Flow Structure in a Precessing Sphere

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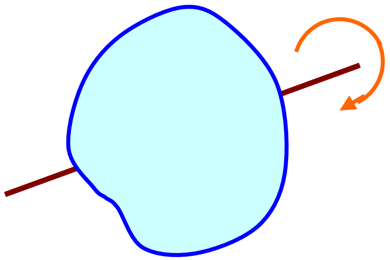
Flow in a Rotating Container



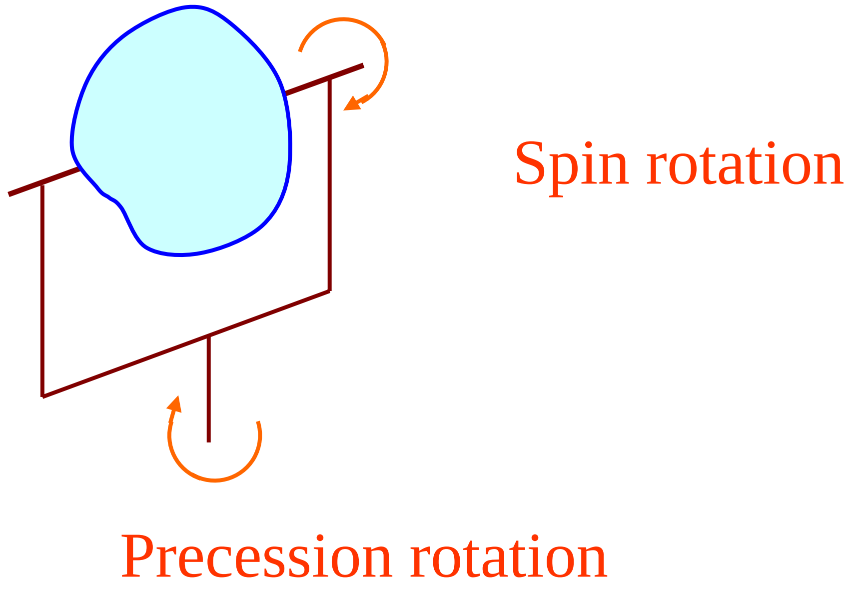
Spin rotation

Flow structure is simple.

Solid-body rotation

It can be proved from NS equation.

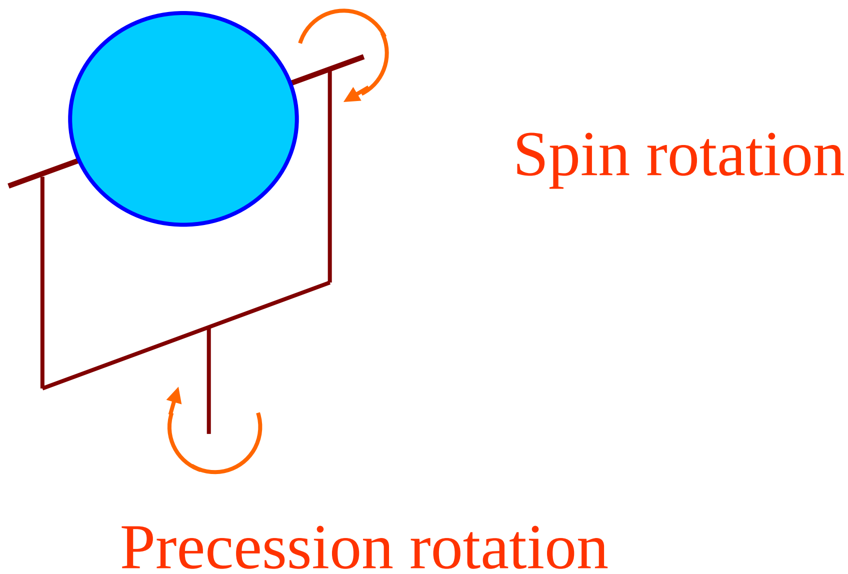
Flow in a Precessing Container



Flow structure is non-trivial.

Flow can be turbulent.

Flow in a Precessing Sphere



Flow structure is non-trivial.

Flow can be turbulent.

Outline

1. Introduction
2. Experiment – State Diagram
3. Numerical Simulation – Stability boundary
   Flow Structure
4. Asymptotic Analysis – Flow Structure
5. Summary

Outline

1．Introduction

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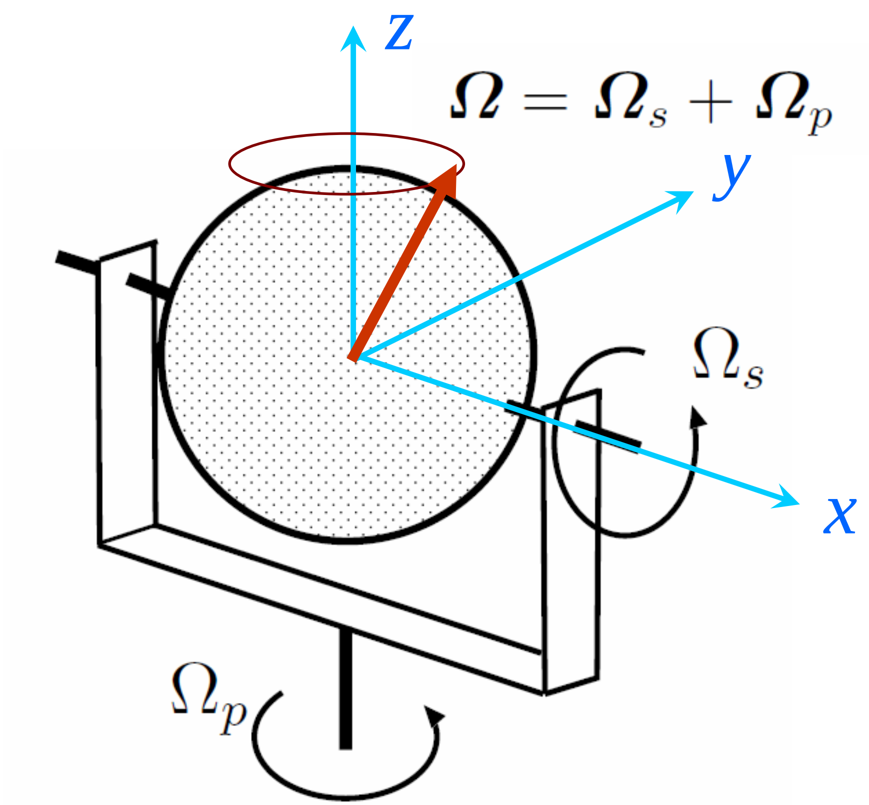
5．Summary

Precessing Sphere

We consider the motion of an incompressible viscous fluid in a precessing sphere, where the spin angular velocity Ω_s and the precession angular velocity Ω_p are perpendicular to each other.



Research on Precessing Sphere/Spheroid/Spherical Shell

Precession of the Earth

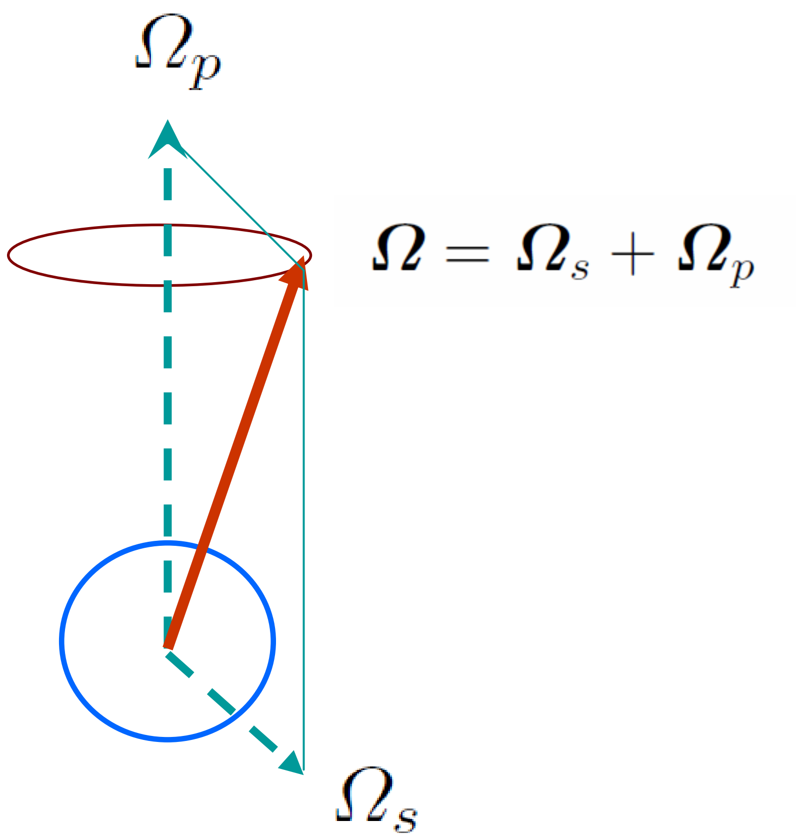
Spin rotation; one day

Precession rotation: 25800 year

Poincare number; 10^{-7}

Precession angle; - 23.5°

Ekman number; $10^{-15}<E<10^{-7}$



Research on Precessing Sphere/Spheroid/Spherical Shell

related to **Geodynamo**

Experiment

Vanyo et al. 1995: Experiments on precessing flows in the Earth's liquid core

Vanyo & Dunn 2000: Core precession:flow structures and energy

Numerical

Lorenzani & Tilgner 2001: Fluid instabilities in precessing spheroidal cavities

Tilgner & Busse 2001: Fluid flows in precessing spherical shells

Tilgner 2005: Precession driven dynamos

Our Motivation

[1] to make a compact turbulence generator

[2] to understand the flow dynamics as one of the standard systems

[3] to contribute to geophysics such as geodynamo

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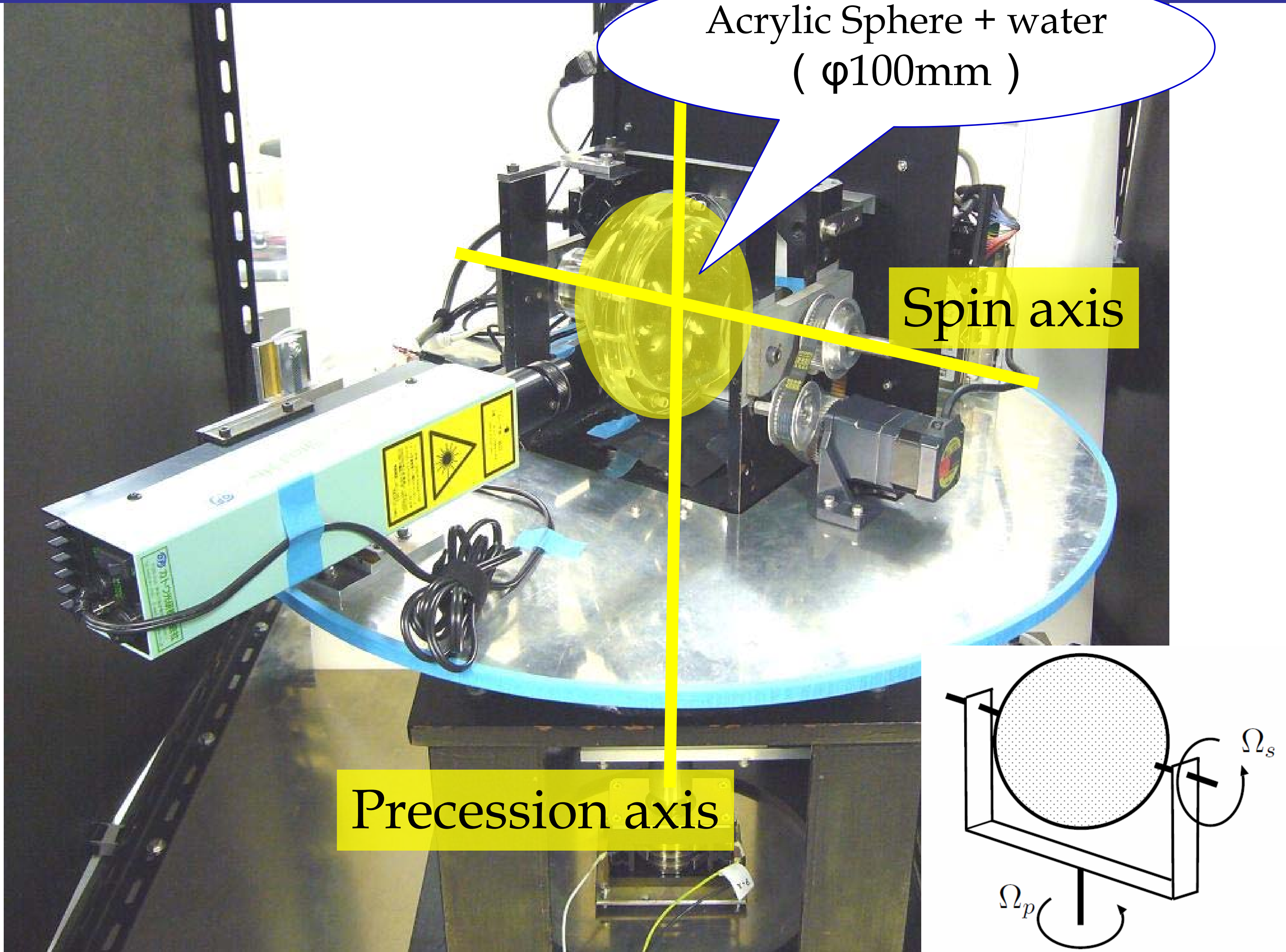
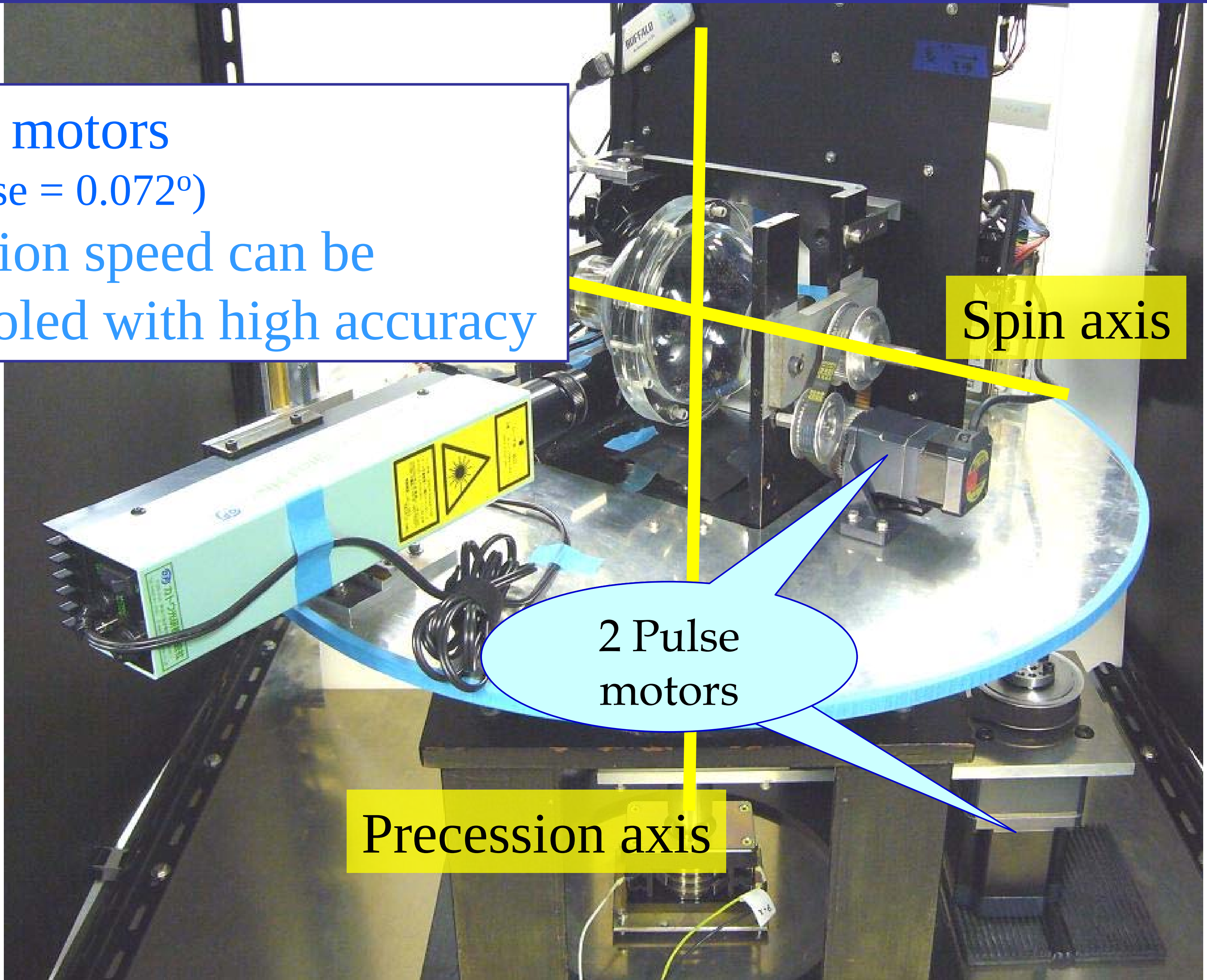
Apparatus



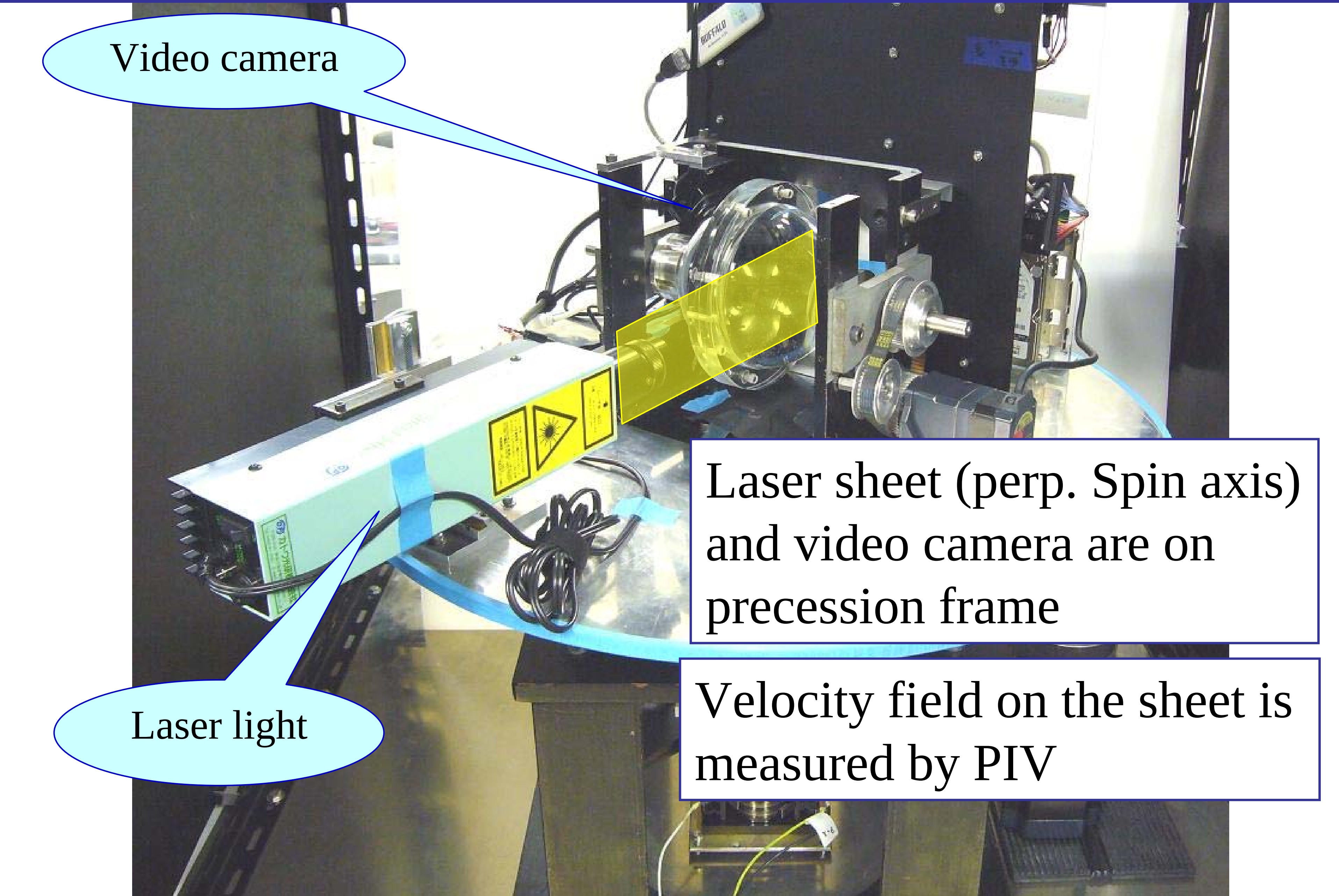
Acrylic Sphere + water
(φ100mm)
Spin axis
Precession axis
Ω_s
Ω_p

Control of rotation speed

Pulse motors
(1 pulse = 0.072º)
Rotation speed can be controled with high accuracy



Visualization / Measurement



Governing Equations (in Precession frame)

Non-dimensional

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\omega} - 2\frac{R_p}{R_s}\hat{\boldsymbol{z}} \times \boldsymbol{u} - \nabla P + \frac{1}{R_s}\nabla^2 \boldsymbol{u}$$

$$\boxed{P = p + \frac{1}{2}|\boldsymbol{u}|^2 - \left(\frac{R_p}{R_s}\right)^2 \frac{1}{2}(\boldsymbol{r} \times \hat{\boldsymbol{z}})^2}$$

