_\nu\varepsilon + (\varepsilon+P)(u_\mu a^\mu + u_\mu u^\mu\bar\nabla_\nu u^\nu) + (u_\mu h^{\mu\nu} + u_\mu u^\mu u^\nu)\bar\nabla_\nu P + P u_\mu K^\mu &=\\ -u^\nu\bar\nabla_\nu\varepsilon - (\varepsilon+P)\bar\nabla_\nu u^\nu + (u^\nu - u^\nu)\bar\nabla_\nu P &=\\ -u^\nu\bar\nabla_\nu\varepsilon - (\varepsilon+P)\bar\nabla_\nu u^\nu &= 0\end{aligned} \tag{B.13}$$

The derivation of the orthogonal projection is straightforward. Now we perform a similar computation to find the different components of the equations of the Blackfold from (3.35) we project along parallel directions to the worldvolume:

$$\begin{aligned}h_{\rho\mu}a^\mu + \frac{1}{n+1}h_{\rho\mu}u^\mu\bar\nabla_\nu u^\nu = \frac{1}{n}h_{\rho\mu}K^\mu + h_{\rho\mu}\bar\nabla^\mu \ln r_o &\Longrightarrow\\ h_{\rho\mu}a^\mu + \frac{1}{n+1}u_\rho\bar\nabla_\nu u^\nu = \bar\nabla_\rho \ln r_o &\end{aligned} \tag{B.14}$$

The other set is found using the orthogonal operator $\perp_{\mu\nu}$:

$$\begin{aligned}\perp^\rho{}_\mu a^\mu + \frac{1}{n+1}\perp^\rho{}_\mu u^\mu\bar\nabla_\nu u^\nu = \frac{1}{n}\perp^\rho{}_\mu K^\mu + \perp^\rho{}_\mu\bar\nabla^\mu \ln r_o &\Longrightarrow\\ K^\rho = n\perp^\rho{}_\mu a^\mu &\end{aligned} \tag{B.15}$$

Taking (3.43) we use Leibniz rule and get:

$$\nabla_{(\mu}[\alpha u_{\nu)}] = [\nabla_{(\mu}\alpha]u_{\nu)} + \alpha[\nabla_{(\mu}u_{\nu)}] = [\nabla_{(\mu}\alpha]u_{\nu)} + \alpha[\omega_{(\mu\nu)} - u_{(\mu}a_{\nu)}] =$$
$$\frac{\alpha}{\alpha}u_{(\mu}[\nabla_{\nu)}\alpha] + \alpha[u_{(\mu}\nabla_{\nu)}\mathrm{ln}\mathcal{T}] = \alpha u_{(\mu}\left[\frac{1}{\alpha}\nabla_{\nu)}\alpha\right] + \alpha[u_{(\mu}\nabla_{\nu)}\mathrm{ln}\mathcal{T}] =$$
$$\alpha u_{(\mu}[\nabla_{\nu)}\mathrm{ln}\alpha] + \alpha u_{(\mu}[\nabla_{\nu)}\mathrm{ln}\mathcal{T}] = \alpha u_{(\mu}\nabla_{\nu)}\mathrm{ln}(\alpha\mathcal{T}) \quad \text{(B.16)}$$

where in the second line we have made use of the antisymmetry of the vorticity tensor.

We derive equations (3.48) and (3.49):

$$k^\mu k^\nu \nabla_{(\mu}k_{\nu)} = k^\mu k^\nu(\nabla_\mu k_\nu + \nabla_\nu k_\mu) = k^\mu[(\nabla_\mu k^\nu)k_\nu + k^\nu(\nabla_\mu k_\nu)] =$$
$$k^\mu[(\nabla_\mu(k^\nu k_\nu)] = -k^\mu\partial_\mu|\mathbf{k}|^2 = -2|\mathbf{k}|(k^\mu\partial_\mu|\mathbf{k}|) = 0 \Longrightarrow$$
$$k^\mu\partial_\mu|\mathbf{k}| = 0 \quad \text{(B.17)}$$

since $\mathbf{k}$ is timelike.

$$a^\mu = u^\nu\nabla_\nu u^\mu = \frac{k^\nu}{|\mathbf{k}|}\nabla_\nu\frac{k^\mu}{|\mathbf{k}|} = \frac{k^\nu}{|\mathbf{k}|}\left[\frac{\nabla_\nu k^\mu}{|\mathbf{k}|} + k^\mu\nabla_\nu\left(\frac{1}{|\mathbf{k}|}\right)\right] =$$
$$\frac{k^\nu}{|\mathbf{k}|}\left[-\frac{\nabla^\mu k_\nu}{|\mathbf{k}|} - \frac{k^\mu}{|\mathbf{k}|^2}\partial_\nu|\mathbf{k}|\right] = -\frac{2k^\nu}{2|\mathbf{k}|}\frac{\nabla^\mu k_\nu}{|\mathbf{k}|} = \frac{\partial^\mu(k^\nu k_\nu)}{2|\mathbf{k}|^2} =$$
$$\frac{\partial^\mu|\mathbf{k}|^2}{2|\mathbf{k}|^2} = \frac{\partial^\mu|\mathbf{k}|}{|\mathbf{k}|} = \partial^\mu\mathrm{ln}|\mathbf{k}| \Longrightarrow$$
$$a^\mu = \partial^\mu\mathrm{ln}|\mathbf{k}| \quad \text{(B.18)}$$

The derivation of the Blackfold paraneters is not hard but lengthy. Let us begin by computing what is inside the integral for the mass taking into account the equations:

$$n^\mu = \frac{1}{R_0}\xi^\mu \quad \text{(B.19)}$$

since n^μ stands for the normal to the spatial slice of the worldvolume it can be chosen to be proportional to the Killing vector field $\frac{\partial}{\partial t}$. Also, a little computation gives:

$$u = (1 - V^2)^{-1/2}\left(\frac{\partial}{\partial t} + \sum_{i=1}^{p} V_i\frac{\partial}{\partial z^i}\right) \quad \text{(B.20)}$$

