Markovianized Single-time Local Energy Transfer:

A K41 compatible Eulerian spectral closure for isotropic turbulence with no tuning parameters

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Overview of this talk

- Basic equations and **notation**
- The turbulence **closure** problem
- Brief review of **Renormalized Perturbation Theory** (RPT)
- Local Energy Transfer theory (LET)
    - **Time-ordered** representation of covariance
    - **Fluctuation dissipation** ansatz
- **Single-time Markovianized** LET
- **Comparisons** with other single-time closures

Basic equations

Incompressible spectral Navier Stokes Eqns (NSE)

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha(\mathbf{k},t) = f_\alpha(\mathbf{k},t) + \lambda M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t)$$

$$k_\alpha u_\alpha(\mathbf{k},t) = 0$$

Book keeping

$$u_\alpha(\mathbf{x},t) = \int d^3 k u_\alpha(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

White noise forcing

$$k_\alpha f_\alpha(\mathbf{k},t) = 0$$

$$\langle f_\alpha(\mathbf{k},t) f_\beta(-\mathbf{k},t')\rangle = P_{\alpha\beta}(\mathbf{k}) w(k,t-t')$$

The closure problem

Covariance Q

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle = \delta(\mathbf{k}+\mathbf{k}')P_{\alpha\beta}(\mathbf{k})Q(k;t,t')$$

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] Q_{\alpha\sigma}(\mathbf{k};t,t') = \lambda M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \, \langle u_\beta(\mathbf{j},t)\, u_\gamma(\mathbf{k}-\mathbf{j},t)\, u_\sigma(-\mathbf{k},t')\rangle \quad *$$

$$\widehat{L}\,\langle UU\rangle = \widehat{M}\,\langle UUU\rangle + \langle f\rangle$$

$$\widehat{L}\,\langle UUU\rangle = \widehat{M}\,\langle UUUU\rangle + \langle f\rangle$$

$$\vdots$$

Ignore tensor arguments and wave vectors

Keep your eyes on Q's G's M's and λ's

Primitive perturbation series

$$u_\alpha(\mathbf{k},t) = u_\alpha^{(0)}(\mathbf{k},t) + \lambda u_\alpha^{(1)}(\mathbf{k},t) + \lambda^2 u_\alpha^{(2)}(\mathbf{k},t) + \dots$$

Navier Stokes eqns

$$\begin{aligned}
\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(0)}(\mathbf{k},t) &= f_\alpha(\mathbf{k},t), \\
\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(1)}(\mathbf{k},t) &= M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j u_\beta^{(0)}(\mathbf{j},t) u_\gamma^{(0)}(\mathbf{k}-\mathbf{j},t), \\
\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(2)}(\mathbf{k},t) &= 2M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j u_\beta^{(0)}(\mathbf{j},t) u_\gamma^{(1)}(\mathbf{k}-\mathbf{j},t), \\
\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(3)}(\mathbf{k},t) &= 2M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j u_\beta^{(2)}(\mathbf{j},t) u_\gamma^{(0)}(\mathbf{k}-\mathbf{j},t) \\
&\quad + M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j u_\beta^{(1)}(\mathbf{j},t) u_\gamma^{(1)}(\mathbf{k}-\mathbf{j},t), \\
&\vdots
\end{aligned}$$

The 'bare' response function

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(0)}(\mathbf{k}, t) = f_\alpha(\mathbf{k}, t)$$

$$\begin{aligned} u_\alpha^{(0)}(\mathbf{k}, t) &= \int dt' \exp\left\{-\nu k^2 (t - t')\right\} f_\alpha(\mathbf{k}, t') \\ &= \int dt' H_{\alpha\beta}^{(0)}(\mathbf{k}; t, t') f_\beta(\mathbf{k}, t') \end{aligned}$$

$$H_{\alpha\beta}^{(0)}(\mathbf{k}; t, t') = P_{\alpha\beta}(\mathbf{k}) H^{(0)}(k; t, t')$$

$$H^{(0)}(k; t, t') = \begin{cases} \exp\left\{-\nu k^2 (t - t')\right\} & t \geq t' \\ 0 & t < t' \end{cases}$$