Modified pressure

$$\nabla \cdot \boldsymbol{u} = 0$$

b. c.

$$\boldsymbol{u} = \hat{\boldsymbol{x}} \times \boldsymbol{r} \qquad \text{on } |\boldsymbol{r}| = 1$$

Control Parameters

Spin Reynolds number

(Reciprocal of Ekman number)

$$R_s = \frac{a^2 \Omega_s}{\nu}$$

Precession Reynolds number

$$R_p = \frac{a^2 \Omega_p}{\nu}$$

When a=5cm, ν=0.01 cm^2/s, $\Omega_s = 2\pi n$,

$$R_s = 1.6 \times 10^4\, n$$

Control Parameters

Reynolds number

$$Re = \frac{a^2 \Omega_s}{\nu} \quad (\equiv R_s)$$

Poincare number

$$\Gamma = \frac{\Omega_p}{\Omega_s} \quad \left(\equiv \frac{R_p}{R_s}\right)$$

Visualized Flow

Re=15,000

Γ=0.00625

Solid –body rotation

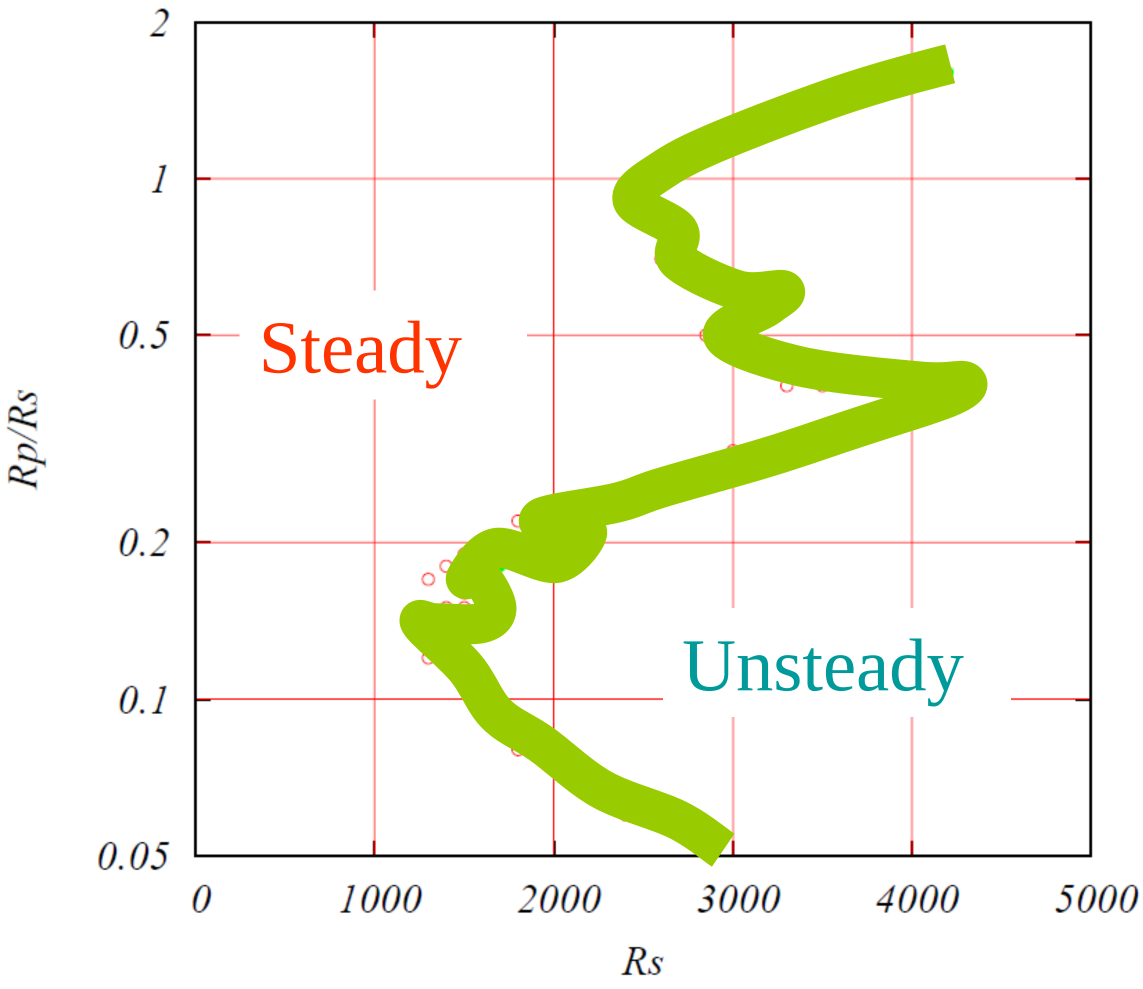
Flow in a Rotating Container

Re=15,000

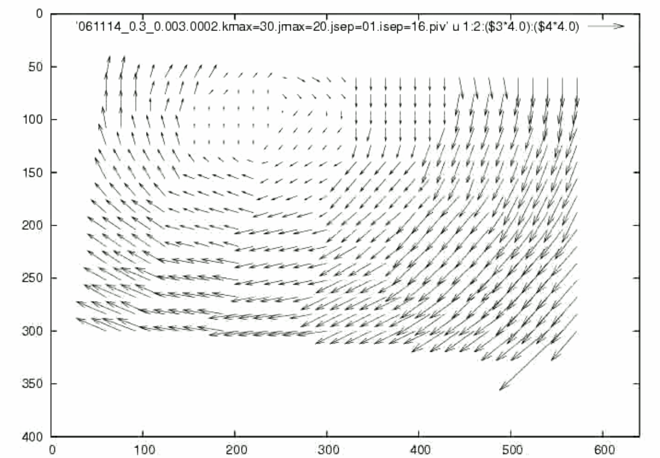
Γ=0.1

Turbulent state

Stability Boundary

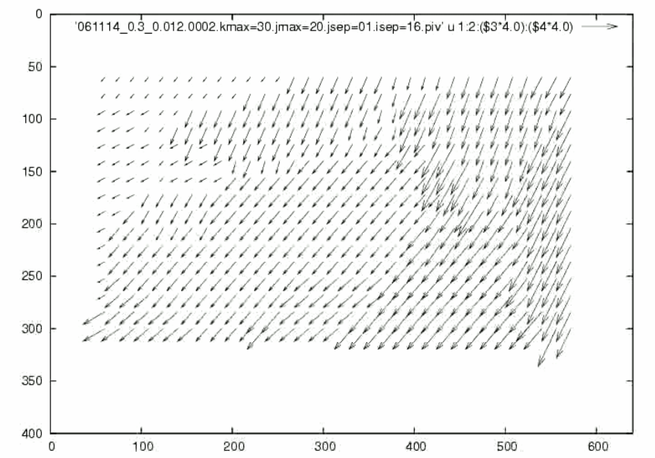


Velocity Field (animation)

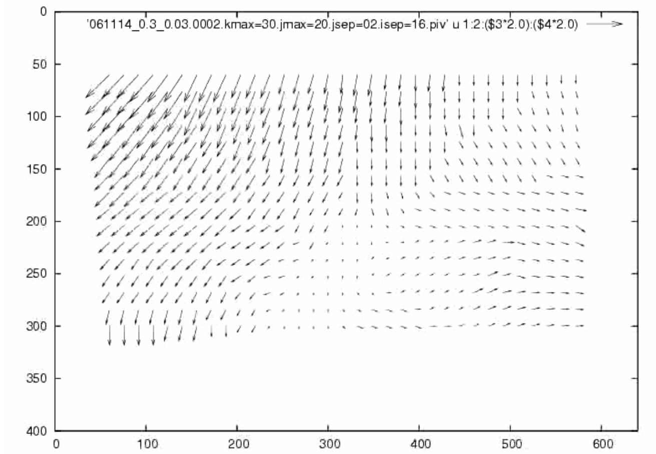


Steady: $R_p/R_s = 0.01$

$R_s \approx 4,500$



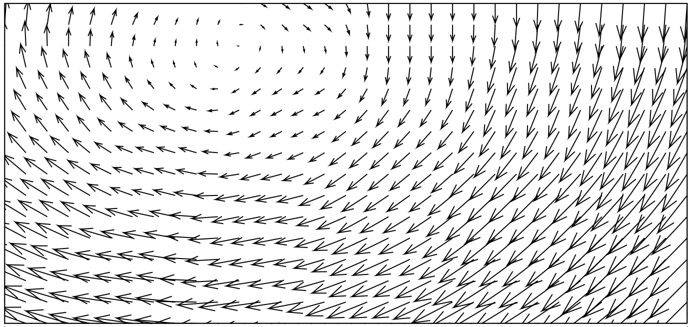
Periodic: $R_p/R_s = 0.04$



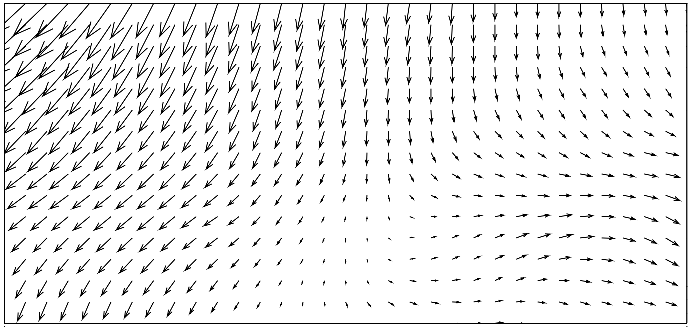
Aperiodic: $R_p/R_s = 0.1$

Velocity Field (snap)

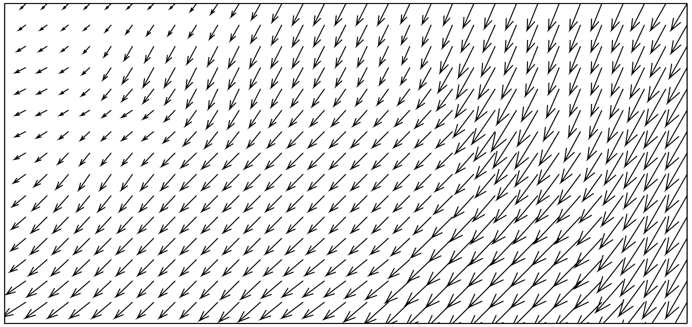
$R_s \approx 4,500$



Steady: $R_p/R_s = 0.01$



Periodic: $R_p/R_s = 0.04$



Aperiodic: $R_p/R_s = 0.1$

Judged from two-time correlation function

Two-Time Correlation of Velocity

$$C_i(\boldsymbol{x},\tau) = \frac{\langle [u_i(\boldsymbol{x},t) - m_i(\boldsymbol{x})][u_i(\boldsymbol{x},t+\tau) - m_i(\boldsymbol{x})] \rangle}{\sigma_i(\boldsymbol{x})^{\ 2}}$$

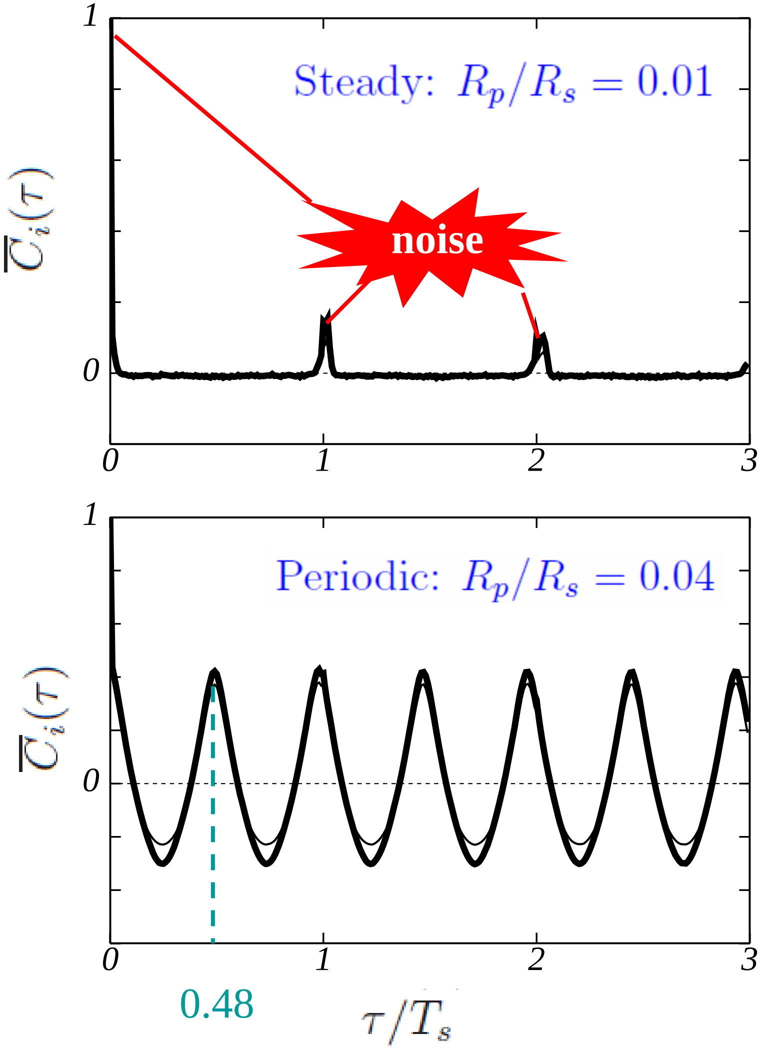
$$(i = 1, 2)$$

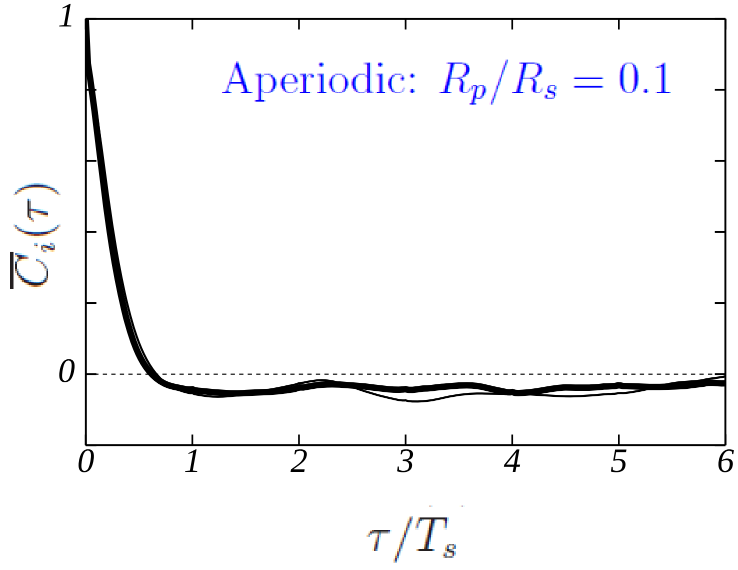
$$m_i(\boldsymbol{x}) = \langle u_i(\boldsymbol{x},t) \rangle$$

$$\sigma_i(\boldsymbol{x})^{\ 2} = \langle (u_i(\boldsymbol{x},t)^2 \rangle - m_i(\boldsymbol{x})^2$$

Two-Time Correlation of Velocity

$$R_s \approx 4,500$$

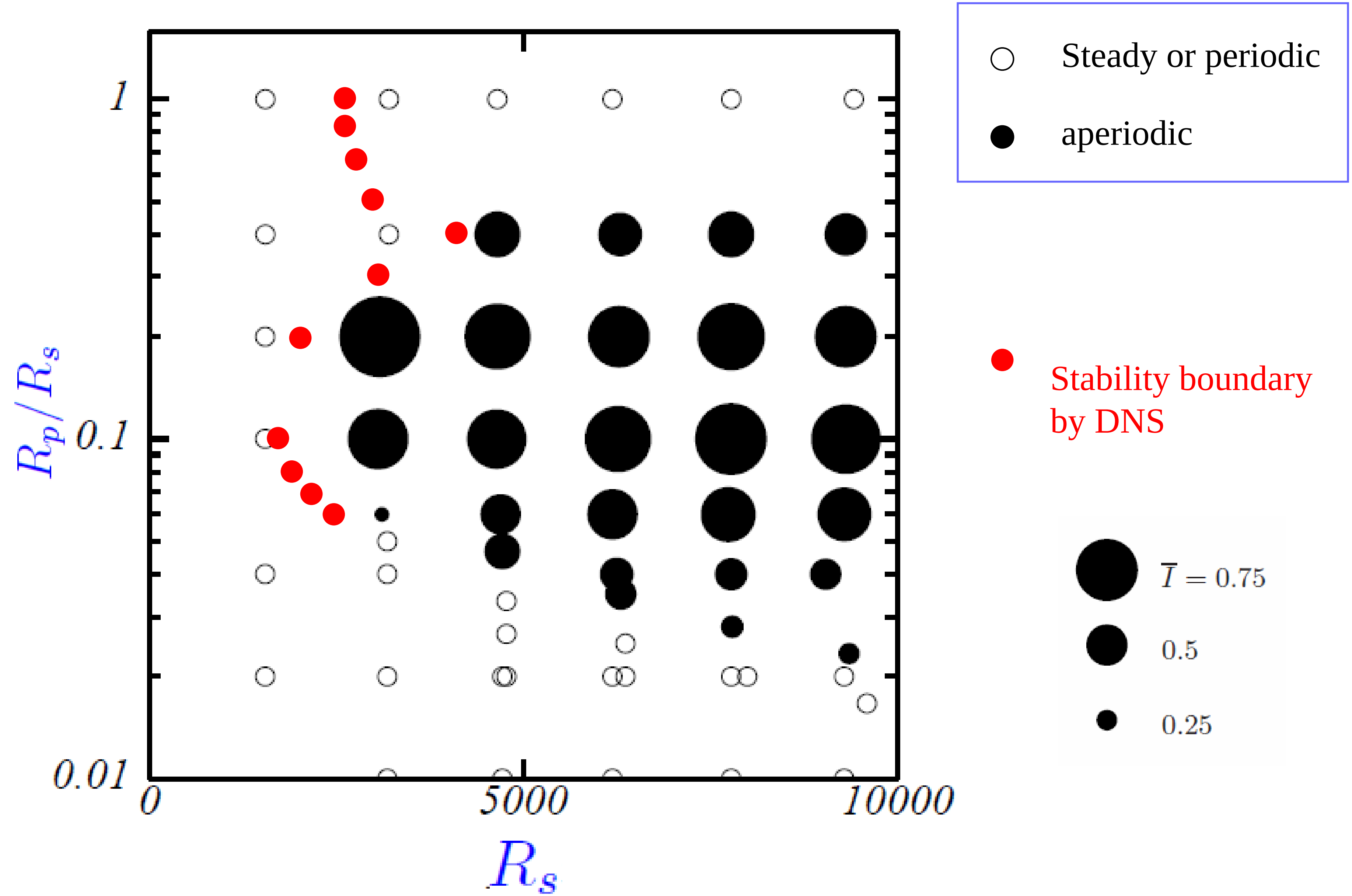




Fluctuation Magnitude

$$I(\boldsymbol{x}) = \sqrt{\frac{\langle |\boldsymbol{u}(\boldsymbol{x},t) - \langle \boldsymbol{u}(\boldsymbol{x},t)\rangle\ |^2\rangle}{\langle |\boldsymbol{u}(\boldsymbol{x},t)|^2\rangle}}$$

Fluctuation Magnitude



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Stability of Steady Flows

By DNS

Governing Equations (in Precession frame)

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\omega} - 2\frac{R_p}{R_s}\widehat{\boldsymbol{z}} \times \boldsymbol{u} - \nabla P + \frac{1}{R_s}\nabla^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

b. c.