Taking into account (3.73) we have:

$$
\begin{aligned}
T_{\mu\nu}n^\mu\xi^\nu = (\varepsilon+P)u_\mu u_\nu \frac{1}{R_0}\xi^\mu\xi^\nu + h_{\mu\nu}\frac{1}{R_0}\xi^\mu\xi^\nu P = \frac{\mathcal{A}_{(n+1)}}{16\pi G} r_o^n \left(\frac{nR_0}{(1-V^2)} - \frac{h_{\mu\nu}\xi^\mu\xi^\nu}{R_0} \right) &= \\
\frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{nR_0}{2\kappa}\sqrt{1-V^2}\right)^n \left(\frac{nR_0}{(1-V^2)} - \frac{h_{\mu\nu}\xi^\mu\xi^\nu}{R_0} \right) &= \\
\frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n R_0^{n+1}(1-V^2)^{\frac{n-2}{2}} \left(n - (1-V^2)\frac{h_{\mu\nu}\xi^\mu\xi^\nu}{R_0^2} \right) &= \\
\frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n R_0^{n+1}(1-V^2)^{\frac{n-2}{2}} \Big(n + (1-V^2) \Big) &\Longrightarrow
\end{aligned}
$$

$$
M = \int_{\mathcal{B}_p} dV_{(p)} T_{\mu\nu}n^\mu\xi^\nu = \frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n \int_{\mathcal{B}_p} dV_{(p)} R_0^{n+1}(1-V^2)^{\frac{n-2}{2}}\Big(n+1-V^2\Big) \tag{B.21}
$$

where we have used the fact that $h_{\mu\nu}\xi^\mu\xi^\nu = \xi^2 = -R_0^2$. A similar computation for the angular momenta gives:

$$
\begin{aligned}
T_{\mu\nu}n^\mu\chi_i^\nu = \frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n R_0^{n-1}(1-V^2)^{\frac{n}{2}} \Big(nu_\mu\xi^\mu u_\nu\chi_i^\nu - h_{\mu\nu}\xi^\mu\chi_i^\nu \Big) &= \\
\frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n R_0^{n-1} n (1-V^2)^{\frac{n-2}{2}} R_i^2\Omega_i &\Longrightarrow
\end{aligned}
$$

$$
J_i = -\int_{\mathcal{B}_p} dV_{(p)} T_{\mu\nu}n^\mu\chi_i^\nu = \frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n n\Omega_i \int_{\mathcal{B}_p} dV_{(p)} R_0^{n-1}(1-V^2)^{\frac{n-2}{2}} R_i^2 \tag{B.22}
$$

where we have made use of the following identities: $u_\mu\xi^\mu = \frac{R_0}{\sqrt{1-V^2}}$ and $u_\mu\chi_i^\mu = \frac{\Omega_i R_i^2}{R_0\sqrt{1-V^2}}$ (no sum).

We prove now the first law (3.79):

$$
\begin{aligned}
\beta\left(M - \sum_{i=1}^p \Omega_i J_i - \frac{\kappa}{8\pi G}\mathcal{A}_H \right) &= \\
\beta\frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n \int_{\mathcal{B}_p} dV_{(p)} R_0^{n+1}(1-V^2)^{\frac{n}{2}}\{(n+1-V^2-nV^2)(1-V^2)^{-1}-n\} &= \\
\beta\frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n \int_{\mathcal{B}_p} dV_{(p)} R_0^{n+1}(1-V^2)^{\frac{n}{2}}\{1\} &= \\
\beta\frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n \int_{\mathcal{B}_p} dV_{(p)} R_0 \left(R_0^2 - \sum_{i=1}^p \Omega_i^2 R_i^2 \right)^{\frac{n}{2}} &\Longrightarrow
\end{aligned}
$$

$$
\beta\left(M - \sum_{i=1}^p \Omega_i J_i - \frac{\kappa}{8\pi G}\mathcal{A}_H \right) = \frac{\mathcal{A}_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n I = \hat{I} \tag{B.23}
$$

where we have repeatedly used $\sum_{i=1}^p \Omega_i^2 R_i^2 = V^2 R_0^2$.

Appendix C

Black Rings

To derive the expression (5.33) we will proceed by explicitly computing both sides and confirming that we arrive at the same result. Let us begin by using the definition of the total tensional energy (5.32). We recall the following useful formulae which will be used throughout:

$$n^a = \frac{\xi^a}{R_0}, \quad u^a = \frac{k^a}{|k|}, \quad k^a = \xi^a + \Omega\chi^a, \quad P = \frac{-\varepsilon}{(n+1)}$$
$$\xi^a\xi_a = -R_0^2, \quad \xi^a\chi_a = 0, \quad |k| = R_0\sqrt{1-V^2}, \quad \gamma^{ab}\gamma_{ab} = \delta^a{}_a = p+1 \tag{C.1}$$