Response fn *H* -- details how the system kicks back

Primitive series for Covariance

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] Q_{\alpha\sigma}\left(\mathbf{k};t,t'\right) = \lambda M_{\alpha\beta\gamma}\left(\mathbf{k}\right) \int d^3j \left\langle u_\beta\left(\mathbf{j},t\right) u_\gamma\left(\mathbf{k}-\mathbf{j},t\right) u_\sigma\left(-\mathbf{k},t'\right)\right\rangle$$

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \nu k^2\right] Q_{\alpha\sigma}(\mathbf{k};t,t') \quad = \quad & \lambda^2 M_{\alpha\beta\gamma}(k) \int d^3j \left\{\left[\int_0^{t'} ds H^{(0)}_{\sigma\rho}(-\mathbf{k};t',s) M_{\rho\delta\epsilon}(-\mathbf{k}) \times \right.\right. \\ & \left. \times\, 2Q^{(0)}_{\beta\delta}(\mathbf{j};t,s) Q^{(0)}_{\epsilon\gamma}(\mathbf{k}-\mathbf{j};t,s)\right] - \\ & - \left[\int_0^{t} ds H^{(0)}_{\beta\rho}(\mathbf{j};t,s) M_{\rho\delta\epsilon}(\mathbf{j}) \times \right. \\ & \left.\left. \times 4Q^{(0)}_{\delta\gamma}(\mathbf{k}-\mathbf{j};t,s) Q^{(0)}_{\epsilon\sigma}(-\mathbf{k};t',s)\right]\right\} + \mathcal{O}(\lambda^3),\end{aligned}$$

Need to choose an equation for the response

General scheme of Renormalized Perturbation Theories (RPT)

- Dressing up 'bare' quantities with 'renormalized' ones
- Using the Wyld Feynman-type diagram scheme -> show to be a partial summation of primitive perturbation series (PS)

Primitive PS ➡ Sum of parts of PS to all orders + PS of renormalized quantities

(Sum of parts of PS to all orders + PS of renormalized quantities) = Renormalized perturbation series

$$
\begin{aligned}
Q &= F[Q^{(0)}, H^{(0)}] \\
H &= Y[Q^{(0)}, H^{(0)}]
\end{aligned}
\qquad
\boxed{
\begin{aligned}
Q^{(0)} &\rightarrow Q \\
H^{(0)} &\rightarrow H
\end{aligned}
}
\qquad
\begin{aligned}
Q &= F[Q, H] \\
H &= Y[Q, H]
\end{aligned}
$$

Truncate at second order - throw away $O(\lambda^3)$ -> Direct interaction

Local Energy Transfer Theory (LET)

- Time-ordering
- Fluctuation-Dissipation relation
- Propagator properties
- K41/Scale invariance
- Single-time Markovian form

New formulation of LET

Serious and more careful treatment of causality by explicit time-ordering (using Heaviside step functions).

Time-ordered covariance

$$Q\left(k;t,t'\right)=\theta\left(t-t'\right)Q\left(k;t,t'\right)+\theta\left(t'-t\right)Q\left(k;t,t'\right)-\delta_{t,t'}Q\left(k;t,t'\right)$$

General - independent from LET

Exponential representation / approximation of 2-t covariance on firmer ground

$$Q(k;t-t') = Q(k)exp\left\{-\omega(k)\left|t-t'\right|\right\}$$

$$H(k;t,t') = \theta(t-t')exp\left\{-\omega(k)(t-t')\right\}$$

$$\lim_{t \to t'} \frac{\partial}{\partial t} Q(k;t,t') = 0$$

Kraichnan,JFM **5** (1959),

Leslie, *Developments in the theory of turbulence* (1973)