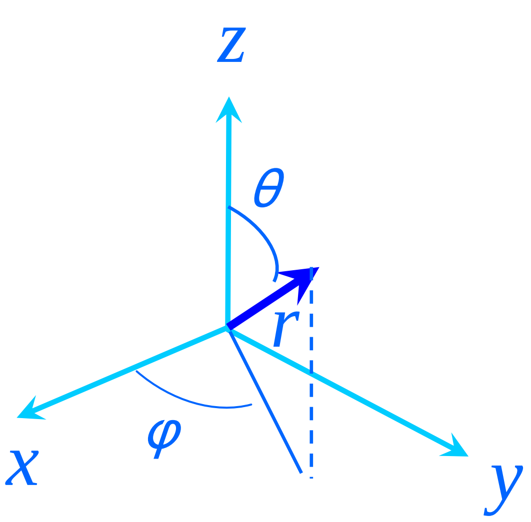
$$\boldsymbol{u} = \widehat{\boldsymbol{x}} \times \boldsymbol{r} \qquad \text{on } |\boldsymbol{r}| = 1$$

Poloidal / Toroidal Representation

① **Incompressible (solenoidal)**

② **Spherical geometry**

$$\boxed{\boldsymbol{u} = \nabla \times (\nabla \times (\boldsymbol{r}U)) + \nabla \times (\boldsymbol{r}W)}$$



$$u_r = -\frac{1}{r}\nabla_\perp^2 U,$$

$$u_\theta = \frac{1}{r}\frac{\partial^2(rU)}{\partial r \partial \theta} + \frac{1}{\sin\theta}\frac{\partial W}{\partial \varphi},$$

$$u_\varphi = \frac{1}{r\sin\theta}\frac{\partial^2(rU)}{\partial r \partial \varphi} - \frac{\partial W}{\partial \theta}$$

Vorticity

$$\boxed{\boldsymbol{\omega} = \nabla \times \nabla \times (\boldsymbol{r}W) + \nabla \times (\boldsymbol{r}(-\nabla^2 U))}$$

$$\omega_r = -\frac{1}{r}\nabla_\perp^2 W$$

$$\omega_\theta = \frac{1}{r}\frac{\partial^2 (rW)}{\partial r \partial \theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}(-\nabla^2 U)$$

$$\omega_\varphi = \frac{1}{r\sin\theta}\frac{\partial^2 (rW)}{\partial r \partial \varphi} - \frac{\partial}{\partial \theta}(-\nabla^2 U)$$

Governing Equations (in Precession frame)

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\omega} - 2\frac{R_p}{R_s}\widehat{\boldsymbol{z}} \times \boldsymbol{u} - \nabla P + \frac{1}{R_s}\nabla^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

b. c.

$$\boldsymbol{u} = \widehat{\boldsymbol{x}} \times \boldsymbol{r} \qquad \text{on } |\boldsymbol{r}| = 1$$

Poloidal / Toroidal Equations

$$
\begin{aligned}
&-\nabla_\perp^2\left(\nabla^2 - R_h\frac{\partial}{\partial t}\right)W \\
&= -2R_v\left[\frac{\partial W}{\partial \varphi} + \frac{1}{r^2}\left\{\nabla_\perp^2\left(r\sin\theta\frac{\partial}{\partial\theta} - r\frac{\partial}{\partial r}r\cos\theta\right) - r\frac{\partial}{\partial r}\left(r\sin\theta\frac{\partial}{\partial\theta} + 2r\cos\theta\right)\right\}U\right] \\
&\quad + \frac{R_h}{r^2\sin^2\theta}\left[\frac{\partial}{\partial\varphi}(r^2\sin\theta N_\theta - r\sin\theta\frac{\partial}{\partial\theta}(r\sin\theta N_\varphi)\right],
\end{aligned}
$$

$$\boldsymbol{N} = \boldsymbol{u} \times \boldsymbol{\omega}$$

$$
\begin{aligned}
&-\nabla_\perp^2\left(\nabla^2 - R_h\frac{\partial}{\partial t}\right)(-\nabla^2 U) \\
&= -2R_v\left[\frac{\partial(-\nabla^2 U)}{\partial \varphi} + \frac{1}{r^2}\left\{\nabla_\perp^2\left(r\sin\theta\frac{\partial}{\partial\theta} - r\frac{\partial}{\partial r}r\cos\theta\right) - r\frac{\partial}{\partial r}\left(r\sin\theta\frac{\partial}{\partial\theta} + 2r\cos\theta\right)\right\}W\right] \\
&\quad + \frac{R_h}{r^2}\left[\nabla_\perp^2(rN_r) - \frac{1}{r^2\sin^2\theta}\left(r\frac{\partial}{\partial r} - 2\right)r\sin\theta\frac{\partial}{\partial\theta}\left(r^2\sin\theta N_\theta\right) - \frac{1}{\sin^2\theta}r\frac{\partial}{\partial r}\frac{\partial}{\partial\varphi}(r\sin\theta N_\varphi)\right]
\end{aligned}
$$

b. c. $U = 0, \quad \dfrac{\partial U}{\partial r} = 0, \quad W = \sin\theta\cos\varphi \quad$ on $r = 1$

Numerical Scheme (Time Integration)

$$-\nabla_\perp^2\left(\nabla^2 - R_h\frac{\partial}{\partial t}\right)W = G$$

$$-\nabla_\perp^2\left(\nabla^2 - R_h\frac{\partial}{\partial t}\right)(-\nabla^2 U) = H$$

$$U = 0, \qquad \frac{\partial U}{\partial r} = 0, \qquad W = \sin\theta\cos\varphi \qquad (\text{on } r = 1)$$

Adams-Bashforth / Crank-Nicolson Scheme $\Delta t = 2\pi/1000$

$$-\nabla_\perp^2\left(\nabla^2 - \frac{2R_h}{\Delta t}\right)W^{t+\Delta t} = \nabla_\perp^2\left(\nabla^2 + \frac{2R_h}{\Delta t}\right)W^t + 3G^t - G^{t-\Delta t}$$

$$-\nabla_\perp^2\left(\nabla^2 - \frac{2R_h}{\Delta t}\right)(-\nabla^2)U^{t+\Delta t} = \nabla_\perp^2\left(\nabla^2 + \frac{2R_h}{\Delta t}\right)(-\nabla^2)U^t + 3H^t - H^{t-\Delta t}$$

$$U^{t+\Delta t} = 0, \qquad \frac{\partial}{\partial r}U^{t+\Delta t} = 0, \qquad W^{t+\Delta t} = \sin\theta\cos\varphi \qquad (\text{on } r = 1)$$

Numerical Scheme (Spatial Differentiation)

Fourier – Legendre – Jacobi Expansion 32×42×85 dealiased

$$U^t(r,\theta,\varphi) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \sum_{j=1}^{[(N-l+1)/2]} \widetilde{U}^t_{jlm} \Phi^l_k(r) \breve{P}_l^{|m|}(\cos\theta) \mathrm{e}^{\mathrm{i}m\varphi}$$

$$W^t(r,\theta,\varphi) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \sum_{j=1}^{[(N-l+1)/2]} \widetilde{W}^t_{jlm} \Phi^l_k(r) \breve{P}_l^{|m|}(\cos\theta) \mathrm{e}^{\mathrm{i}m\varphi}$$

$$\boxed{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left((1-r^2)r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)\Phi^l_k - \frac{l(l+1)}{r^2}\Phi^l_k + k(k+3)\Phi^l_k = 0 \qquad (0 \le r \le 1,\ 0 \le l \le k,\ \ k+l = \text{even})}$$

Orthogonal Relation

$$\int_0^1 \Phi^l_n(r)\Phi^l_{n'}(r) r^2 \mathrm{d}r = \delta_{nn'}$$

Matsushima & Marcus (1995)

Poloidal / Toroidal Equations

$$-\nabla_\perp^2\left(\nabla^2 - R_h\frac{\partial}{\partial t}\right)W$$

$$= -2R_v\left[\frac{\partial W}{\partial\varphi} + \frac{1}{r^2}\left\{\nabla_\perp^2\left(r\sin\theta\frac{\partial}{\partial\theta} - r\frac{\partial}{\partial r}r\cos\theta\right) - r\frac{\partial}{\partial r}\left(r\sin\theta\frac{\partial}{\partial\theta} + 2r\cos\theta\right)\right\}U\right]$$

$$+\frac{R_h}{r^2\sin^2\theta}\left[\frac{\partial}{\partial\varphi}(r^2\sin\theta N_\theta - r\sin\theta\frac{\partial}{\partial\theta}(r\sin\theta N_\varphi)\right],$$

$\boldsymbol{N} = \boldsymbol{u} \times \boldsymbol{\omega}$

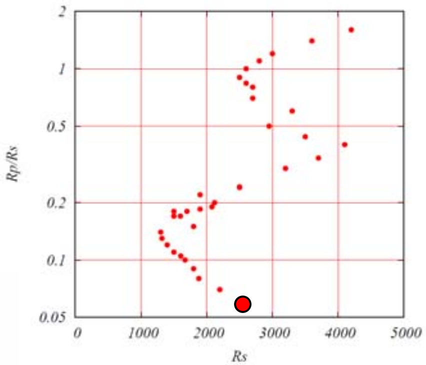
$$-\nabla_\perp^2\left(\nabla^2 - R_h\frac{\partial}{\partial t}\right)(-\nabla^2 U)$$

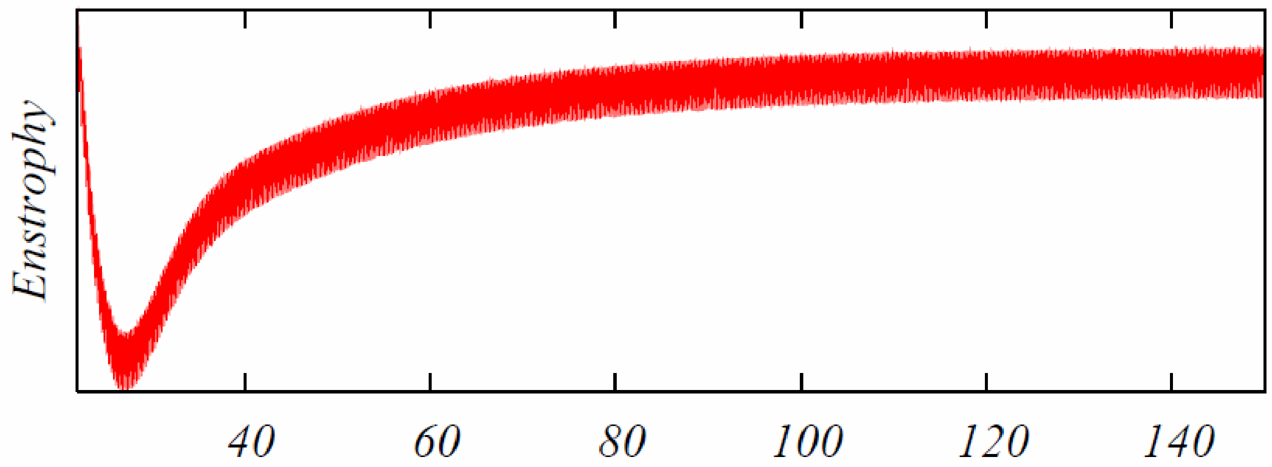
$$= -2R_v\left[\frac{\partial(-\nabla^2 U)}{\partial\varphi} + \frac{1}{r^2}\left\{\nabla_\perp^2\left(r\sin\theta\frac{\partial}{\partial\theta} - r\frac{\partial}{\partial r}r\cos\theta\right) - r\frac{\partial}{\partial r}\left(r\sin\theta\frac{\partial}{\partial\theta} + 2r\cos\theta\right)\right\}W\right]$$

$$+\frac{R_h}{r^2}\left[\nabla_\perp^2(rN_r) - \frac{1}{r^2\sin^2\theta}\left(r\frac{\partial}{\partial r} - 2\right)r\sin\theta\frac{\partial}{\partial\theta}\left(r^2\sin\theta N_\theta\right) - \frac{1}{\sin^2\theta}r\frac{\partial}{\partial r}\frac{\partial}{\partial\varphi}(r\sin\theta N_\varphi)\right]$$

b. c. $\quad U = 0, \quad \dfrac{\partial U}{\partial r} = 0, \quad W = \sin\theta\cos\varphi \quad$ on $r = 1$