We will be using these formulae throughout the calculation. Let us begin by computing the integrand:

$$\begin{aligned}
-R_0\left(\gamma^{ab} + n^a n^b\right)T_{ab} = -R_0\left(T^a{}_a + \frac{1}{R_0^2}\,\xi^a\xi^b((\varepsilon+P)\frac{k_a}{|k|}\frac{k_b}{|k|} + \gamma_{ab}P)\right) = \\
-R_0\left((-(\varepsilon+P) + P(D-n-2)) + \frac{(\xi^a k_a)^2}{(R_0|k|)^2}(\varepsilon+P) + \frac{\xi^a\xi_a}{R_0^2}P\right) = \\
-R_0\left(P(n+1) - P + P(D-n-2) + \frac{(\xi^a\xi_a)^2}{(R_0^2\sqrt{1-V^2})^2}(-P(n+1)+P) + \frac{\xi^a\xi_a}{R_0^2}P\right) = \\
-R_0\left(P(D-2) - nP\frac{(-R_0^2)^2}{(R_0^4(1-V^2))} + \frac{-R_0^2}{R_0^2}P\right) = -R_0\left(P(D-2) - \frac{nP}{(1-V^2)} - P\right) = \\
-R_0P\left((D-2) - \left(\frac{n}{(1-V^2)} + 1\right)\right) = -R_0P\left((D-2) - \left(\frac{n+1-V^2}{(1-V^2)}\right)\right)
\end{aligned} \tag{C.2}$$

Now let us proceed to compute the other side of the equation. First, recall that we have:

$$s = -P4\pi r_o, \quad T = \frac{|k|n}{4\pi r_o} = \frac{R_0\sqrt{1-V^2}n}{4\pi r_o} \tag{C.3}$$

By using the expressions (3.60), (3.72), (3.74) and (3.75) we can compute the integrand of the RHS of (5.33):

$$
\begin{aligned}
(D-3)M-(D-2)\Bigg(TS+\sum_{i=1}^{p}\Omega_i J_i\Bigg) \propto -(D-3)PR_0(1-V^2)^{-1}(n+1-V^2)-\\
(D-2)\Big\{-P4\pi r_o\frac{1}{\sqrt{1-V^2}}\frac{R_0 n\sqrt{1-V^2}}{4\pi r_o}-R_0^{-1}nP(1-V^2)^{-1}V^2R_0^2\Big\}=\\
-\frac{PR_0}{(1-V^2)}\{(D-3)(n+1-V^2)+(D-2)(-n(1-V^2)-nV^2)\}=\\
-\frac{PR_0}{(1-V^2)}\{(D-3)(n+1-V^2)-(D-2)n\}=\\
-\frac{PR_0}{(1-V^2)}\{(D-3)(1-V^2)-n\}=-PR_0\Big\{(D-3)-\Big(\frac{n+1-V^2-1+V^2}{(1-V^2)}\Big)\Big\}=\\
-PR_0\Big\{(D-3)-\Big(\frac{n+1-V^2}{(1-V^2)}\Big)+1\Big\}=-PR_0\Big\{(D-2)-\Big(\frac{n+1-V^2}{(1-V^2)}\Big)\Big\}
\end{aligned}
\tag{C.4}
$$

This is nothing but (C.2). Since the integrals are both over $\mathcal{B}_p$ we see that both sides of (5.33) are indeed equal.

We will derive now (5.40). Recall that $n^a=\xi^a$ and $\xi^a u_a=-\cosh\sigma$:

$$
\begin{aligned}
M=\int_{S_R^1} dV_1\; T_{ab}n^a\xi^b=\int_{S_R^1} dV_1\{(\varepsilon+P)u_a u_b\xi^a\xi^b+P\xi_a\xi^a\}=\\
\int_{S_R^1} dV_1\{\varepsilon\cosh^2\sigma+P(\cosh^2\sigma-1)\}=\int_{S_R^1} dV_1\{\varepsilon\cosh^2\sigma+P\sinh^2\sigma\}=\\
\int_{S_R^1} dV_1\Big\{\frac{\mathcal{A}_{(2)}2r_o}{16\pi G}\cosh^2\sigma+\frac{\mathcal{A}_{(2)}r_o}{16\pi G}\sinh^2\sigma\Big\}=\frac{\mathcal{A}_{(2)}r_o}{16\pi G}(\cosh^2\sigma+1)\int_{S_R^1} dV_1=\\
\frac{3\mathcal{A}_{(2)}r_o}{16\pi G}2\pi R=\frac{3\pi}{2G}r_o\;R
\end{aligned}
\tag{C.5}
$$

In order to get the metric of a Blackstring (5.27) we must rearrange the terms in (5.26). Here we compute the dz^2 term which requires little manipulation. From (5.26), we collect the terms that go with dz^2:

$$
\begin{aligned}
\frac{1}{\hat{f}}\left(f-\frac{r_o^2}{r^2}\sinh^2\sigma\cosh^2\sigma\right)=\frac{1}{\hat{f}}\left(1+\frac{r_o^2}{r^2}(\sinh^2\sigma-\cosh^2\sigma)-\frac{r_o^2}{r^2}\sinh^2\sigma\cosh^2\sigma\right)=\\
\frac{1}{\hat{f}}\hat{f}\left(1+\frac{r_o}{r}\sinh^2\sigma\right)=\left(1+\frac{r_o}{r}\sinh^2\sigma\right)
\end{aligned}
\tag{C.6}
$$

Appendix D

Gregory-Laflamme Instability

Let us compute explicitly the limit $r \to \infty$ of the equations (6.18). Taking this limit makes $V \to 1$ so we are left with:

$$\begin{aligned} h_+ &\approx h_- \frac{(2\Omega^2 + m^2)}{m^2} \\ \partial_r h &\approx \frac{\Omega(h_+ + h_-)}{2} \\ \partial_r h_- &\approx \frac{m^2 h}{\Omega} \end{aligned} \tag{D.1}$$

which reduces to the pair:

$$\begin{aligned} \partial_r h \approx \frac{\Omega}{2}\left(h_- \frac{(2\Omega^2 + m^2)}{m^2} + h_-\right) &= \frac{\Omega}{m^2}(\Omega^2 + m^2)h_- \\ \partial_r h_- &\approx \frac{m^2 h}{\Omega} \end{aligned} \tag{D.2}$$

By differentiating with respect to r in the first equation and substituting the second into the result, we obtain a SHO type of equation for h:

$$\partial_r^2 h \approx (\Omega^2 + m^2)h \tag{D.3}$$

We see that the solutions shown in the sixth chapter satisfy these equations. One can easily get h_+ from the constraint once h and h_- are known.