Time-Series of Enstrophy

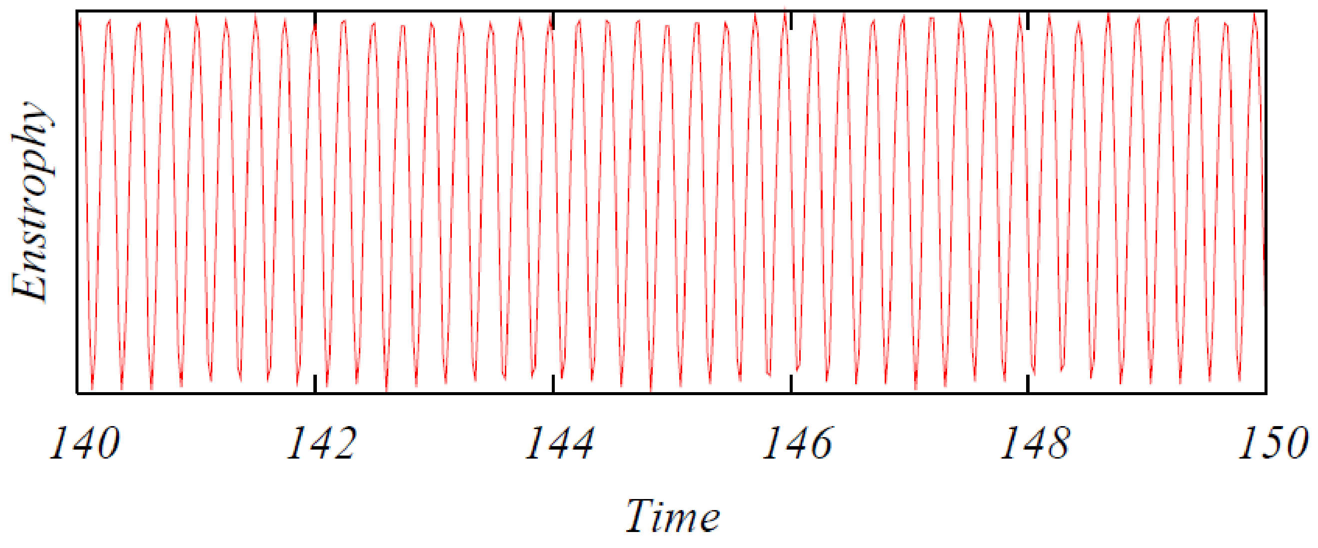




$R_s = 2450$

$R_p = 147$

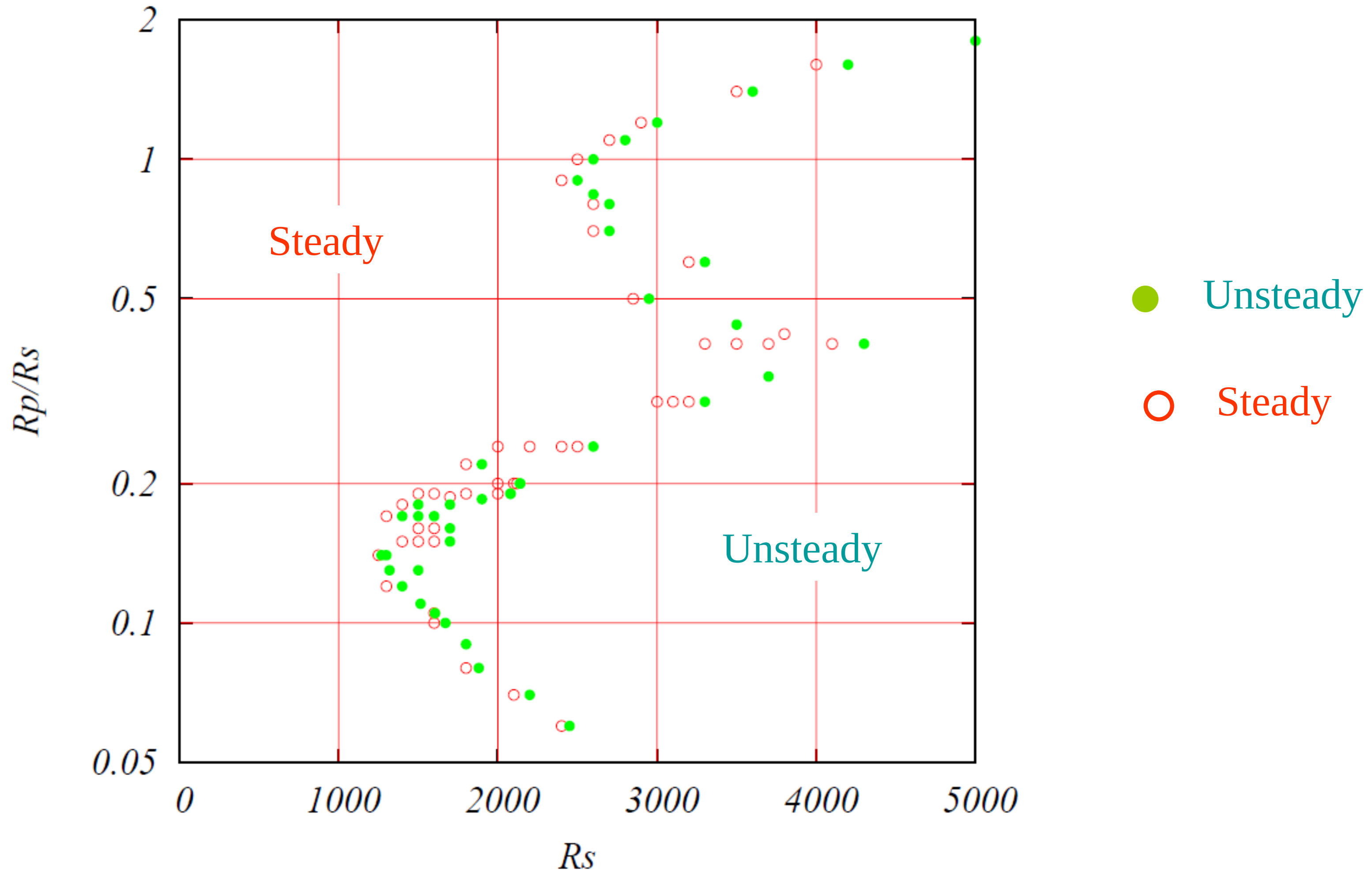
γ= 0.06



λ= 0.25

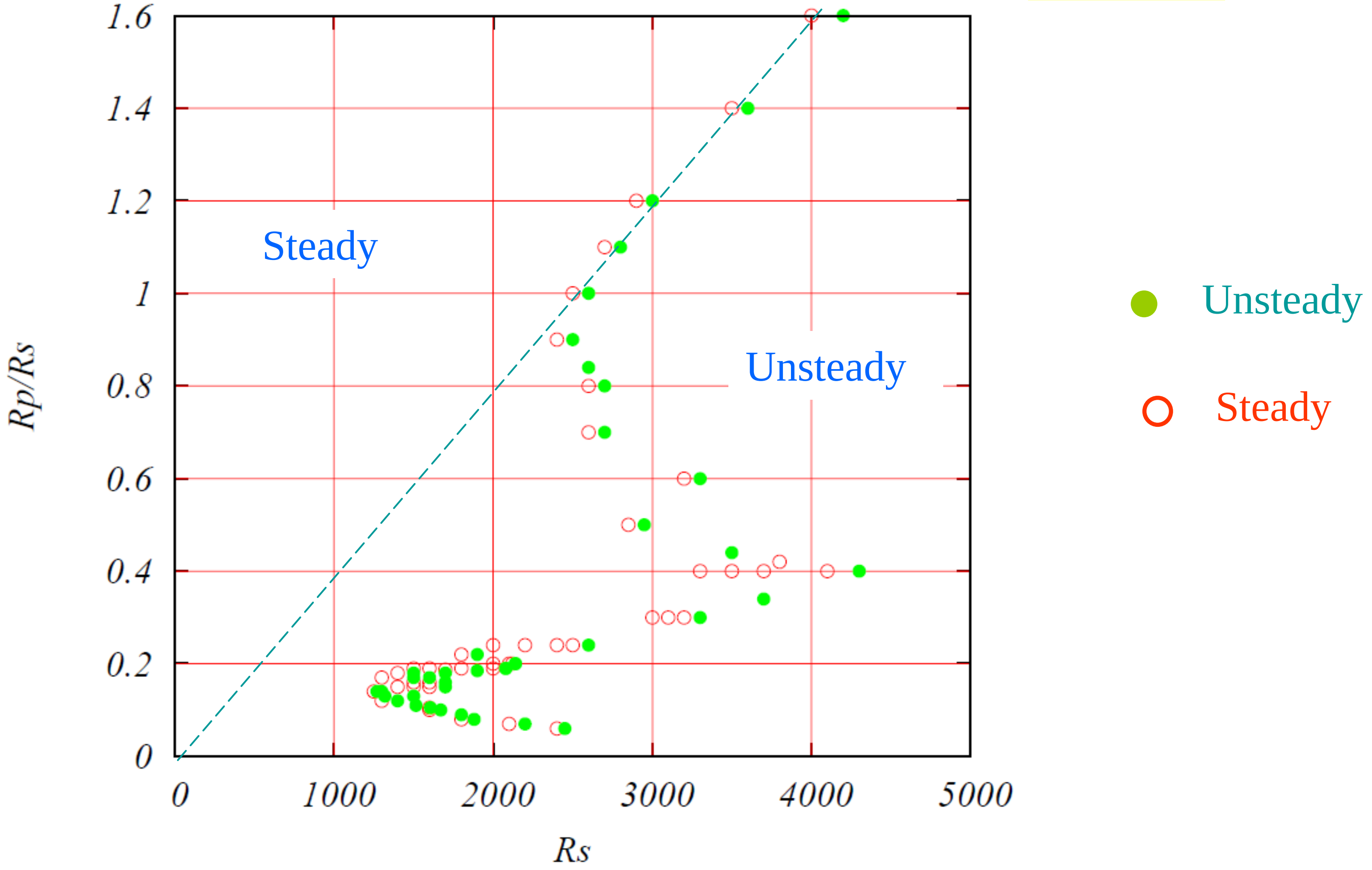
Stability Boundary

in progress

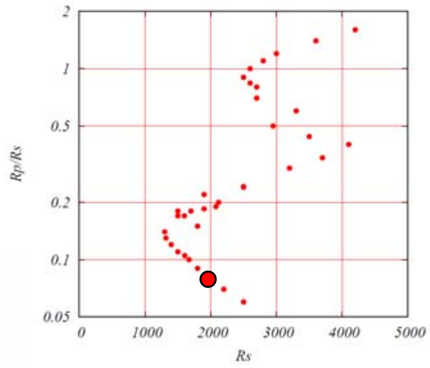


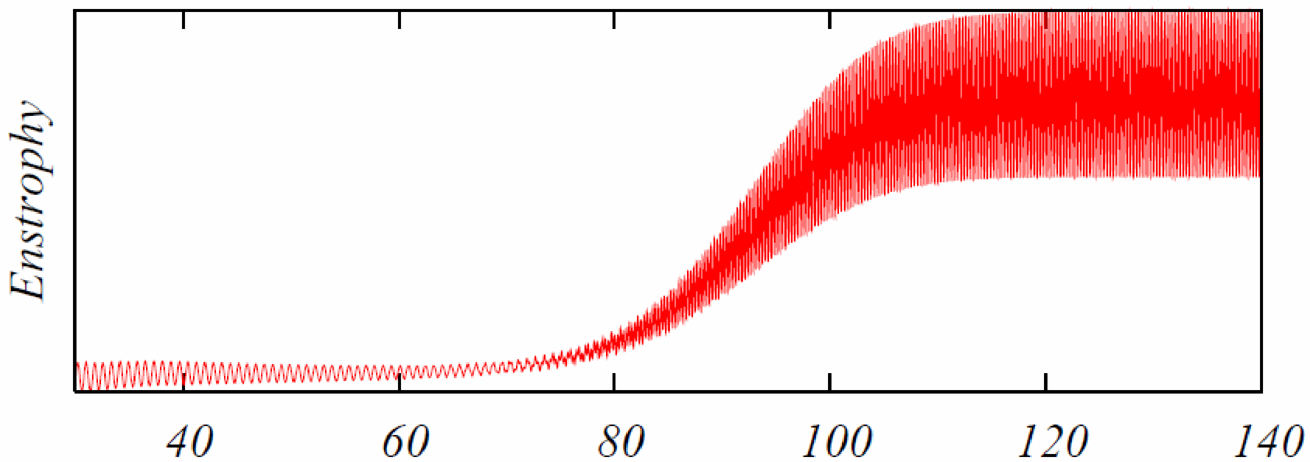
Stability Boundary

in progress



Time-Series of Enstrophy

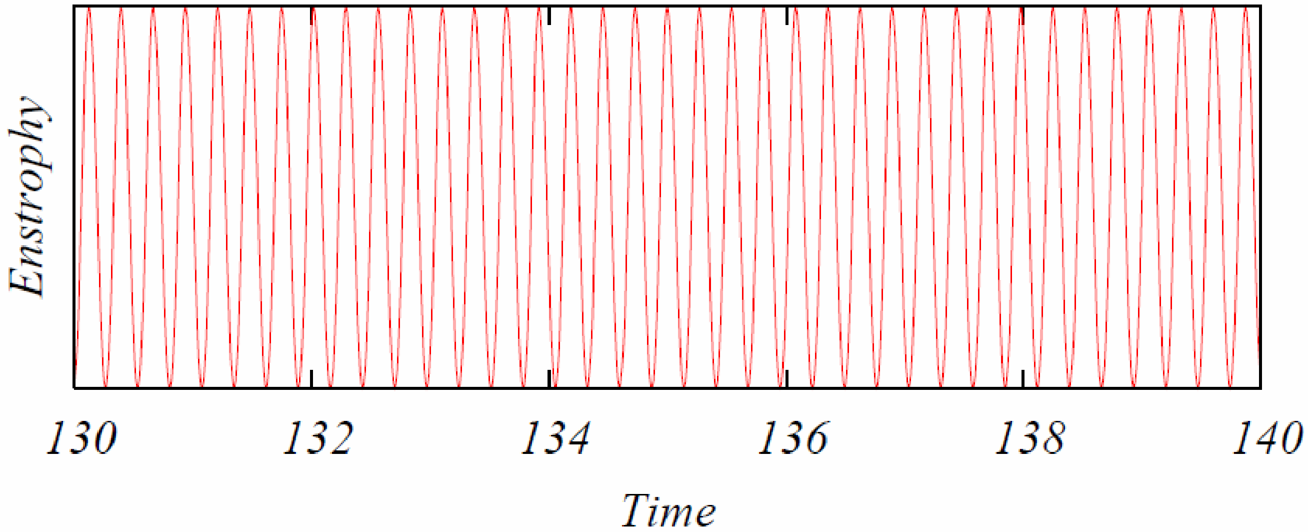




$R_s = 1880$

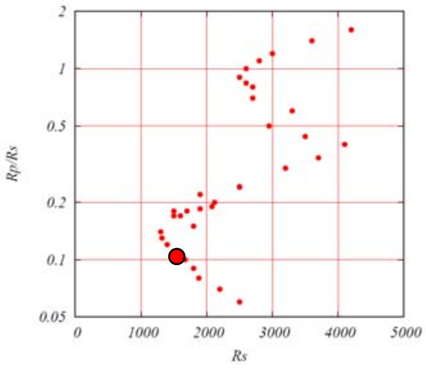
$R_p = 150.4$

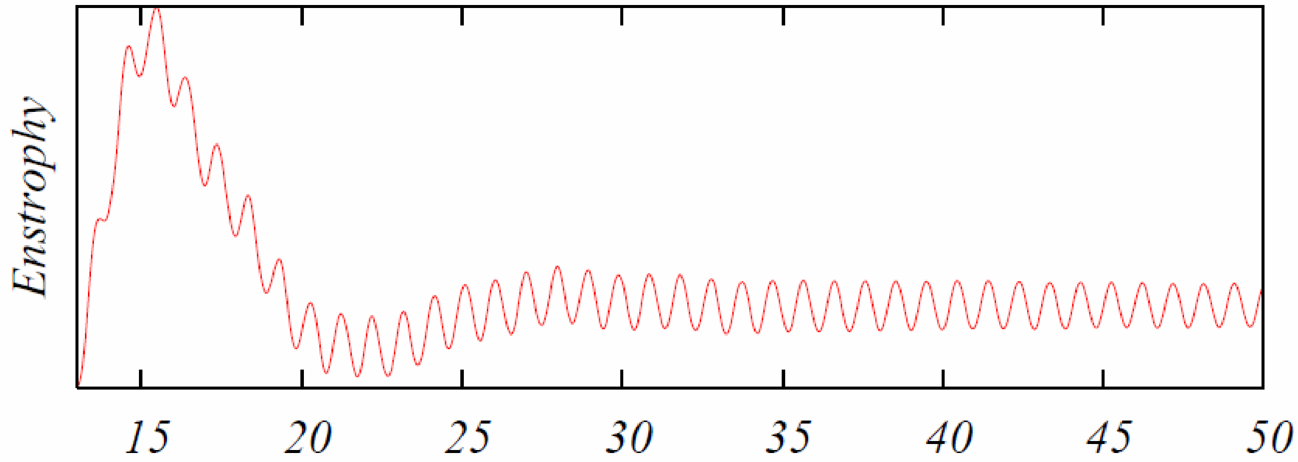
$\gamma = 0.08$



$\lambda = 0.27$

Time-Series of Enstrophy

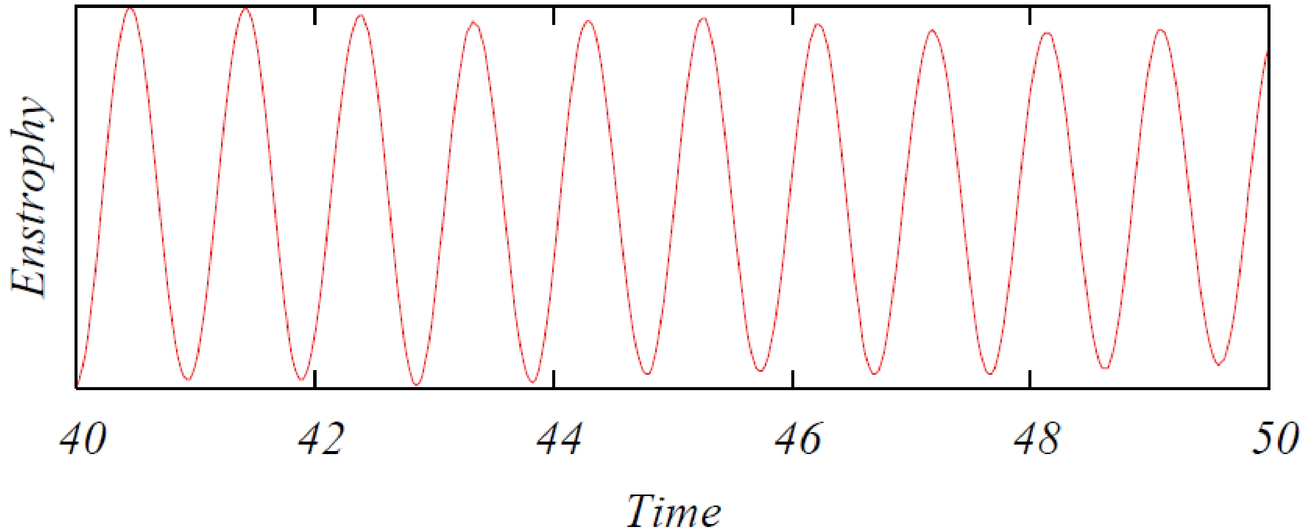




$R_s = 1515$

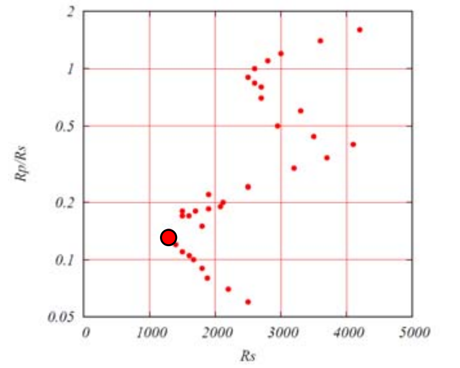
$R_p = 166.65$

$\gamma = 0.11$



$\lambda = 0.96$

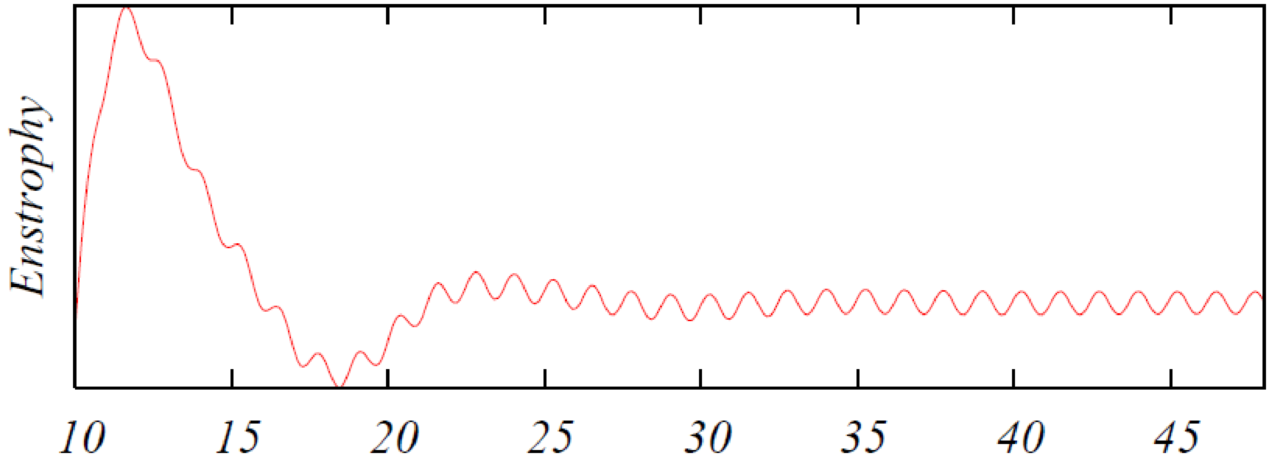
Time-Series of Enstrophy

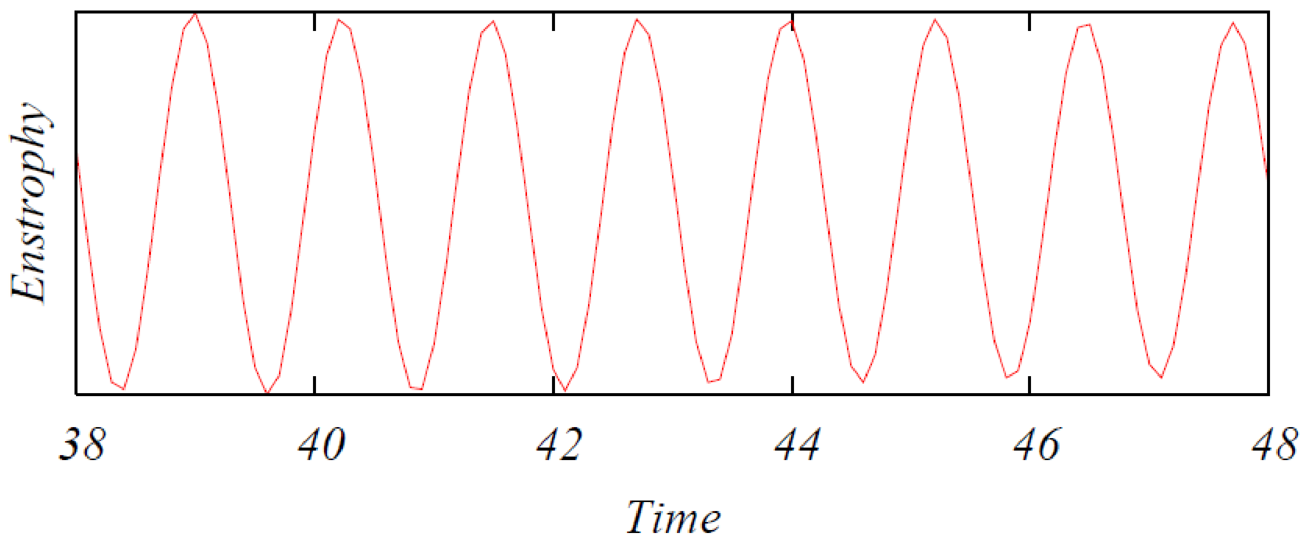


$R_s = 1270$

$R_p = 177.8$

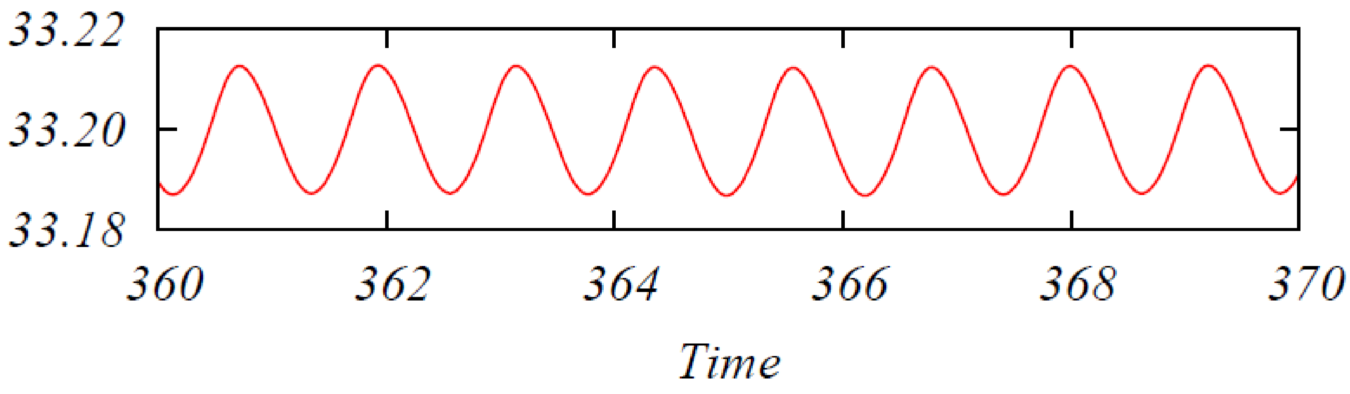
$\gamma = 0.14$

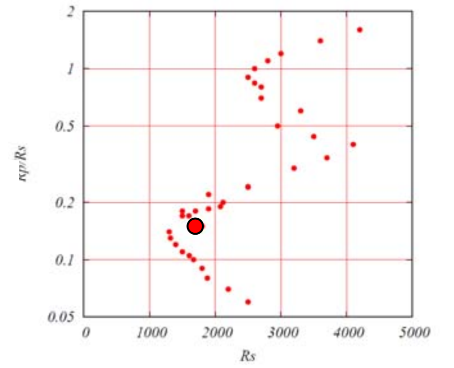




$\lambda = 1.24$

Time-Series of Enstrophy

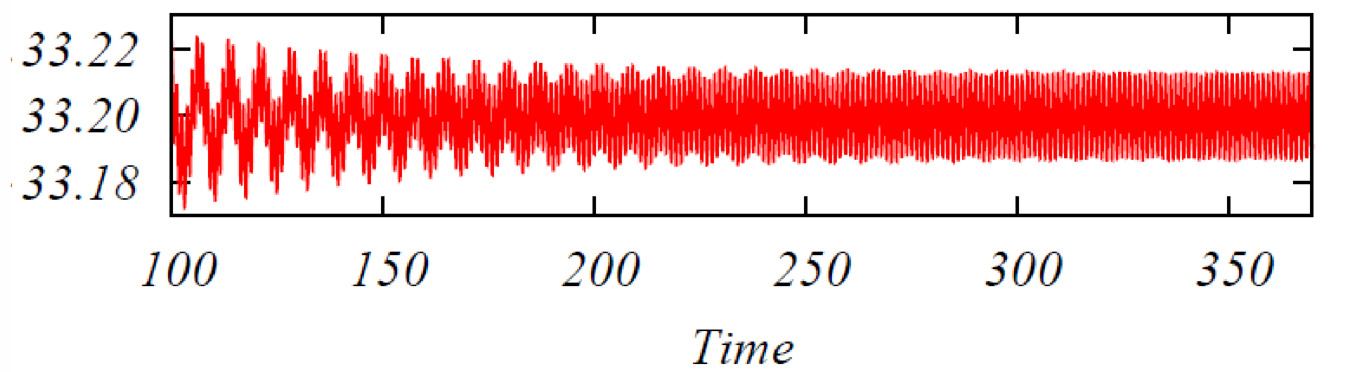




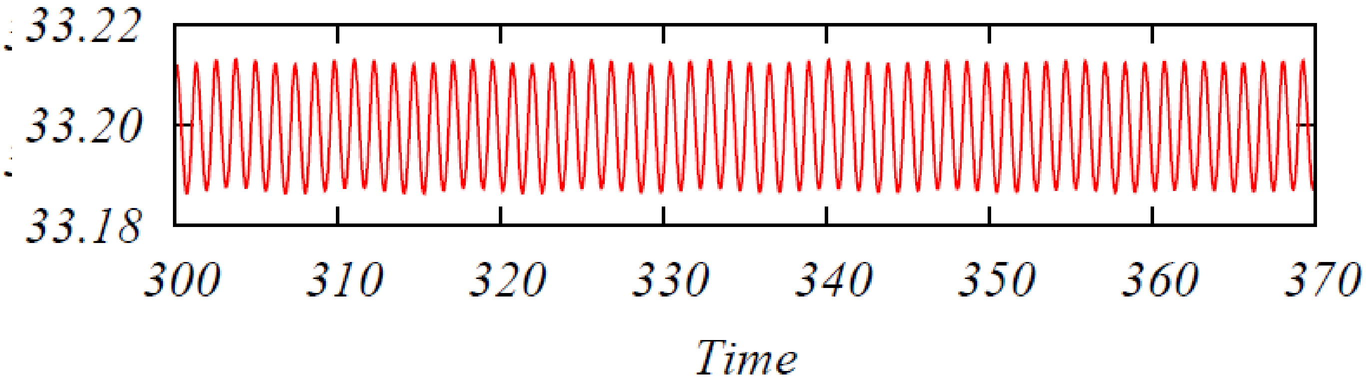
$R_s = 1700$

$R_p = 255$

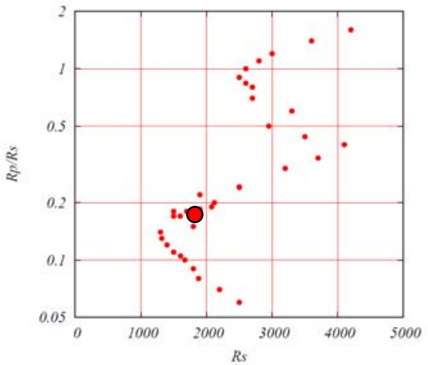
$\gamma = 0.15$

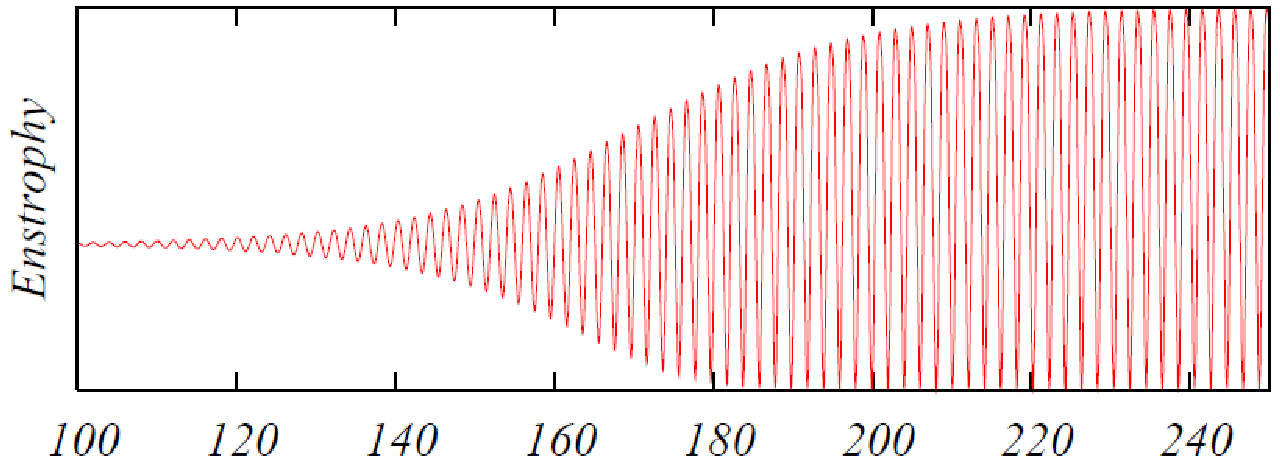


$\lambda = 1.21$



Time-Series of Enstrophy

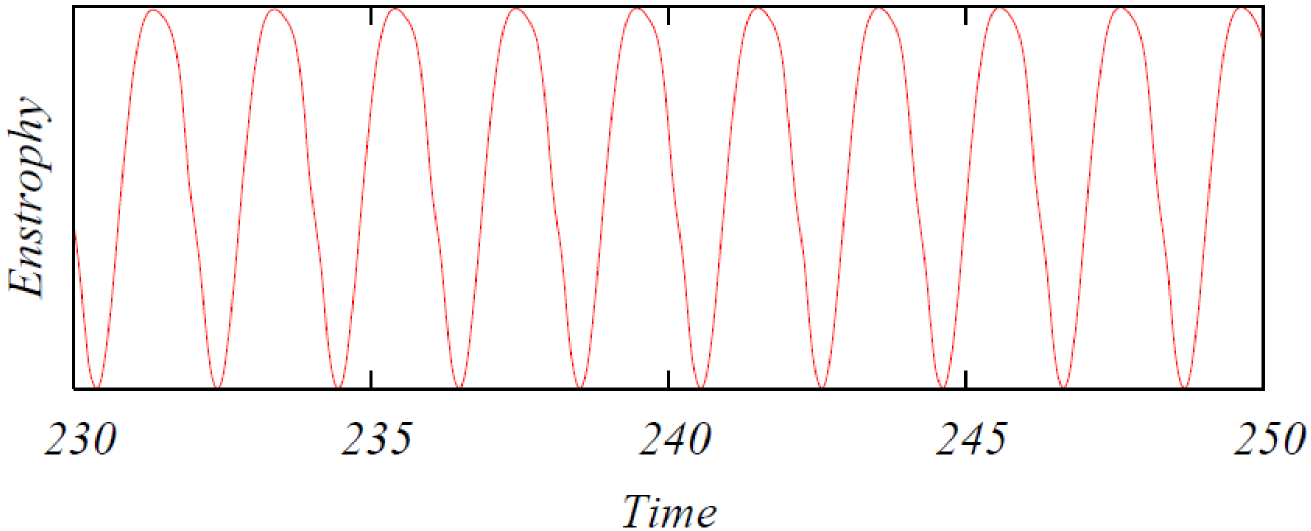




$R_s = 1900$

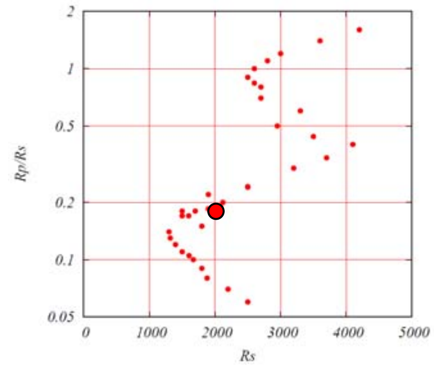
$R_p = 351.5$

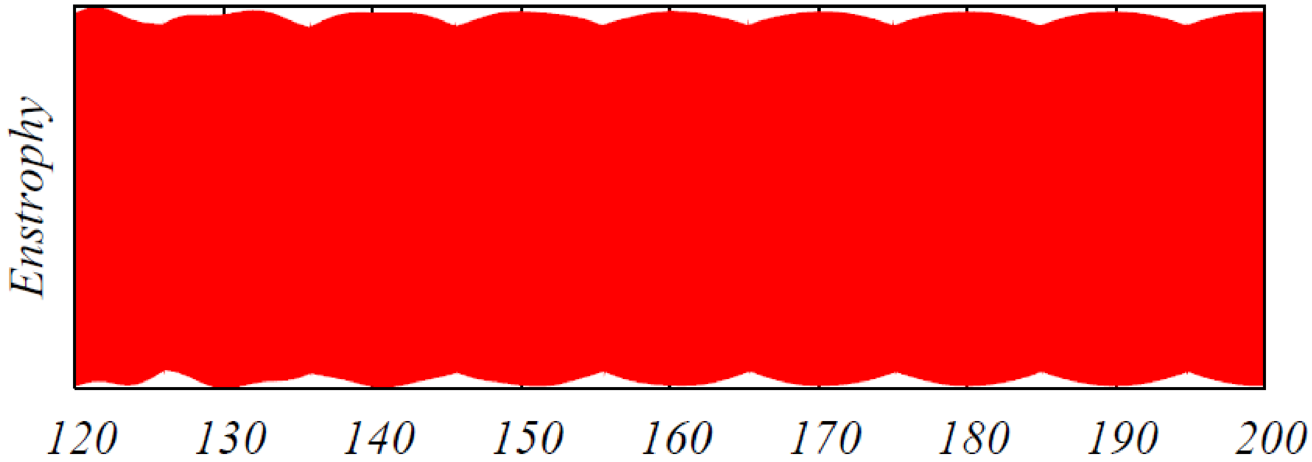
$\gamma = 0.185$



$\lambda = 2.03$

Time-Series of Enstrophy

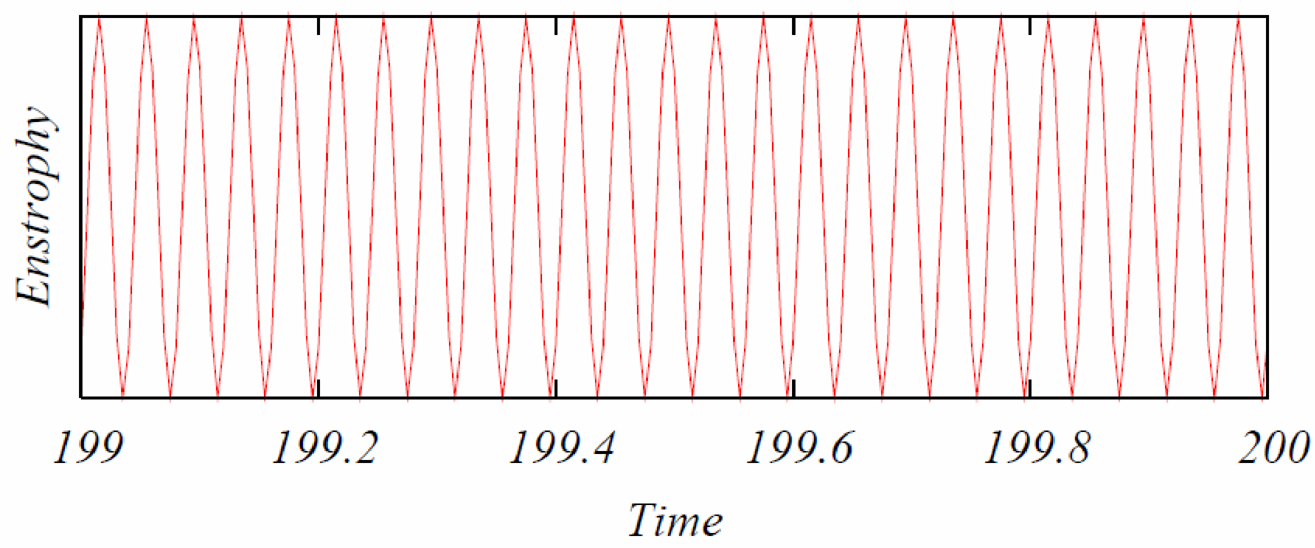




R_s = 2080

R_p = 395.2

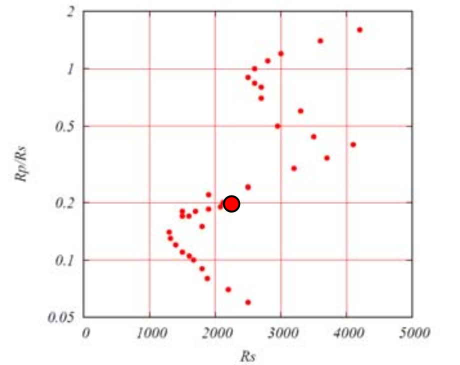
γ= 0.19

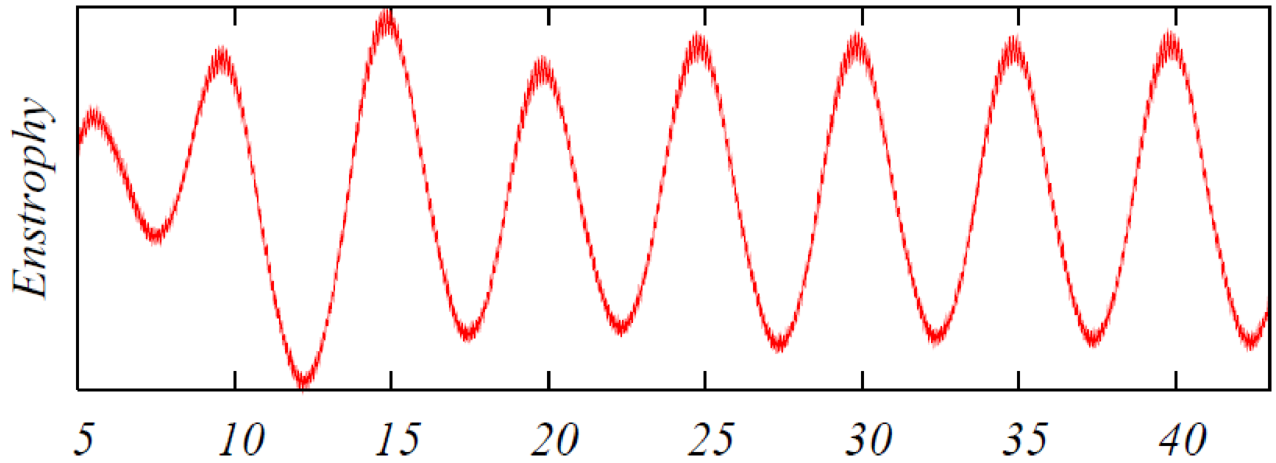


λ= 10.0

λ= 0.04

Time-Series of Enstrophy

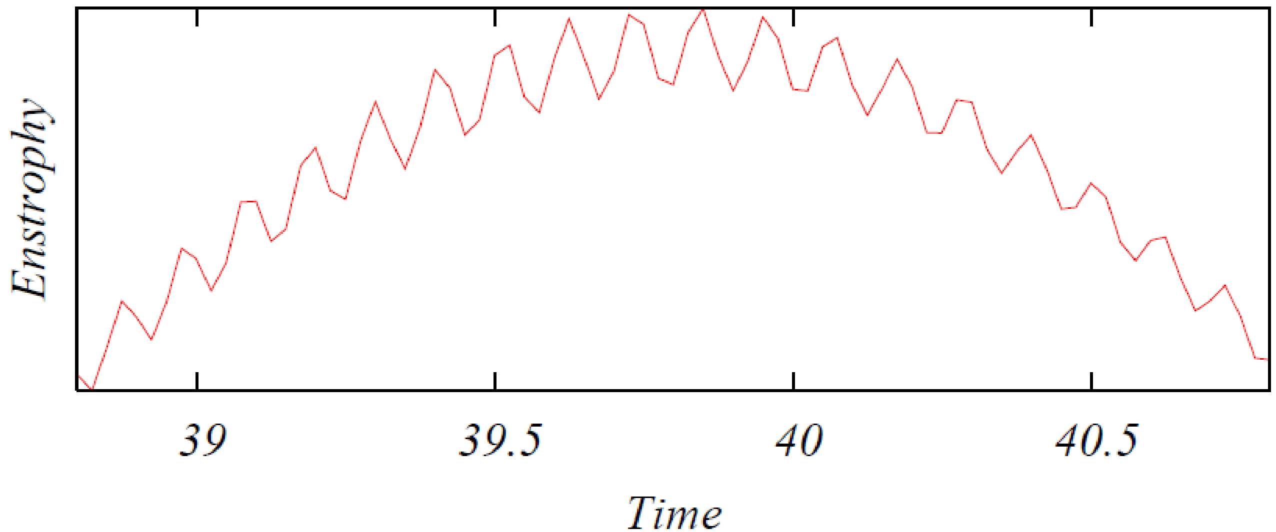




$R_s = 2140$

$R_p = 428$

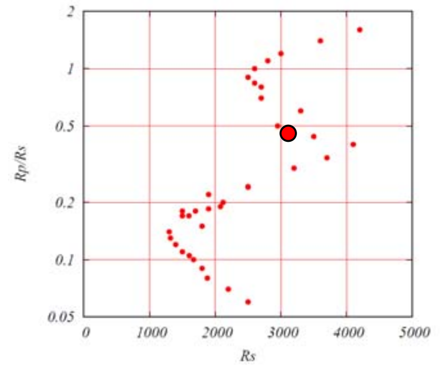
$\gamma = 0.2$

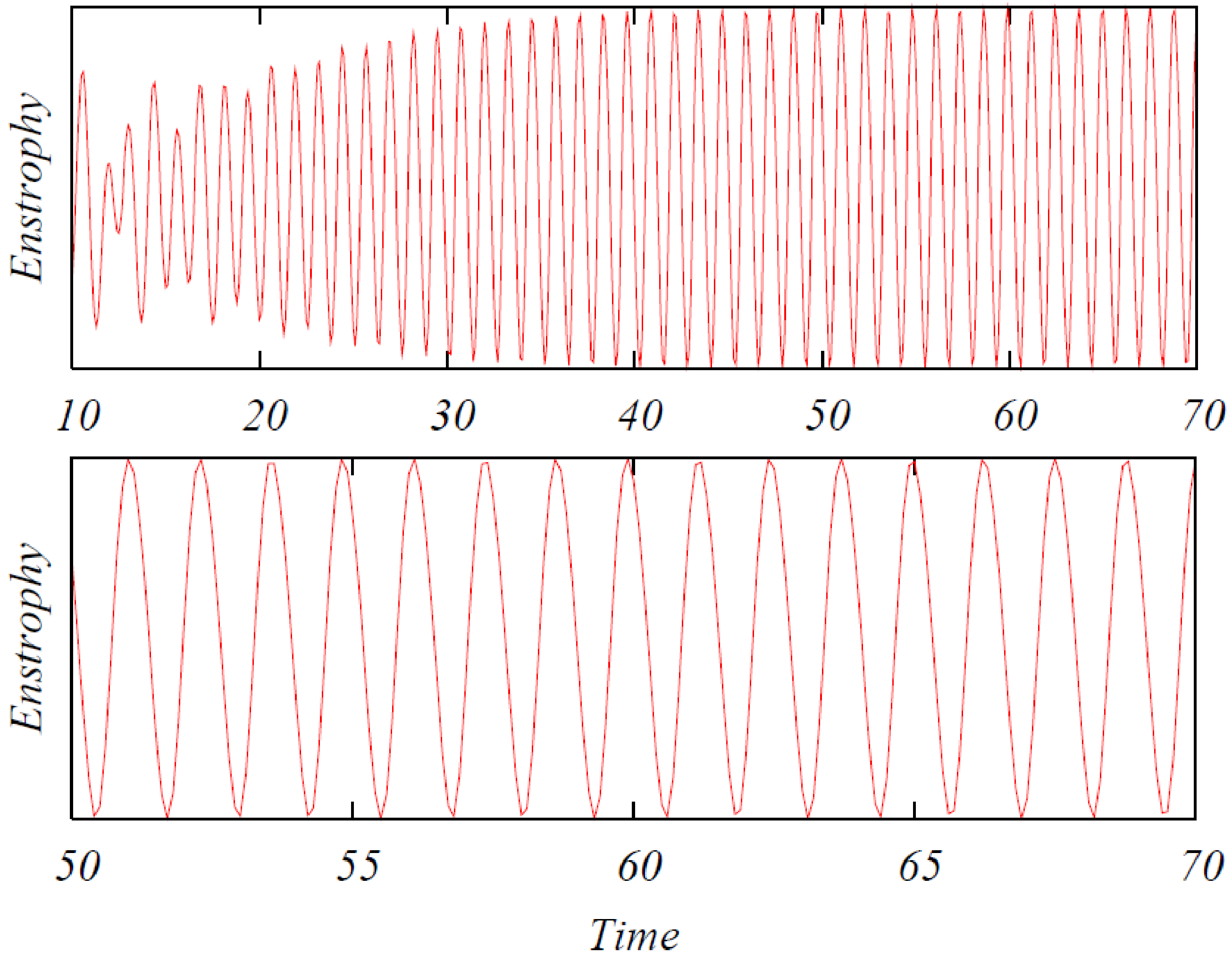


$\lambda = 5.0$

$\lambda = 0.10$

Time-Series of Enstrophy





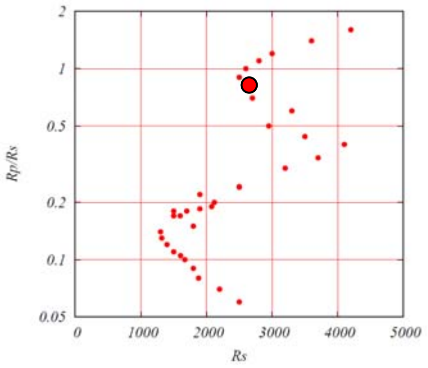
R_s = 2950

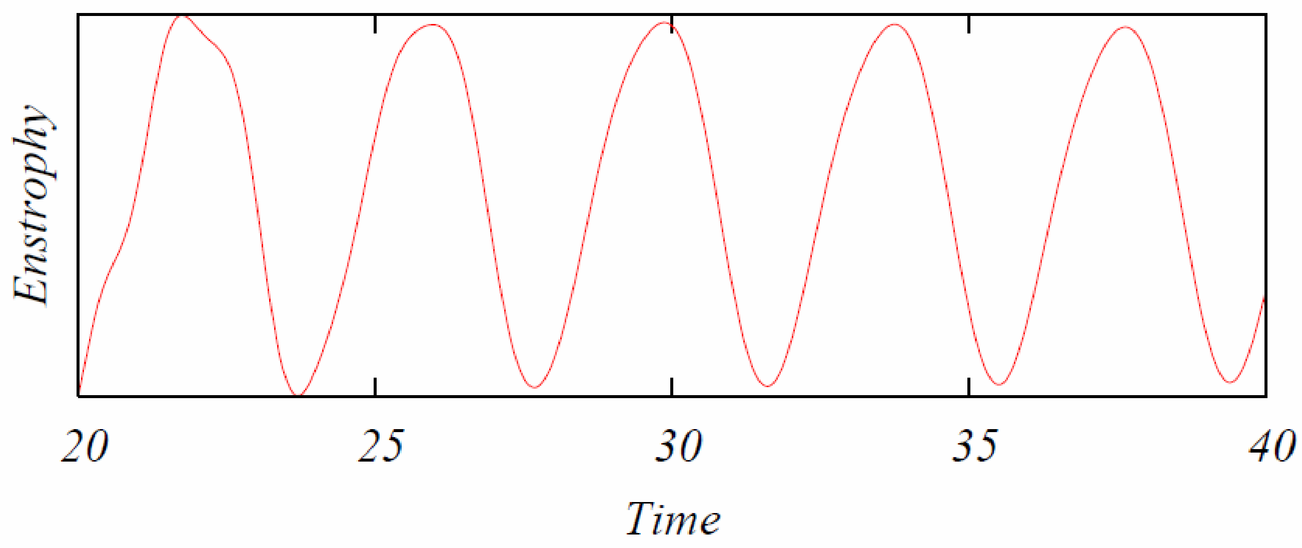
R_p = 1475

γ= 0.5

λ= 1.27

Time-Series of Enstrophy





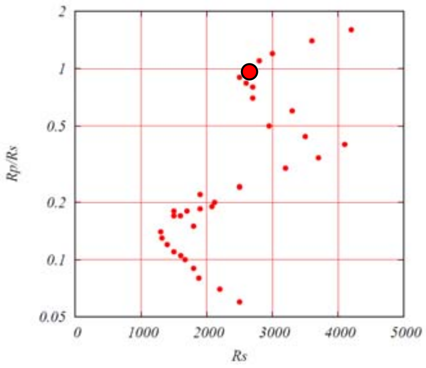