Now, we will derive the approximate position of the apparent horizon. For this, the first step was to compute the radial geodesic equation for the perturbed metric of the black string. Rather than using the general formulae for the geodesic equations which are second order differential equations we will use the condition imposed on the tangent vector of a null geodesic which yields a first order differential equation, thus:

$$(g_{\mu\nu} + h_{\mu\nu})\dot{x}^{\mu}\dot{x}^{\nu} = 0 \tag{D.4}$$

where $\dot{x}^{\mu} = \frac{\partial x^{\mu}}{\partial \tau}$. Since we are interested in the radial geodesics we must set $\dot{\theta} = \dot{\varphi} = \dot{z} = 0$. Recall that we are working with the metric (6.28):

$$ds^2 = -\frac{(r - r_+)}{r} du^2 + 2dudr + r^2 d\Omega_2^2 + dz^2 \tag{D.5}$$

So we have:

$$g_{uu} = -\frac{(r - r_+)}{r}, \quad g_{ur} = 1, \quad g_{rr} = 0 \tag{D.6}$$

and including the perturbation, the geodesic equation looks like:

$$(g_{uu} + h_{uu})\dot{u}\dot{u} + 2(g_{ur} + h_{ur})\dot{u}\dot{r} + (g_{rr} + h_{rr})\dot{r}\dot{r} = 0 \tag{D.7}$$

multiplying now by $\left(\frac{\partial \tau}{\partial u}\right)^2$ and substituting the values of the unperturbed metric we finally get:

$$h_{uu} - \frac{(r - r_+)}{r} + 2(1 + h_{ur})\frac{dr}{du} + h_{rr}\left(\frac{dr}{du}\right)^2 = 0 \tag{D.8}$$

Now, we solve for the $\frac{dr}{du}$ which is not being multiplied by any perturbation element and substitute the rest of them by its unperturbed value as given by (6.29), $\frac{dr}{du} = \frac{r-r_+}{2r}$. We get:

$$\frac{dr}{du} \approx \frac{r - r_+}{2r} - \frac{r - r_+}{2r} h_{ur} - \frac{1}{2} h_{uu} - \frac{(r - r_+)^2}{8r^2} h_{rr} \tag{D.9}$$

Now we require the above equation to be equal to zero, rearranging and multiplying by $2r$ we get:

$$(r - r_+) - (r - r_+)h_{ur} - r h_{uu} - \frac{(r - r_+)^2}{4r_+} h_{rr} = 0 \tag{D.10}$$

Again, we solve for the first r we find in the equation and take the limit $r \to r_+$ for the rest:

$$r \approx r_+(1 + h_{uu}) + \lim_{r \to r_+}\left[(r - r_+)h_{ur} + \frac{(r - r_+)^2}{4r_+} h_{rr}\right] \tag{D.11}$$

Now we recall that the perturbations have a factor $e^{i\mu z}$ and that they are finite and regular at the horizon as well as at infinity. Taking the real part of them gives in the above equation:

$$r \approx r_+ + (A + B + C)\cos(\mu z) \tag{D.12}$$

where A, B and C stand for the results when the limit has been taken for the perturbation, we can group them into a single constant to finally get:

$$r \approx r_+ + const.\ \cos(\mu z) \tag{D.13}$$

Let us continue with the Blackfold approach. We present the derivation of various formulae appearing in section 6.4. The perturbed extrinsic equations are:

$$T^{\mu\nu}\delta K_{\mu\nu}{}^{\rho} = 0 \tag{D.14}$$

by using (6.41) we can expand:

$$(T^{tt}\partial_t\partial_t + T^{ti}\partial_t\partial_i + T^{ij}\partial_i\partial_j)\xi^{\rho} = 0 \tag{D.15}$$

now, substituting (6.40):

$$((\varepsilon + \delta\varepsilon)\partial_t\partial_t + ((\varepsilon + P)v^i)\partial_t\partial_i + \left(P + \frac{dP}{d\varepsilon}\delta\varepsilon\right)\sum_{j=1}^{p}\partial_j\partial_j)\xi^{\rho} = 0 \tag{D.16}$$

retaining only the terms linear in any of the perturbations we are left with (6.42):

$$(\varepsilon\partial_t^2 + P\sum_{j=1}^{p}\partial_i^2)\xi^{\rho} = 0 \tag{D.17}$$

The intrinsic equations are:

$$\partial_t T^{tt} + \partial_i T^{it} = 0, \quad \partial_t T^{tj} + \partial_i T^{ij} = 0 \tag{D.18}$$

conveniently derivating both expressions, upon substraction we get:

$$\partial_t^2 T^{tt} - \partial_i\partial_j T^{ij} = 0 \tag{D.19}$$

which upon substituting (6.40):

$$\partial_t^2(\varepsilon + \delta\varepsilon) - \sum_{j=1}^{p}\partial_j\partial_j\left(P + \frac{dP}{d\varepsilon}\delta\varepsilon\right) = 0 \tag{D.20}$$

the zeroth order perturbation part is satisfied and we are thus left with the first order piece which gives the evolution of the perturbation:

$$\left(\partial_t^2 - \frac{dP}{d\varepsilon}\sum_{j=1}^{p}\partial_j\partial_j\right)\delta\varepsilon = 0 \tag{D.21}$$

this is (6.46). In order to get the dispersion elation we substitute (6.47), (6.50), (6.51) and recall that we must take $\delta r_o \sim \delta\varepsilon$:

$$\left(\partial_t^2 + \frac{1}{n+1}\sum_{j=1}^{p}\partial_j^2\right)\delta\varepsilon = \left(\Omega^2 + \frac{1}{n+1}i^2\mu^2\right)\delta\varepsilon = 0 \tag{D.22}$$

which yields:

$$\Omega^2 = \frac{1}{n+1}\mu^2 \tag{D.23}$$

and we get (6.52) straightforwardly. In order to prove (6.54) we first need to express certain quantities in terms of ε. Using the expression for a neutral Blackfold (3.11), (3.12) and (3.16) we can rewrite:

$$\mathcal{T} = \frac{n}{4\pi}\left(\frac{\mathcal{A}_{(n+1)}}{16\pi G}(n+1)\right)^{\frac{1}{n}}\varepsilon^{-\frac{1}{n}}, \quad s = \frac{\mathcal{A}_{(n+1)}}{4G}\left(\frac{\mathcal{A}_{(n+1)}}{16\pi G}(n+1)\right)^{-\frac{n+1}{n}}\varepsilon^{\frac{n+1}{n}} \tag{D.24}$$

and from the first equation we easily get:

$$\frac{d\mathcal{T}}{d\varepsilon} = -\frac{1}{4\pi}\left(\frac{\mathcal{A}_{(n+1)}}{16\pi G}(n+1)\right)^{\frac{1}{n}}\varepsilon^{-\frac{n+1}{n}} \tag{D.25}$$

multiplying:

$$s\left|\frac{d\mathcal{T}}{d\varepsilon}\right| = \frac{1}{4\pi}\left(\frac{\mathcal{A}_{(n+1)}}{16\pi G}(n+1)\right)^{\frac{1}{n}}\varepsilon^{-\frac{n+1}{n}}\frac{\mathcal{A}_{(n+1)}}{4G}\left(\frac{\mathcal{A}_{(n+1)}}{16\pi G}(n+1)\right)^{-\frac{n+1}{n}}\varepsilon^{\frac{n+1}{n}} \tag{D.26}$$

cancelling out terms we are left with:

$$s\left|\frac{d\mathcal{T}}{d\varepsilon}\right| = \frac{1}{(n+1)} \tag{D.27}$$

this immediately give (6.54).

We now move forward to derive equations (6.60). It is not hard but rather lengthy a calculation. The equations of motion are found from the conservation of the stress-energy tensor:

$$\partial_t T^{tt} + \partial_i T^{it} = 0 \tag{D.28}$$

$$\partial_t T^{tj} + \partial_i T^{ij} = 0 \tag{D.29}$$

Let us begin by computing (D.28). The tt component of the stress-energy tensor is given by:

$$T^{tt} = \varepsilon u^t u^t + P\Pi^{tt} - \zeta\theta\Pi^{tt} - 2\eta\sigma^{tt} \tag{D.30}$$

since $u^t = 1$ we have $\Pi^{tt} = \eta^{tt} + u^t u^t = -1 + 1 = 0$ so we are just left with:

$$T^{tt} = \varepsilon - 2\eta\sigma^{tt} \tag{D.31}$$

Let us compute σ^{tt}:

$$\begin{aligned}
&\sigma^{tt} = \Pi^{tc}\Pi^{td}\partial_{(c}u_{d)} - \tfrac{\theta}{p}\Pi^{tt} = \Pi^{tc}\Pi^{td}\partial_{(c}u_{d)} = (\eta^{tc} + u^c)(\eta^{td} + u^d)\tfrac{1}{2}(\partial_c u_d + \partial_d u_c) \\
&= \tfrac{1}{2}\{\eta^{tc}\eta^{td}\partial_c u_d + \eta^{tc}u^d\partial_c u_d + u^c\eta^{td}\partial_c u_d + u^c u^d \partial_c u_d \\
&+\eta^{tc}\eta^{td}\partial_d u_c + \eta^{tc}u^d\partial_d u_c + u^c\eta^{td}\partial_d u_c + u^c u^d\partial_d u_c\} \\
&= \tfrac{1}{2}\{\partial^t u^t + u^d\partial^t u_d + u^c\partial_c u^t + u^c u^d\partial_c u_d + \partial^t u^t + u^d\partial_d u^t + u^c\partial^t u_c + u^c u^d\partial_d u_c\} \\
&= u^d\partial^t u_d + u^c u^d\partial_d u_c = u^t\partial^t u_t + 2u^j\partial_t u_j + u^t u^t\partial_t u_t + u^t u^i\partial_i u_t + u^j u^i\partial_i u_j \\
&= 2u^j\partial_t u_j + u^j u^i\partial_i u_j
\end{aligned} \tag{D.32}$$

recall that $u^t = -u_t = 1$ so that all its derivatives vanish. So, (D.31) finally looks like:

$$T^{tt} = \varepsilon - 2\eta(2u^j\partial_t u_j + u^j u^i\partial_i u_j) \tag{D.33}$$

Note that: $-1 = u^a u_a = u^t u_t + u^i u_i = -1 + u^i u_i \Longrightarrow u^i u_i = 0$ and hence $0 = \partial_j(u^j u^i u_i) = (\partial_j u^j)u^i u_i + 2u^j u^i\partial_j u_i \Longrightarrow u^j u^i\partial_j u_i = 0$ so we get:

$$T^{tt} = \varepsilon - 4\eta u^j\partial_t u_j \tag{D.34}$$

when we introduce the perturbation we get:

$$T^{tt} = \varepsilon + \delta\varepsilon - 4\eta\delta u^j\partial_t\delta u_j \tag{D.35}$$