$R_s = 2700$

$R_p = 2160$

$\gamma = 0.8$

$\lambda = 3.94$

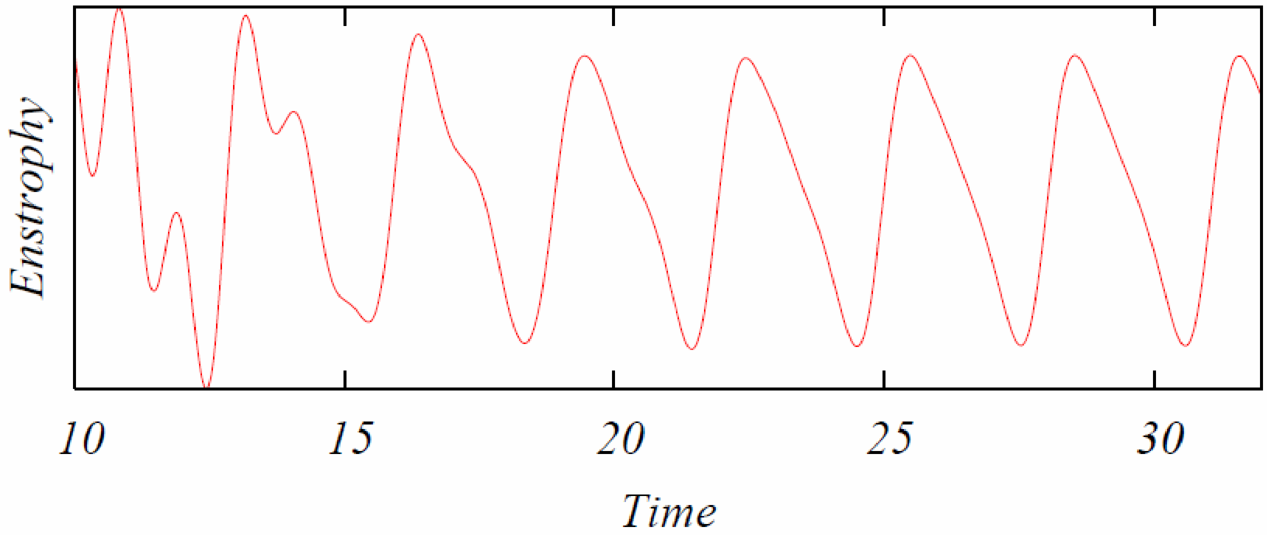
Time-Series of Enstrophy



$R_s = 2600$

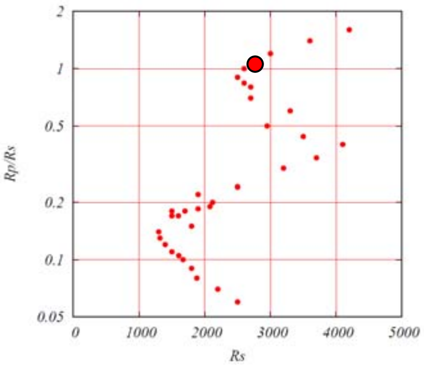
$R_p = 2600$

$\gamma = 1$



$\lambda = 2.79$

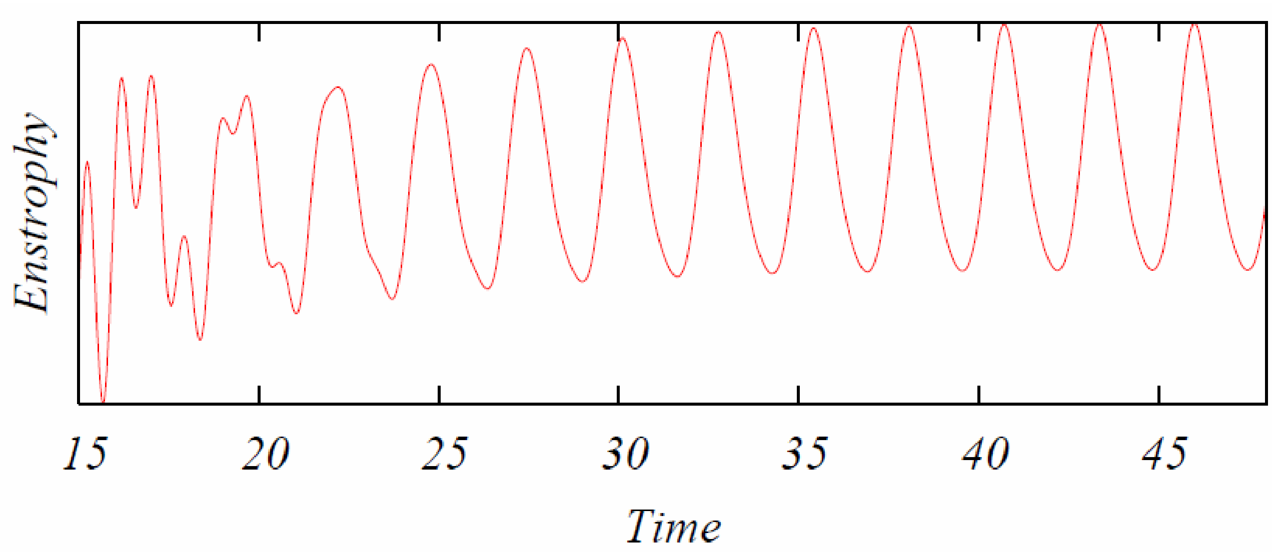
Time-Series of Enstrophy



$R_s = 2800$

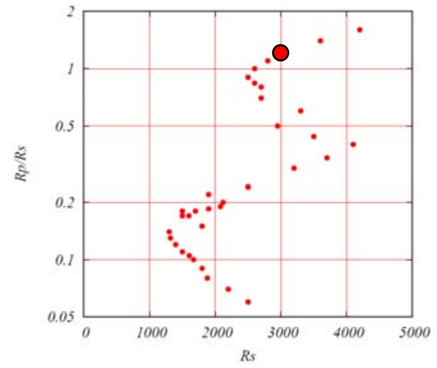
$R_p = 3080$

$\gamma = 1.1$



$\lambda = 2.64$

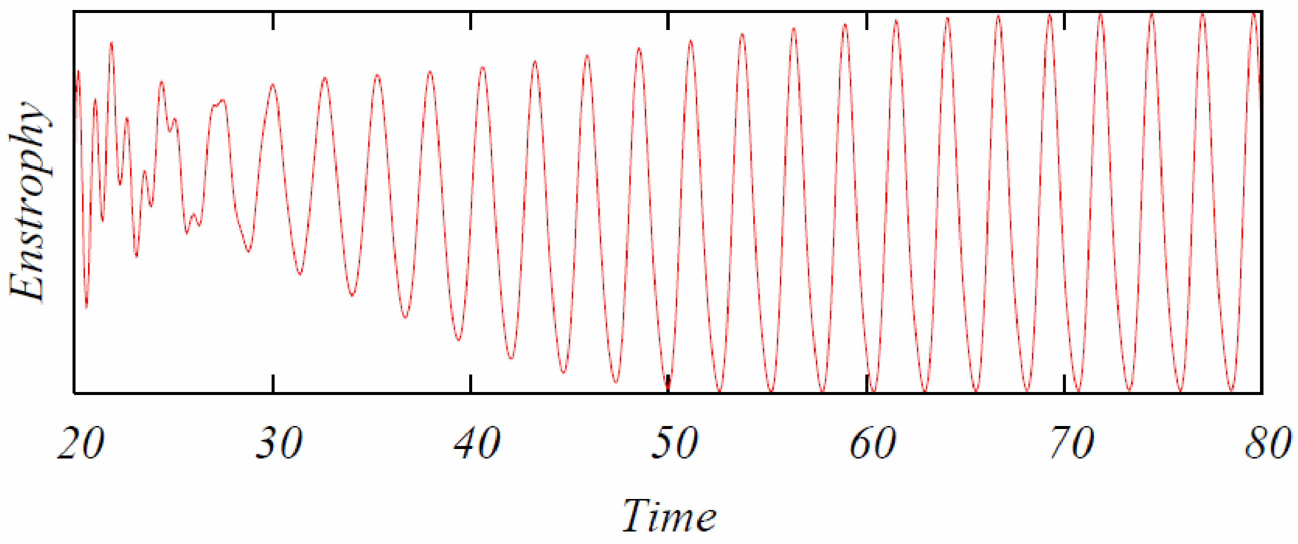
Time-Series of Enstrophy



$R_s = 3000$

$R_v = 3600$

$\gamma = 1.2$



$\lambda = 2.58$

Outline

1．Introduction

2．Experiment – State Diagram

3．Numerical Simulation – Stability boundary

Flow Structure

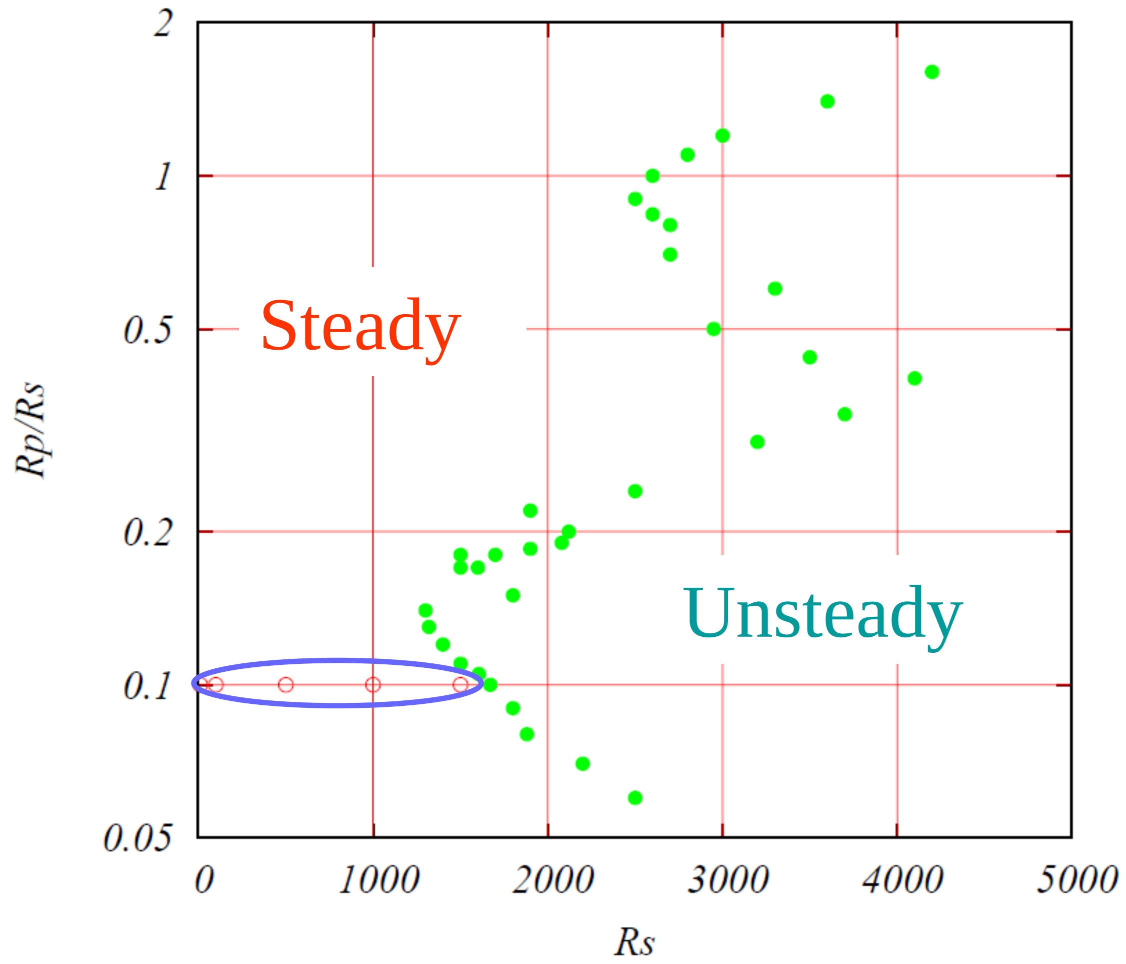
4．Asymptotic Analysis – Flow Structure

5．Summary

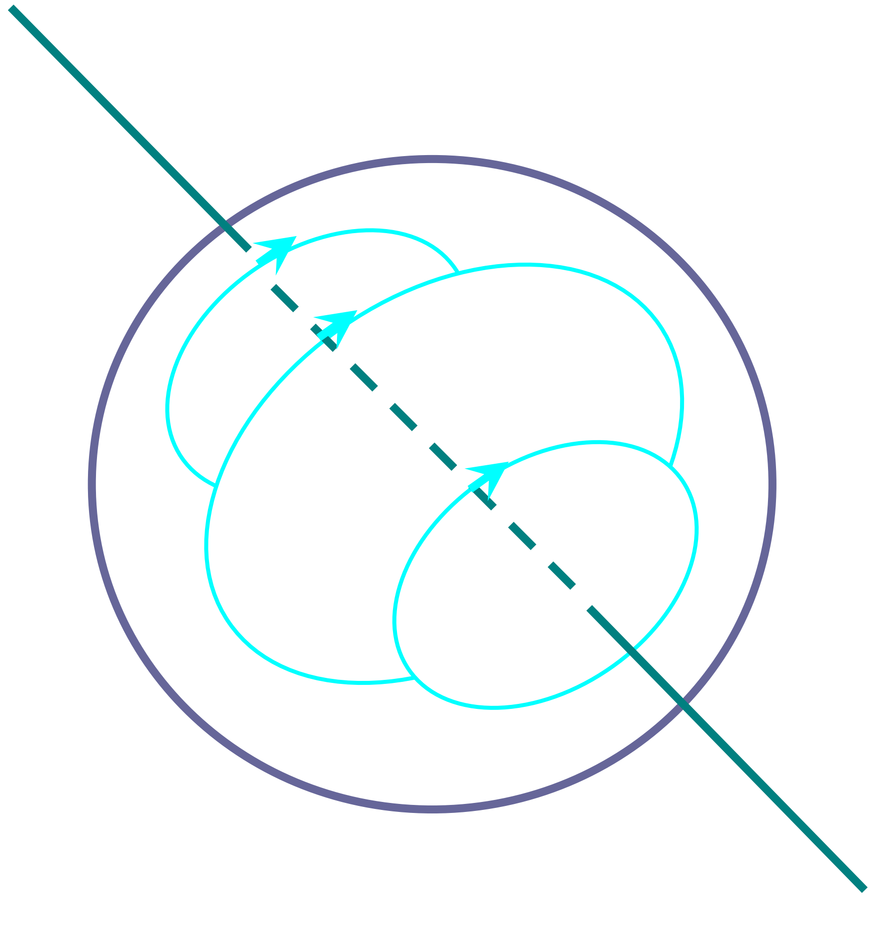
Flow Structure

Steady States

Stability Curve



Solid-Body Rotation

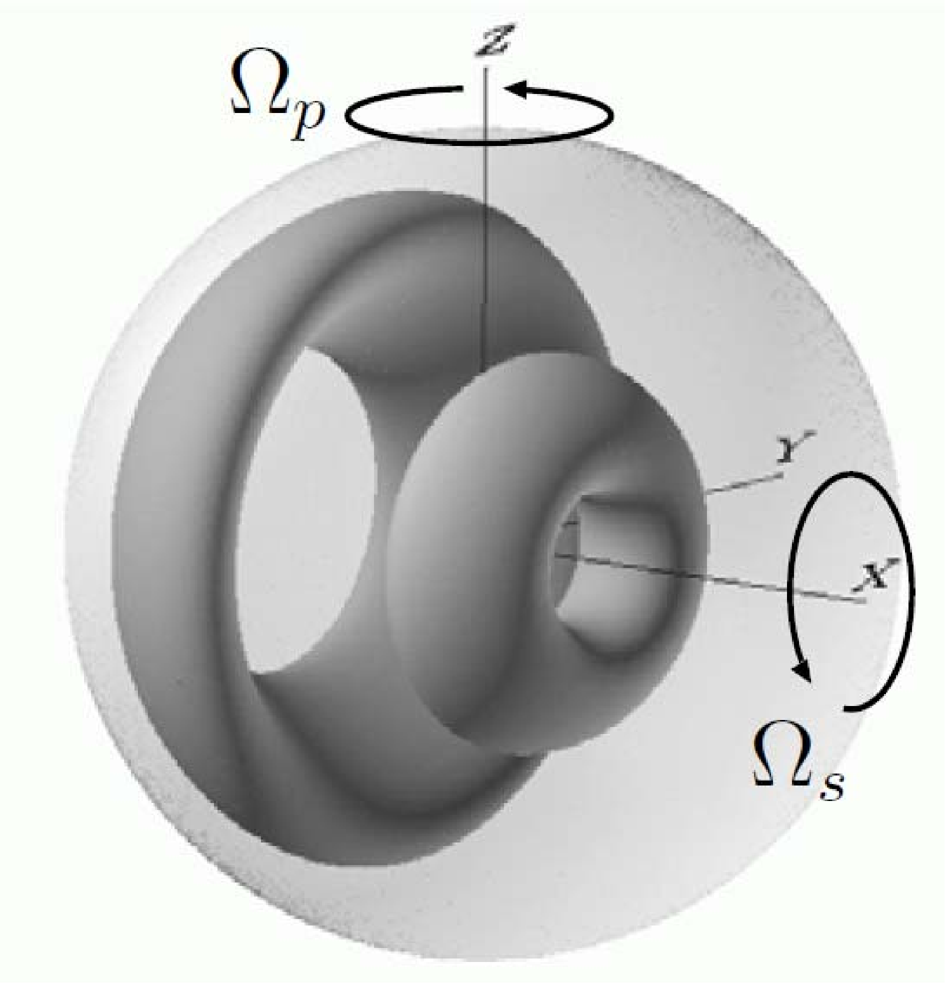


$\Gamma << 1$

or

$Re << 1$

Streamline Tori

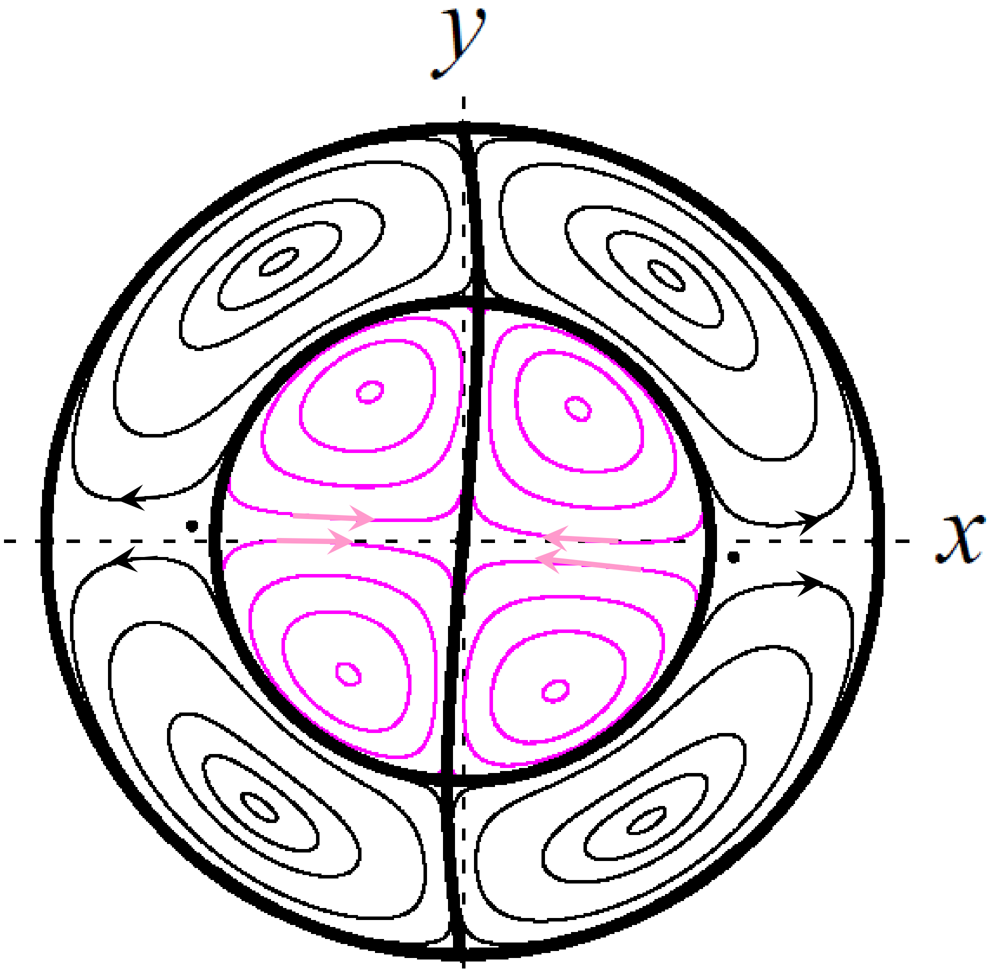


Rs=10

Rp=1

The whole surface of a torus is covered by a single streamline.