We continue computing the ti components:

$$T^{ti} = \varepsilon u^t u^i + P\,\Pi^{ti} - \zeta\theta\Pi^{ti} - 2\eta\sigma^{ti} \tag{D.36}$$

we now have $\Pi^{ti} = u^i$ so we can compute the components σ^{ti}:

$$\begin{aligned}
&\sigma^{ti} = \Pi^{tc}\Pi^{id}\partial_{(c}u_{d)} - \tfrac{\theta}{p}\Pi^{ti} = (\eta^{tc} + u^c)(\eta^{id} + u^i u^d)\partial_{(c}u_{d)} - \tfrac{\theta}{p}u^i \\
&= (\eta^{tt} + u^t)(\eta^{it} + u^i u^t)\partial_{(t}u_{t)} + (\eta^{tt} + u^t)(\eta^{ij} + u^i u^j)\partial_{(t}u_{j)} \\
&+(\eta^{tl} + u^l)(\eta^{it} + u^i u^t)\partial_{(l}u_{t)} + (\eta^{tl} + u^l)(\eta^{ij} + u^i u^j)\partial_{(l}u_{j)} - \tfrac{\theta}{p}u^i \\
&= u^l u^i\partial_{(l}u_{t)} + u^l(\eta^{ij} + u^i u^j)\partial_{(l}u_{j)} - \tfrac{\theta}{p}u^i \\
&= u^l(\delta^{ij} + u^i u^j)\tfrac{1}{2}(\partial_l u_j + \partial_j u_l) - \tfrac{\theta}{p}u^i \\
&= \tfrac{1}{2}\{u^l(\partial_l u^i + \partial^i u_l) + u^l u^i u^j(\partial_l u_j + \partial_j u_l)\} - \tfrac{\theta}{p}u^i \\
&= \tfrac{1}{2}u^l\partial_l u^i - \tfrac{\theta}{p}u^i = \{\tfrac{1}{2}u^l\partial_l - \tfrac{\theta}{p}\}u^i
\end{aligned} \tag{D.37}$$

so (D.36) is:

$$T^{ti} = \varepsilon u^i + P\,u^i - \zeta\theta u^i - 2\eta\left\{\frac{1}{2}u^l\partial_l - \frac{\theta}{p}\right\}u^i \tag{D.38}$$

when we introduce the perturbation we get:

$$T^{ti} = \left[(\varepsilon + \delta\varepsilon) + (P + c_L^2\delta\varepsilon) - \zeta\delta\theta - 2\eta\left\{\frac{1}{2}\delta u^l\partial_l - \frac{\delta\theta}{p}\right\}\right]\delta u^i \tag{D.39}$$

Plugging into (D.28) the results (D.35) and (D.39), retaining only terms to first order in the perturbation we get:

$$\partial_t \delta\varepsilon + (\partial_i \delta u^i)\varepsilon + P\partial_i \delta u^i = 0 \tag{D.40}$$

recalling the form of the perturbation (6.59) it is straightforward to get:

$$\omega\delta\varepsilon + (\varepsilon + P)\mu_i \delta v^i = 0 \tag{D.41}$$

But for the higher order terms, this is nothing but (6.60). Let us continue with (D.29). We have already computed one of the ingredients of this equation and the other has been simplified in [49] to:

$$T^{ij} = P\ \delta^{ij} - \eta\left(2\partial^{(i}u^{j)} - \frac{2}{p}\delta^{ij}\partial^l u_l\right) - \zeta\delta^{ij}\partial^l u_l \tag{D.42}$$

after introducing the perturbation the expresion is modified to give:

$$T^{ij} = (P + c_L^2\delta\varepsilon)\ \delta^{ij} - \eta\left(2\partial^{(i}\delta u^{j)} - \frac{2}{p}\delta^{ij}\partial^l \delta u_l\right) - \zeta\delta^{ij}\partial^l \delta u_l \tag{D.43}$$

Again, plugging into (D.28) the results (D.39) and (D.43), retaining only terms to first order in the perturbation we get:

$$\varepsilon\partial_t\delta u^i + P\partial_t\delta u^i + c_L^2\delta^{ij}\partial_j\delta\varepsilon - 2\eta\partial_j\partial^{(j}\delta u^{i)} + \eta\frac{2}{p}\delta^{ij}\partial_j\partial^l\delta u_l - \zeta\delta^{ij}\partial_j\partial^l\delta u_l = 0 \tag{D.44}$$

which, after using (6.59) gives, up to higher order terms, the second equation of (6.60):

$$i\omega(\varepsilon + P)\delta v^j + ic_L^2\mu^j\delta\varepsilon + \eta\mu^2\delta v^j + \mu^j\left(\left(1 - \frac{2}{p}\right)\eta + \zeta\right)\mu_i\delta v^i = 0 \tag{D.45}$$

Finally, in order to get the dispersion relation (6.62) we need to proceed in various steps. First, multiply (6.61) by ω and then make the substitution $\omega = -i\Omega$, we are left with:

$$\Omega^2 + \Omega\frac{\mu^2}{sT}\left(2\left(1 - \frac{1}{p}\right)\eta + \zeta\right) + c_L^2\mu^2 = 0 \tag{D.46}$$

the solution for the second degree equation is:

$$\Omega = \frac{1}{2}\left\{ - \frac{\mu^2}{sT}\left(2\left(1 - \frac{1}{p}\right)\eta + \zeta\right) \pm \sqrt{\left[\frac{\mu^2}{sT}\left(2\left(1 - \frac{1}{p}\right)\eta + \zeta\right)\right]^2 - 4c_L^2\mu^2}\right\} \tag{D.47}$$

we can now rewrite the following piece of the solution:

$$\frac{1}{2}\sqrt{\left[\frac{\mu^2}{sT}\left(2\left(1 - \frac{1}{p}\right)\eta + \zeta\right)\right]^2 - 4c_L^2\mu^2} = \frac{1}{2}\mu\sqrt{\frac{\mu^2}{s^2T^2}\left(2\left(1 - \frac{1}{p}\right)\eta + \zeta\right)^2 - 4c_L^2} \tag{D.48}$$