Cross-Section of Streamline Tori

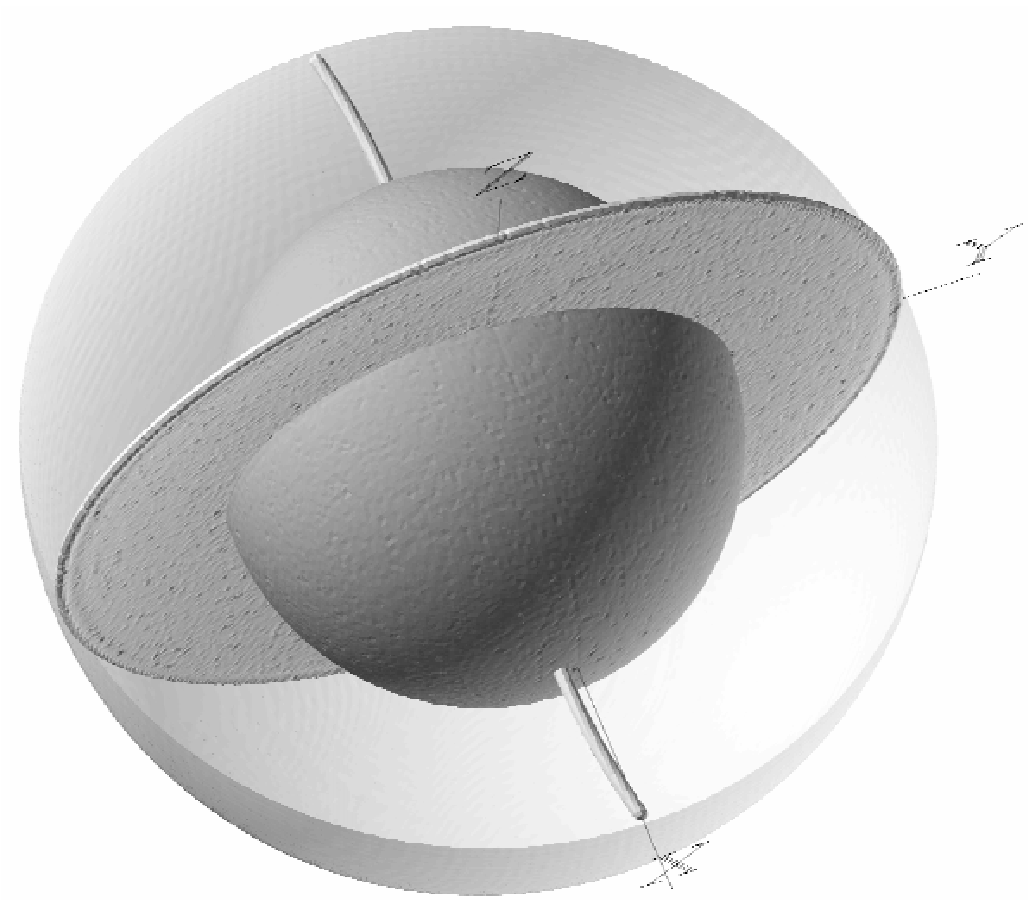


Rs=10

Rp=1

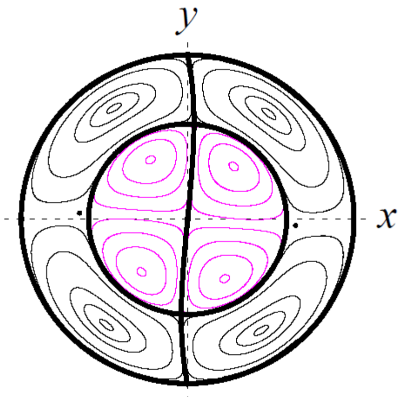
Spin axis

Separatrix Surfaces

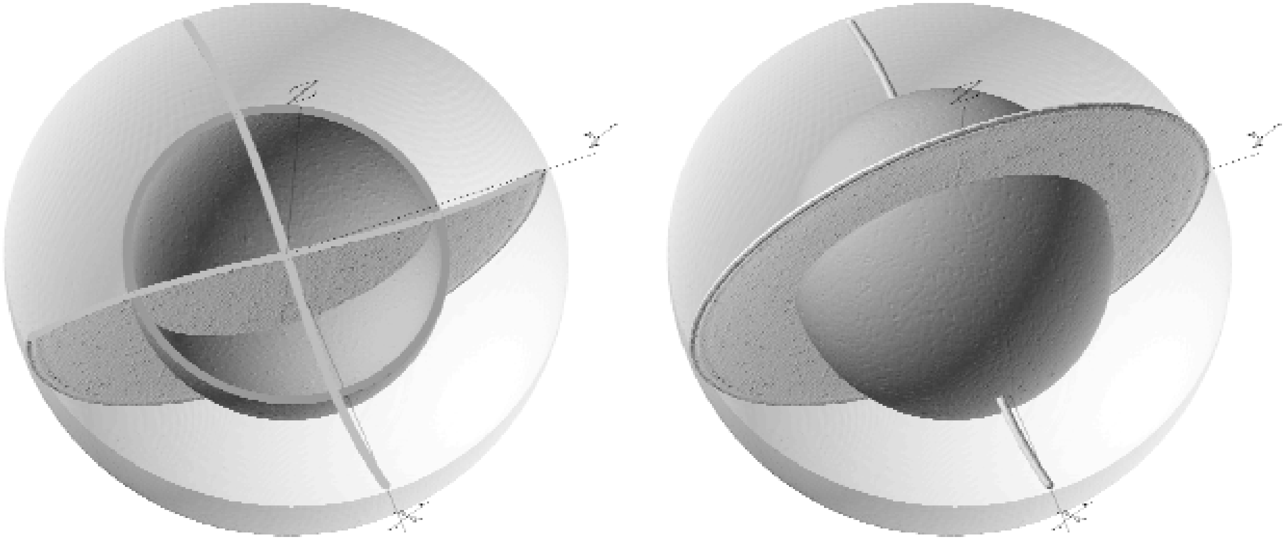


Rs=10

Rp=1



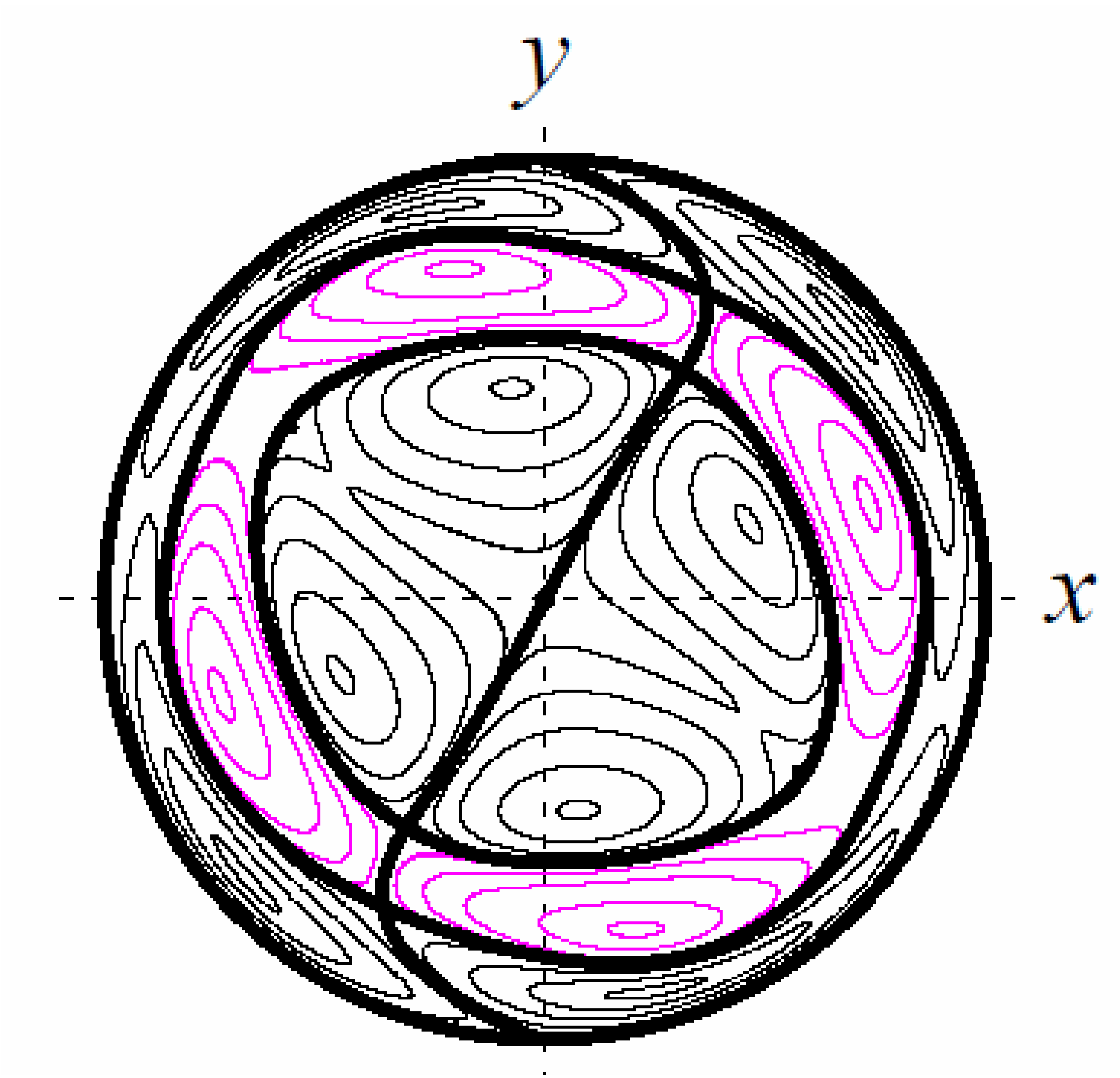
Separatrix Surfaces



Rs=10

Rp=1

Cross-Section of Streamline Tori

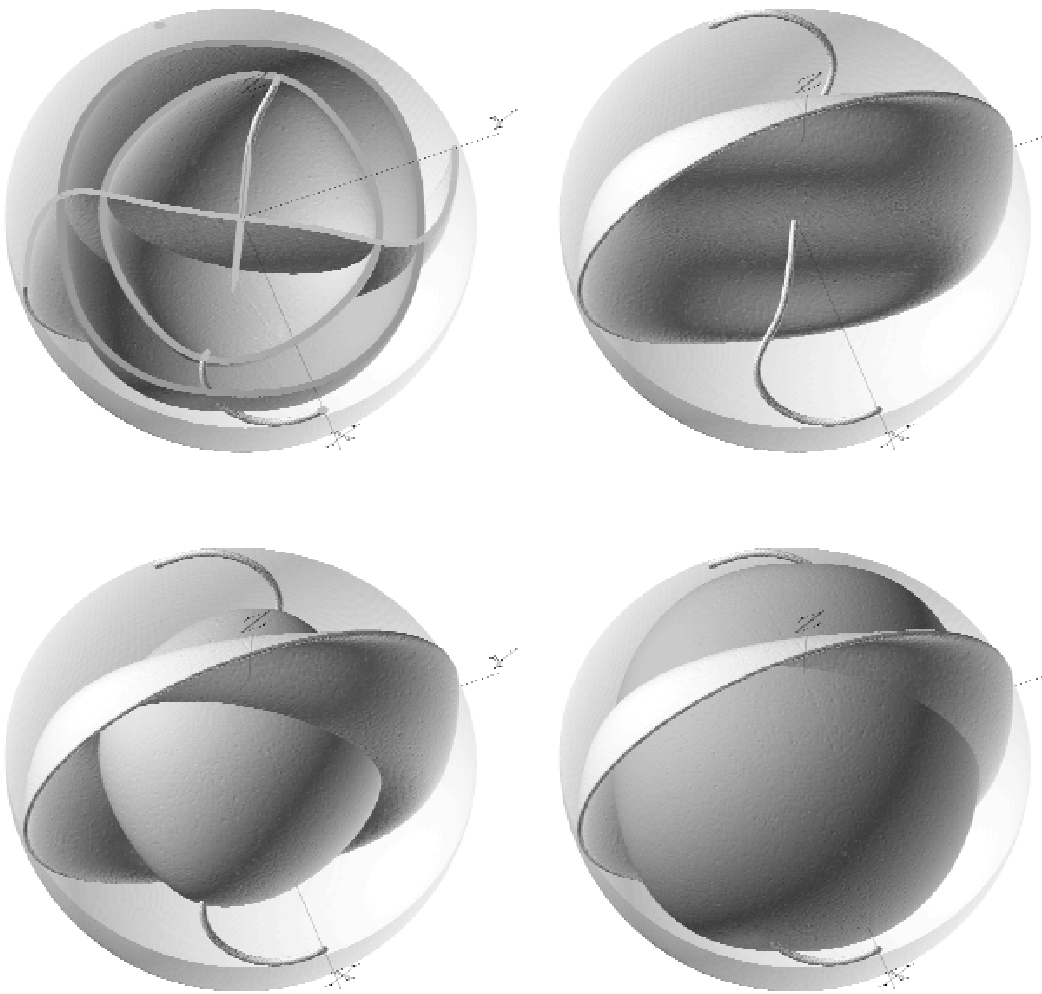


Rs=200

Rp=20

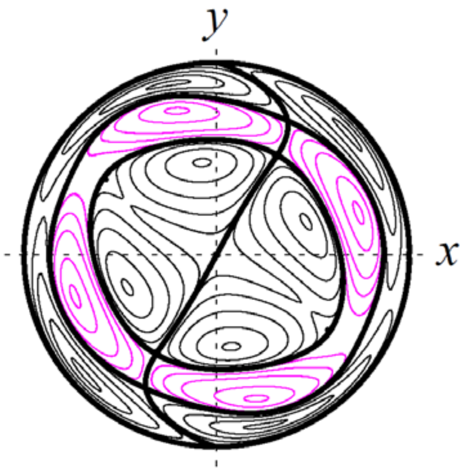
Spin axis

Separatrix Surfaces

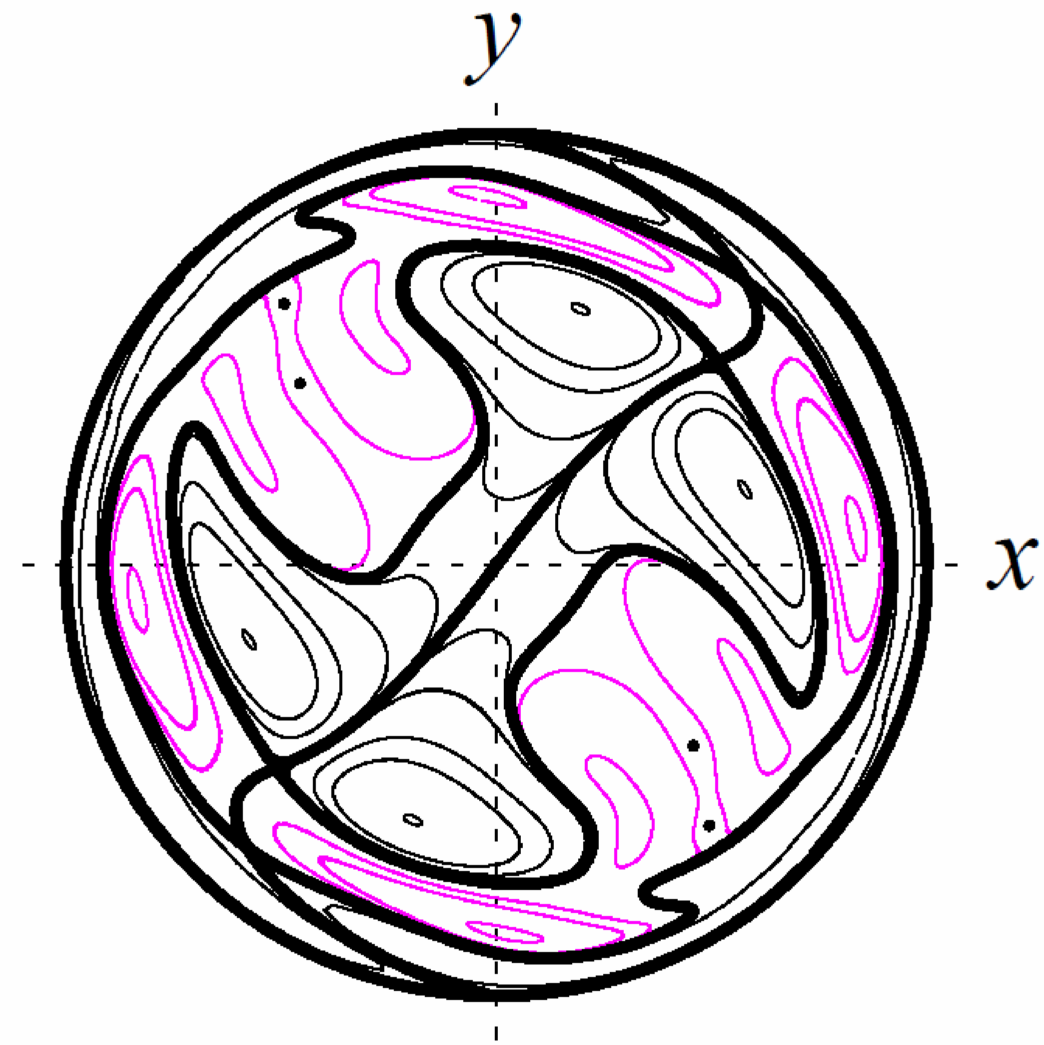


Rs=200

Rp=20



Cross-Section of Streamline Tori

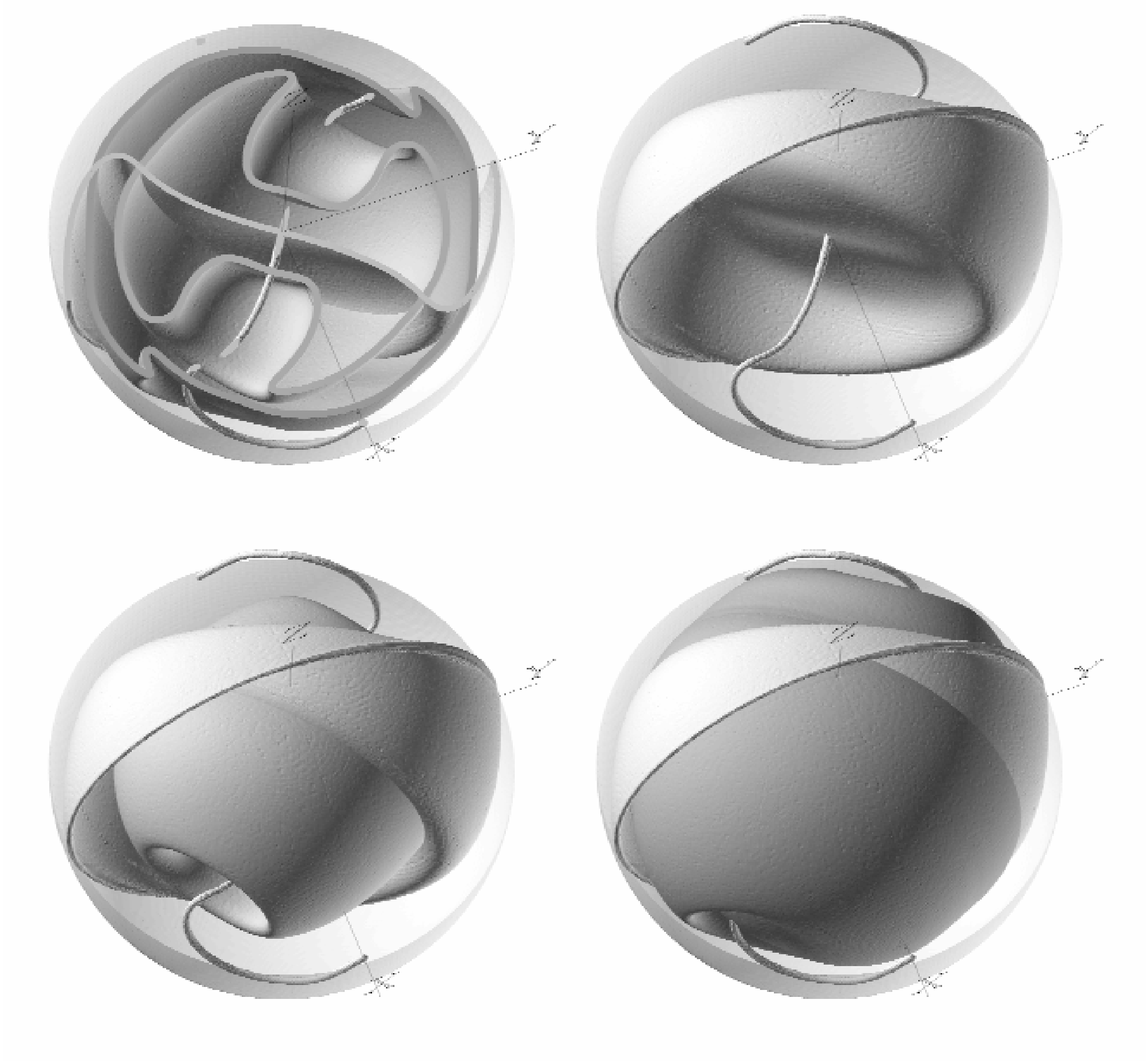


Rs=500

Rp=50

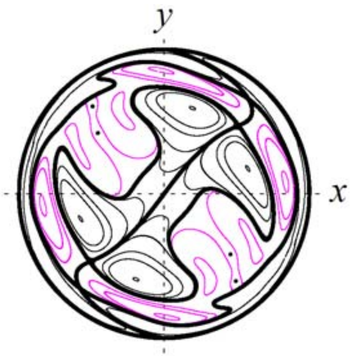
Spin axis

Separatrix Surfaces

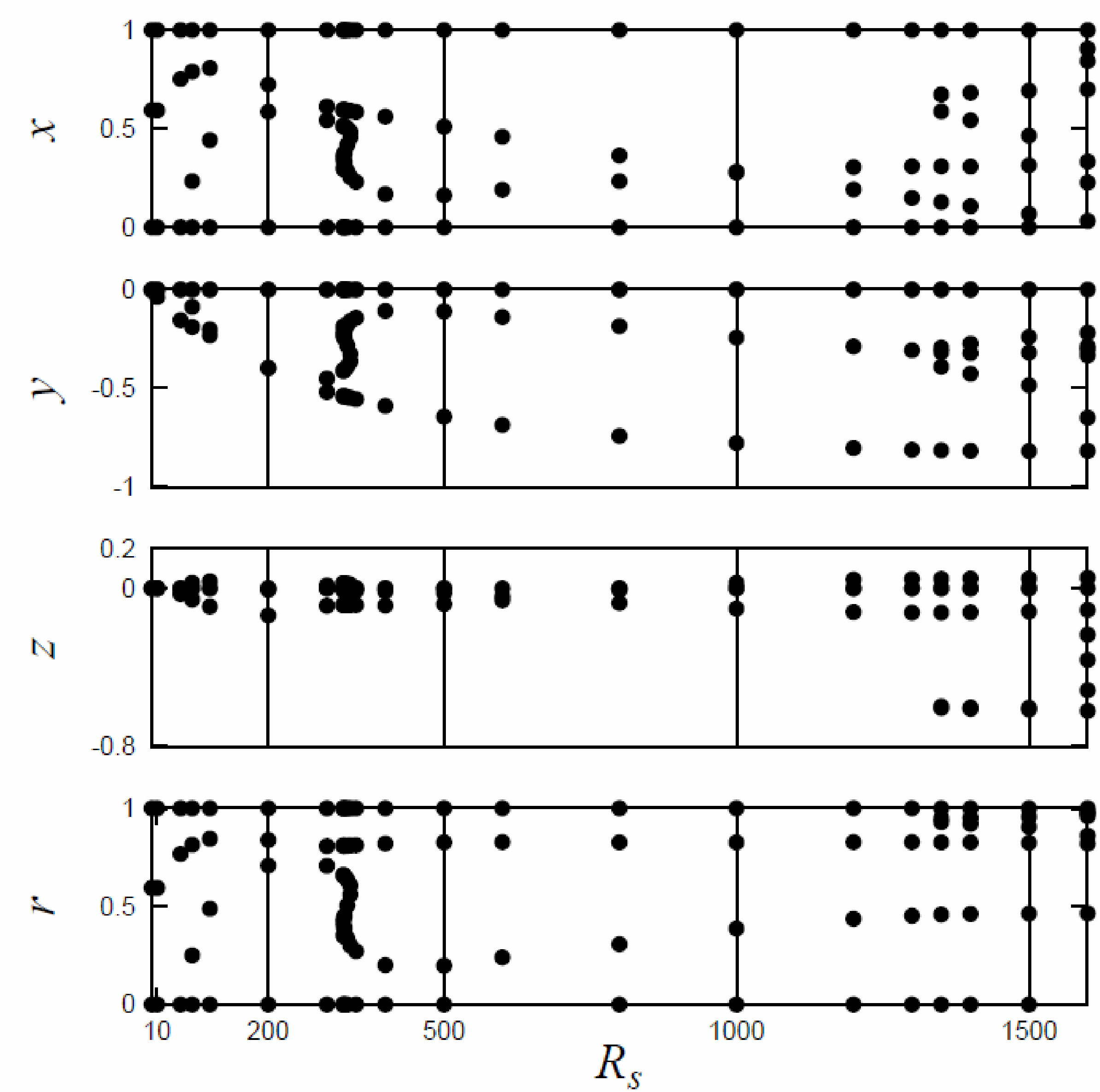


Rs=500

Rp=50



Location of Stagnation Points



Outline

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Flow Structure

4．Asymptotic Analysis – Flow Structure

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Asymptotic Analysis

In the double limit of
small Reynolds numbers
and large times

Low-Reynolds-Number Flow

$$R_s \frac{\partial \boldsymbol{u}}{\partial t} = R_s \boldsymbol{u} \times \boldsymbol{\omega} - 2\Gamma R_s \hat{\boldsymbol{z}} \times \boldsymbol{u} - R_s \nabla P + \nabla^2 \boldsymbol{u}$$