we now expand the square root in a Taylor series about $\mu = o$ and we get:

$$\frac{1}{2}\mu\sqrt{\frac{\mu^2}{s^2T^2}\left(2\left(1-\frac{1}{p}\right)\eta+\zeta\right)^2-4c_L^2} = \mu\sqrt{-c_L^2}+\mathcal{O}(\mu^3) \tag{D.49}$$

so the solution (D.47) is:

$$\Omega = -\frac{\mu^2}{2sT}\left(2\left(1-\frac{1}{p}\right)\eta+\zeta\right) \pm \left(\mu\sqrt{-c_L^2}+\mathcal{O}(\mu^3)\right) \tag{D.50}$$

and since we need $\Omega > 0$ we pick up the plus sign to get (6.62):

$$\Omega = \mu\sqrt{-c_L^2} - \left(\left(1-\frac{1}{p}\right)\eta+\frac{\zeta}{2}\right)\frac{\mu^2}{sT}+\mathcal{O}(\mu^3) \tag{D.51}$$

The derivation of (6.67) is straightforward, we just need to plugg (6.66) into (D.18) and retain terms to first order in the perturbation. In order to derive the dispersion relation (6.70) we first use the generic form of the perturbation (6.68) and we get the following system of equations for the perturbation coefficients:

$$\begin{aligned}&(\Omega + \tfrac{1}{2}ik\ \mathrm{tanh}\sigma)\delta\hat{\varepsilon} + (\varepsilon+P)ik\delta\hat{u}^z = 0\\ &\left(\tfrac{\mathrm{tanh}\sigma}{2}\Omega - \tfrac{1}{2}ik + \tfrac{\mathrm{tanh}^2\sigma}{2}ik\right)\delta\hat{\varepsilon} + (2(\varepsilon+P)\mathrm{tanh}\sigma\ ik + (\varepsilon+P)\Omega)\delta\hat{u}^z = 0\end{aligned} \tag{D.52}$$

the determinant of the coefficients can be read off from the above and is simply:

$$\begin{vmatrix} (\Omega + \frac{1}{2}ik\ \mathrm{tanh}\sigma) & (\varepsilon+P)ik \\ \left(\frac{\mathrm{tanh}\sigma}{2}\Omega - \frac{1}{2}ik + \frac{\mathrm{tanh}^2\sigma}{2}ik\right) & (\varepsilon+P)(2\ \mathrm{tanh}\sigma\ ik+\Omega) \end{vmatrix}$$

as we said in the corresponding section, making this equal to zero gives the dispersion relation (6.70).

The derivation of (6.74) proceeds in a similar fashion. From (D.15) and (6.66) we have:

$$\begin{aligned}&[(\varepsilon+\delta\varepsilon)\partial_t\partial_t + ((\varepsilon+\delta\varepsilon+P+\delta P)(u^i+\delta u^i))\partial_t\partial_i +\\ &((\varepsilon+\delta\varepsilon+P+\delta P)(u^iu^j+\delta u^iu^j+u^i\delta u^j)+(P+\delta P)\delta^{ij})\partial_i\partial_j]\xi^\rho = 0\end{aligned} \tag{D.53}$$

If we retain only terms to first order in the perturbation we are left with (6.74):

$$[\varepsilon\partial_t^2 + (\varepsilon+P)u^i\partial_i\partial_t + (\varepsilon+P)u^iu^j\partial_i\partial_j + P\partial_i\partial^i]\xi^\rho = 0 \tag{D.54}$$

Let us proceed to derive equation (6.100), first we need to project equation (6.93) to obtain (6.99):

$$\begin{aligned}
&{\Pi_a}^b \nabla_b \mathrm{ln} r_o^{n+1} = \theta {\Pi_a}^b u_b + (n+1){\Pi_a}^b a_b \\
&{\Pi_a}^b (n+1)\nabla_b \mathrm{ln} r_o = (n+1) a_a \\
&{\Pi_a}^b \nabla_b \mathrm{ln} r_o = a_a
\end{aligned} \tag{D.55}$$

Bearing this in mind and using (6.96), (6.97) and (6.98) we can readily compute (6.100):

$$\begin{aligned}
&\nabla_a \phi = {\Pi_a}^b \nabla_b \phi = \tfrac{1}{2}{\Pi_a}^b \nabla_b \mathrm{ln}\left(1 - \tfrac{r_o^n}{R^n}\right) = \tfrac{-n}{2f(R)} \tfrac{r_o^{n-1}}{R^n} {\Pi_a}^b \nabla_b r_o = \\
&\tfrac{-n}{2f(R)} \tfrac{r_o^n}{R^n} {\Pi_a}^b \tfrac{\nabla_b r_o}{r_o} = \tfrac{-n}{2} \tfrac{r_o^n}{f(R)R^n} {\Pi_a}^b \nabla_b \mathrm{ln} r_o = \tfrac{-n}{2} \tfrac{r_o^n}{R^n - r_o^n} a_a = \tfrac{-n}{2} \tfrac{r_o^n - R^n + R^n}{R^n - r_o^n} a_a = \\
&\tfrac{-n}{2}\left(\tfrac{R^n}{R^n - r_o^n} - 1\right) a_a = \tfrac{-n}{2}\left(\tfrac{1}{f(R)} - 1\right) a_a
\end{aligned} \tag{D.56}$$

Now, let us compute the value of the Critical Point. We must require the prefactor of a_a in the above expression to be equal to unity:

$$\begin{aligned}
&\tfrac{-n}{2}\left(\tfrac{1}{f(R)} - 1\right) = 1 \Longrightarrow \tfrac{1}{f(R)} = \tfrac{2}{n} + 1 \Longrightarrow 1 - \tfrac{r_o^n}{R^n} = \tfrac{n}{n+2} \Longrightarrow \\
&\tfrac{r_o^n}{R^n} = \tfrac{2}{n+2} \Longrightarrow R_c = r_o \left(\tfrac{2+n}{2}\right)^{\frac{1}{n}}
\end{aligned} \tag{D.57}$$

By using the above equation we can rewrite (6.103):

$$\frac{\hat{a}_a}{a_a} = \frac{\hat{c}_L^2}{c_L^2} = \left[1 - \frac{n}{2}\left(\frac{1}{f(R)} - 1\right)\right] = -(n+1)\hat{c}_L^2 \tag{D.58}$$

from which we can solve for $\hat{c}_L^2$:

$$\begin{aligned}
&\hat{c}_L^2 = \tfrac{-1}{n+1}\left\{1 - \tfrac{n}{2}\left(\tfrac{1}{f(R)} - 1\right)\right\} = \tfrac{-1}{n+1}\tfrac{1}{2f(R)}\left\{(2+n)\left(1 - \tfrac{r_o^n}{R^n}\right) - n\right\} = \\
&\tfrac{-1}{n+1}\tfrac{1}{2f(R)R^n}((2+n)(R^n - r_o^n) - nR^n) = \tfrac{-1}{n+1}\tfrac{1}{f(R)}\left\{1 - \left(\tfrac{2+n}{n} r_o^n\right)\tfrac{1}{R_c}\right\} \\
&\Longrightarrow \hat{c}_L^2 = \tfrac{-1}{n+1}\left(\tfrac{1-(R/R_c)^n}{f(R)}\right)
\end{aligned} \tag{D.59}$$

In order to obtain the constraint equations measured on the wall we first note that when actiong on a scalar function f we have $\nabla_a f = \hat{\nabla}_a f$. Also recall that $c_L^2 = \frac{-1}{n+1}$ equations (6.94) and (6.103):

$$\begin{aligned}
&\hat{\nabla}_a \mathrm{ln} r_o^{n+1} = \nabla_a \mathrm{ln} r_o^{n+1} = \theta u_a + (n+1)a_a = \hat{\theta}\hat{u}_a + (n+1)\tfrac{-\hat{a}_a}{\hat{c}_L^2(n+1)} \Longrightarrow \\
&\hat{\nabla}_a \mathrm{ln} r_o^{n+1} = \hat{\theta}\hat{u}_a - \tfrac{1}{\hat{c}_L^2}\hat{a}_a
\end{aligned} \tag{D.60}$$

which could have also been derived by requiring the conservation of the stress-energy tensor:

$$\hat{\nabla}_a \hat{T}^{ab} = 0 \tag{D.61}$$

To close this section we are going to compute the value of the R for which the 1^{st} Order Phase Transition takes place. The pessure for the Blackbrane was given in (6.88) and using the expresion for the energy density (6.87) we can rewrite it in a more useful form for our purposes:

$$\hat{P}^{(BB)} = \frac{\mathcal{A}_{(n+1)}}{8\pi G}\left\{(n+1)R^n\sqrt{f(R)} + \frac{n r_o^n}{2\sqrt{f(R)}}\right\} \tag{D.62}$$

the stress-energy tensor for Minkowski can be found in [52]:

$$T_{ab}^{(\mathbb{M})} = \frac{\mathcal{A}_{(n+1)}}{8\pi G}(n+1)R^n \hat{h}_{ab} \tag{D.63}$$

and if we compare it to the standard expression of a perfect fluid we come to the conclusion that:

$$\varepsilon^{(\mathbb{M})} = -P^{(\mathbb{M})}, \quad P^{(\mathbb{M})} = \frac{\mathcal{A}_{(n+1)}}{8\pi G}(n+1)R^n \tag{D.64}$$

We can rewrite now the pressure of the Blackbrane as:

$$\hat{P}^{(BB)} = P^{(\mathbb{M})}\sqrt{f(R)} + \frac{\mathcal{A}_{(n+1)}}{16\pi G}\frac{n r_o^n}{\sqrt{f(R)}} \tag{D.65}$$

If we requirw that for certain R_1 both pressures are the same $\hat{P}^{(BB)} = P^{(\mathbb{M})}$ we must solve for R_1 in the following equation:

$$(n+1)R_1^n\left(1 - \sqrt{f(R_1)}\right)\sqrt{f(R_1)} = \frac{1}{2} n r_o^n \tag{D.66}$$

Let us try an ansatz of the same form as R_c:

$$R_1^n = A(n)\, r_o^n \tag{D.67}$$

the equation now becomes:

$$A\left[1 - \sqrt{\left(1 - \frac{1}{A}\right)}\right]\sqrt{1 - \frac{1}{A}} = \frac{n}{2(n+1)} \tag{D.68}$$

we now define a new function to get a simpler expression to deal with:

$$1 - \frac{1}{A} \equiv B^2 \Longrightarrow A = \frac{1}{1 - B^2} \tag{D.69}$$

upon substitution we are left with:

$$\frac{(1-B)B}{1-B^2} = \frac{B}{1+B} = \frac{n}{2(n+1)} \tag{D.70}$$

solving for B is straightforward, after a little manipulation we get:

$$B = \frac{n}{2+n} \tag{D.71}$$

which gives an expression for A:

$$A = \frac{(2+n)^2}{4(1+n)} \tag{D.72}$$

plugging this into (D.63) and using the definition of R_c (6.101) we get (6.111):

$$R_1 = \left(\frac{n+2}{2(n+1)}\right)^{\frac{1}{n}} R_c \tag{D.73}$$

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