$$P = p + \frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}\Gamma^2 (\boldsymbol{r} \times \hat{\boldsymbol{z}})^2$$

$$\Gamma = \frac{R_p}{R_s} = \frac{\Omega_p}{\Omega_s} = O(1)$$

$$\boxed{R_s \ll 1 \quad R_p \ll 1}$$

$$\boldsymbol{u} = \boldsymbol{u}^{(0)} + R_s \boldsymbol{u}^{(1)} + {R_s}^2 \boldsymbol{u}^{(2)} + \cdots,$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}^{(0)} + R_s \boldsymbol{\omega}^{(1)} + {R_s}^2 \boldsymbol{\omega}^{(2)} + \cdots,$$

$$P = P^{(0)} + R_s P^{(1)} + {R_s}^2 P^{(2)} + \cdots$$

Low-Reynolds-Number Flow

0th Order

$$u_r^{(0)} = 0,$$
$$u_\theta^{(0)} = 0,$$
$$u_\varphi^{(0)} = r \sin\theta,$$

Solid-body rotation

1st Order

$$u_r^{(1)} = 0,$$
$$u_\theta^{(1)} = \frac{\Gamma}{10}(1 - r^2) r \sin\varphi,$$
$$u_\varphi^{(1)} = \frac{\Gamma}{10}(1 - r^2) r \cos\theta \cos\varphi,$$

Differential rotation around *y*-axis

Low-Reynolds-Number Flow

2^{nd} Order

$$u_r^{(2)} = \frac{\mathit{\Gamma}}{420} r(1-r^2)^2 \sin\theta \cos\varphi (\mathit{\Gamma} \sin\theta \sin\varphi + \cos\theta),$$

$$u_\theta^{(2)} = -\frac{\mathit{\Gamma}}{2520} r(7r^2-3)(1-r^2)(\mathit{\Gamma} \sin 2\theta \sin\varphi + \cos 2\theta) \cos\varphi,$$

$$u_\varphi^{(2)} = -\frac{\mathit{\Gamma}}{2520} r(7r^2-3)(1-r^2)(\mathit{\Gamma} \sin\theta \cos 2\varphi - \cos\theta \sin\varphi)$$

$$-\frac{\mathit{\Gamma}^2}{1400} r(9-5r^2)(1-r^2) \sin\theta$$

Low-Reynolds-Number Flow

3rd Order

$$u_r^{(3)} = \left[\frac{\mathit{\Gamma}^3(58-15r^2)}{623700} + \frac{\mathit{\Gamma}(5-3r^2)}{249480}\right] r(1-r^2)^2 \sin 2\theta \sin\varphi$$

$$+\frac{\mathit{\Gamma}^2(10-3r^2)r(1-r^2)^2}{113400}(3\cos^2\theta - 1)$$

$$-\frac{\mathit{\Gamma}^2(148-69r^2)r(1-r^2)^2}{1247400}\sin^2\theta\cos 2\varphi,$$

Low-Reynolds-Number Flow

3rd Order

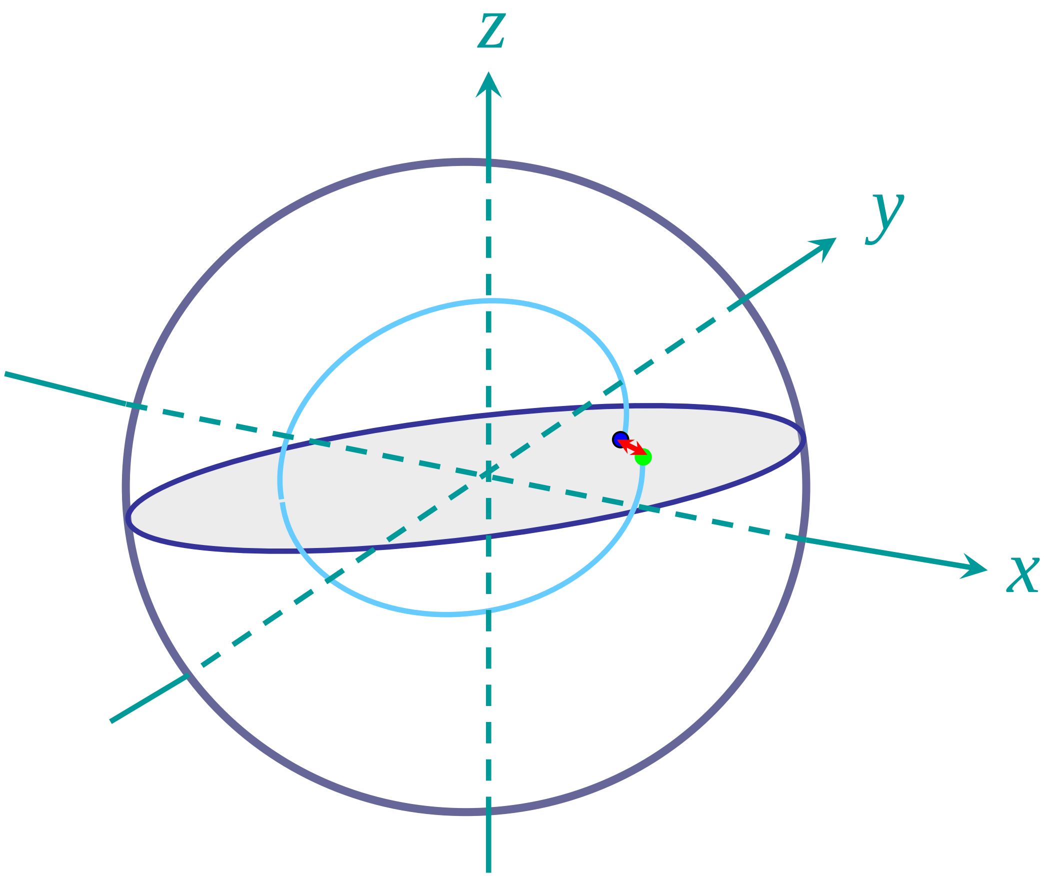
$$
\begin{aligned}
u_\theta^{(3)} = &-\frac{\Gamma^3}{249480} r^3(1-r^2)(13-9r^2)\sin^2\theta\sin 3\varphi \\
&-\frac{\Gamma^3}{12474000} r(1-r^2)(4884-4555r^2+1275r^4)\sin\varphi \\
&+\frac{\Gamma^3}{2494800} r(1-r^2)(232-663r^2+195r^4)\cos 2\theta\sin\varphi \\
&-\frac{\Gamma^2}{2494800} r(1-r^2)(148-287r^2+87r^4)\sin 2\theta\cos 2\varphi \\
&-\frac{\Gamma^2}{226800} r(1-r^2)(30-85r^2+27r^4)\sin 2\theta \\
&-\frac{\Gamma}{2494800} r(1-r^2)(99-250r^2+135r^4)\sin\varphi \\
&+\frac{\Gamma}{249480} r(1-r^2)^2(5-3r^2)\cos 2\theta\sin\varphi,
\end{aligned}
$$

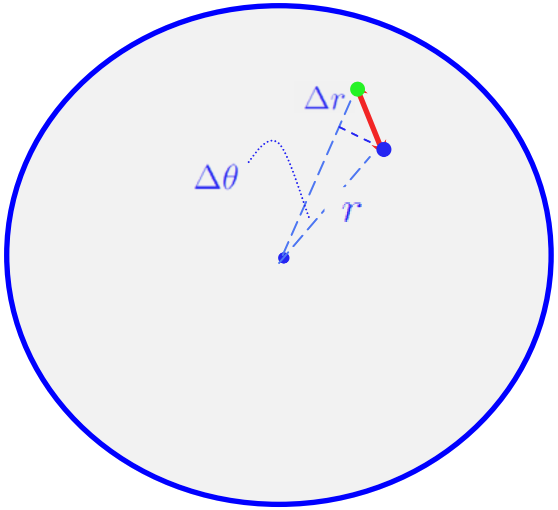
Low-Reynolds-Number Flow

3rd Order

$$
\begin{aligned}
u_\varphi^{(3)} = &-\frac{\Gamma^3}{249480} r^3(1-r^2)(13-9r^2)\sin^2\theta\cos\theta\cos 3\varphi \\
&-\frac{\Gamma^3}{997920} r^3(1-r^2)(13-9r^2)\cos 3\theta\cos\varphi \\
&-\frac{\Gamma^3}{24948000} r(1-r^2)(7448-2805r^2+825r^4)\cos\theta\cos\varphi \\
&+\frac{\Gamma^2}{249480} r^3(1-r^2)(13-9r^2)\sin 3\theta\sin 2\varphi \\
&+\frac{\Gamma^2}{623700} r(1-r^2)(74-241r^2+111r^4)\sin\theta\sin 2\varphi \\
&+\frac{\Gamma}{249480} r^3(1-r^2)(13-9r^2)\cos 3\theta\cos\varphi \\
&-\frac{\Gamma}{2494800} r(1-r^2)(49-40r^2+15r^4)\cos\theta\cos\varphi.
\end{aligned}
$$

Poincare Map of Streamline





$$\Delta r = -R_s{}^3 \frac{\pi \Gamma^2}{16200} r_0 (1 - r_0{}^2)^2 (1 - 3 r_0{}^2)(3\cos^2\theta_0 - 1) + O(R_s{}^4)$$

$$\Delta\theta = R_s{}^3 \frac{\pi \Gamma^2}{32400} (1 - r_0{}^2)(3 - 22 r_0{}^2 + 27 r_0{}^4) \sin 2\theta_0 + O(R_s{}^4)$$

Cross-Section of a Streamline Torus

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{2r(1-r^2)(1-3r^2)(3\cos^2\theta - 1)}{(3 - 22r^2 + 27r^4)\sin 2\theta}$$

$$R_s \ll 1$$

$$R_p \ll 1$$

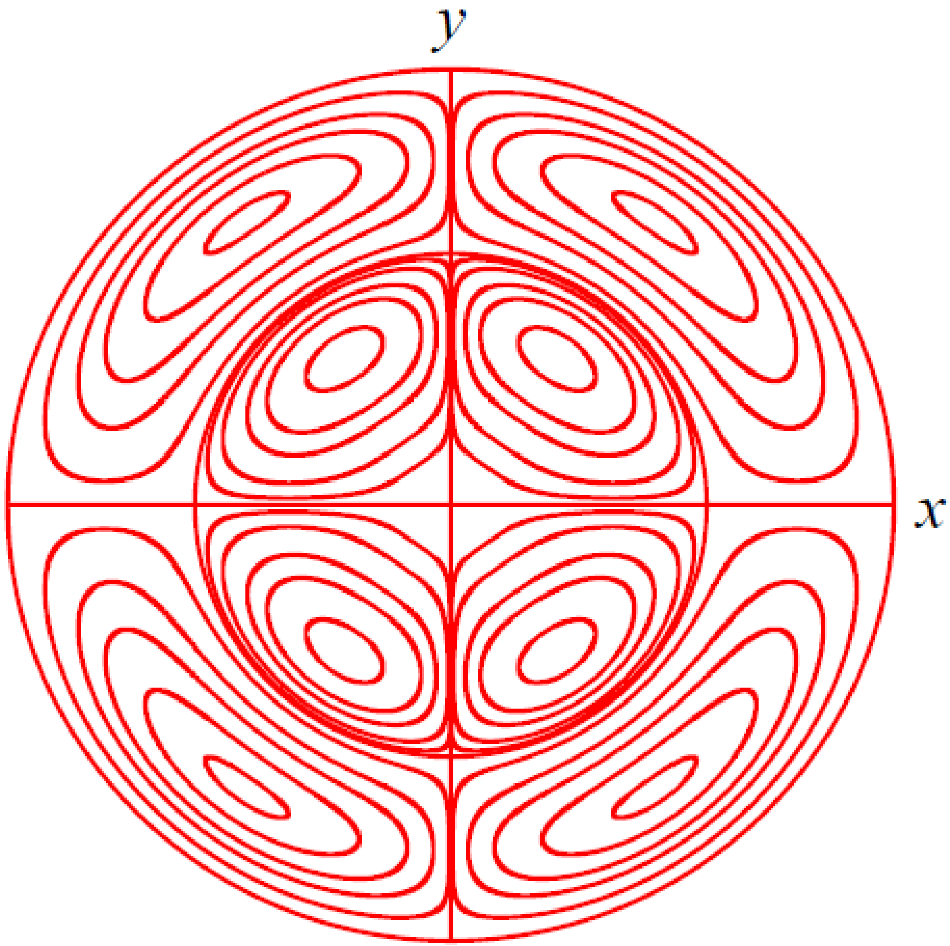
$$t = O(R_s {R_p}^2)$$

$$r_0 \to r,\ \theta_0 \to \theta,\ \Delta r/\Delta\theta \to \mathrm{d}r/\mathrm{d}\theta$$

$$\boxed{r^3(1-r^2)^2(\tfrac{1}{3} - r^2)(1 - \cos^2\theta)\cos\theta = \mathrm{const.}}$$

$$(1 - x^2 - y^2 - z^2)^2(\tfrac{1}{3} - x^2 - y^2 - z^2)(y^2 + z^2)x = \mathrm{const.}$$

Cross-Section of a Streamline Torus



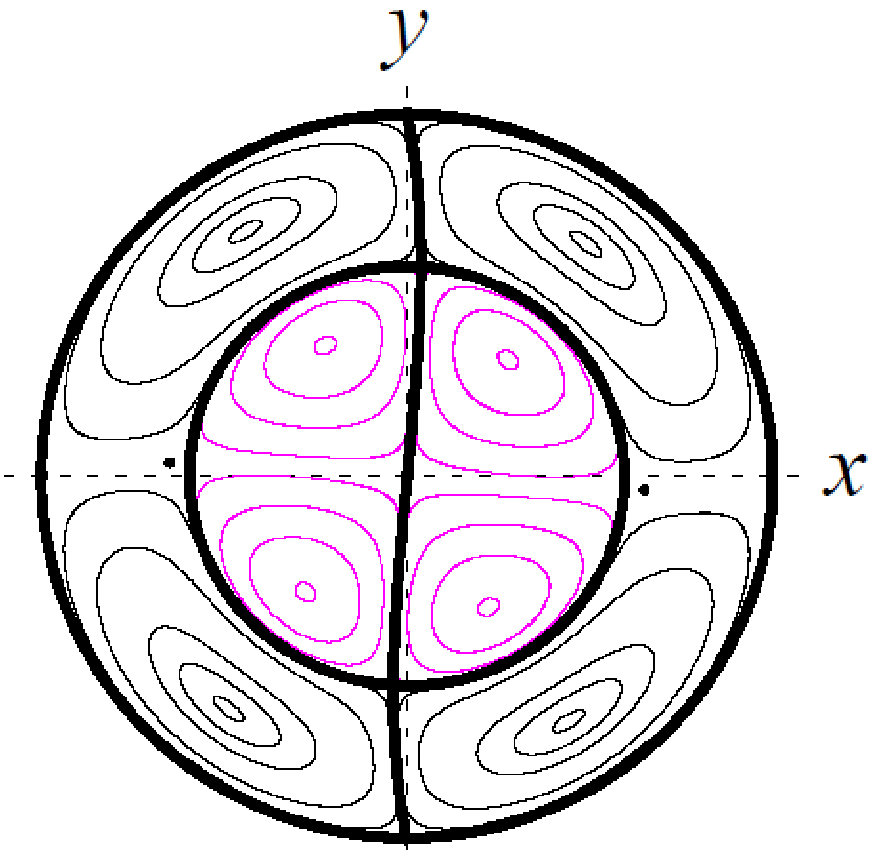
$$R_s \ll 1$$

$$R_p \ll 1$$

$$t = O(R_s {R_p}^2)$$

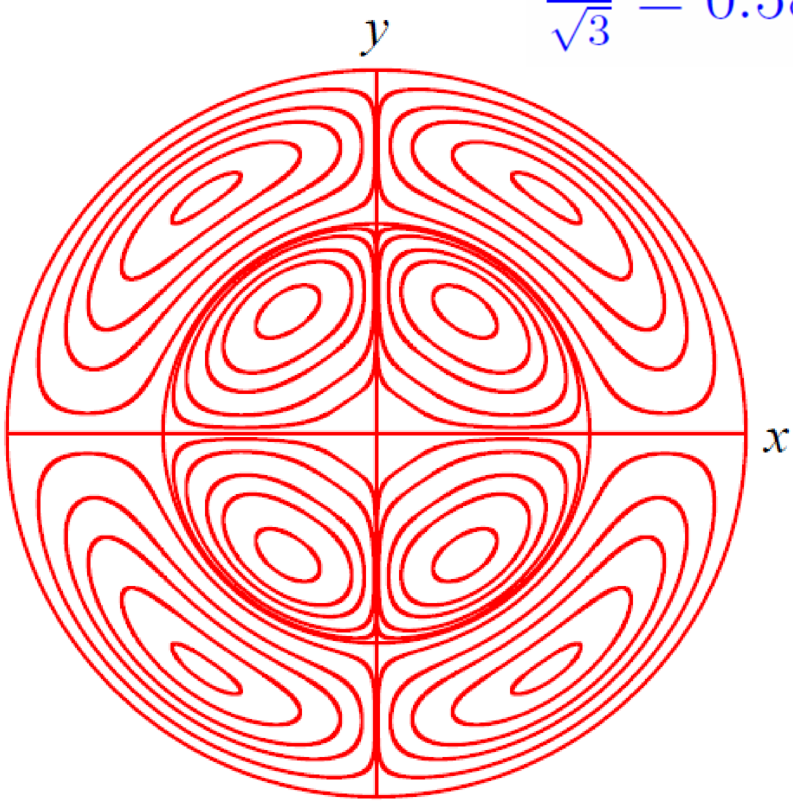
$$r^3(1-r^2)^2(\tfrac{1}{3}-r^2)(1-\cos^2\theta)\cos\theta = \text{const.}$$

Comparison: DNS & Theory



Rs=10, *Rp*=1

$$\frac{1}{\sqrt{3}} = 0.58$$



$$r^3(1-r^2)^2(\tfrac{1}{3}-r^2)(1-\cos^2\theta)\cos\theta = \text{const.}$$

Summary

① The state diagram of flows in a precessing sphere was constructed experimentally.

② The stability curve of steady flow was revealed partially by DNS.

③ The toral structure of streamlines was observed by DNS. So far no chaotic streamline has been found.

④ An analytical expression was obtained for the streamline tori in the double limit of small Reynolds numbers and large times.

Future Problems

① Complete the stability curve

② Clarify the characteristics of critical modes

③ Perform the linear stability analysis

④ Raise the speed of the numerical code

⑤ Examine the turbulence characteristics, such as intensity, mixing, etc.