LET Ansatz

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] Q_{\alpha\sigma}\left(\mathbf{k}; t, t'\right) = \lambda M_{\alpha\beta\gamma}\left(\mathbf{k}\right) \int d^3 j \left\langle u_\beta\left(\mathbf{j}, t\right) u_\gamma\left(\mathbf{k}-\mathbf{j}, t\right) u_\sigma\left(-\mathbf{k}, t'\right)\right\rangle$$

$$\begin{aligned} Q_{\alpha\sigma}\left(\mathbf{k}; t, t'\right) \quad = \quad & H^{(0)}_{\alpha\varepsilon}\left(\mathbf{k}; t, s\right) Q_{\epsilon\sigma}\left(\mathbf{k}; s, t'\right) + \\ & + \left[\lambda \int_s^t dt'' H^{(0)}_{\alpha\epsilon}\left(\mathbf{k}; t, t''\right) M_{\epsilon\beta\gamma}\left(\mathbf{k}\right) \times \right. \\ & \left. \times \int d^3 j \left\langle u_\beta\left(\mathbf{j}, t''\right) u_\gamma\left(\mathbf{k}-\mathbf{j}, t''\right) u_\sigma\left(-\mathbf{k}, t'\right)\right\rangle \right] \end{aligned}$$

From definition of $\langle u_\alpha(\mathbf{k}, t) u_\beta(\mathbf{k}', t')\rangle = Q_{\alpha\beta}\left(\mathbf{k}'; t, t'\right) \delta(\mathbf{k}+\mathbf{k}')$

$$Q_{\alpha\sigma}\left(\mathbf{k}; t, t'\right) = Q^{(0)}_{\alpha\sigma}\left(\mathbf{k}; t, t'\right) + \lambda^2 Q^{(2)}_{\alpha\sigma}\left(\mathbf{k}; t, t'\right) \cdots$$

$$Q^{(0)}_{\alpha\sigma}\left(\mathbf{k};t,t'\right)=\theta\left(t-s\right)H^{(0)}_{\alpha\epsilon}\left(\mathbf{k};t,s\right)Q^{(0)}_{\epsilon\sigma}\left(\mathbf{k};s,t'\right)$$

$$\begin{aligned}&Q_{\alpha\sigma}\left(\mathbf{k};t,t'\right)\\&=\left[H^{(0)}_{\alpha\varepsilon}\left(\mathbf{k};t,s\right)+\frac{1}{Q_{\epsilon\sigma}\left(\mathbf{k};s,t'\right)}\lambda\int_s^t dt''H^{(0)}_{\alpha\epsilon}\left(\mathbf{k};t,t''\right)M_{\epsilon\beta\gamma}\left(\mathbf{k}\right)\times\right.\\&\times\left.\int d^3j\left\langle u_\beta\left(\mathbf{j},t''\right)u_\gamma\left(\mathbf{k}-\mathbf{j},t''\right)u_\sigma\left(-\mathbf{k},t'\right)\right\rangle\right]Q_{\epsilon\sigma}\left(\mathbf{k};s,t'\right)\end{aligned}$$

$$Q_{\alpha\sigma}\left(\mathbf{k};t,t'\right)=\theta\left(t-s\right)H_{\alpha\epsilon}\left(\mathbf{k};t,s\right)Q_{\epsilon\sigma}\left(\mathbf{k};s,t'\right)$$

LET Ansatz: FDR out of equilibrium

LET Propagator ansatz

$$\theta(t-t')\,Q(k;t,t') = \theta(t-t')\,\theta(t-s)\,H(k;t,s)\,Q(k;s,t')$$

Fluctuation Dissipation Relation

$$\theta(t-t')\,Q(k;t,t') = \theta(t-t')\,H(k;t,t')\,Q(k;t',t')$$

Important: *H* is a functional of *Q* (and others)

LET equations

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)Q\left(k;t,t'\right)$$

2-time

$$= \int d^3j L\left(\mathbf{k},\mathbf{j}\right)\left\{\int_0^{t'} ds H\left(k;t',s\right)Q\left(j;t,s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right.$$

$$\left.-\int_0^{t} ds H\left(j;t,s\right)Q\left(k;s,t'\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right\}$$

1-time

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right)$$

$$= 2\int d^3j L\left(\mathbf{k},\mathbf{j}\right)\int_0^{t} ds H\left(k;t,s\right)H\left(j;t,s\right)H\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\times$$

$$\left[Q\left(j;s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)-Q\left(k;s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)\right]$$

LET response

$$\theta(t-t')\left(\frac{\partial}{\partial t}+\nu k^2\right)\theta(t-t')\,H(k;t,t')-\theta(t-t')\,H(k;t,t')\,\delta(t-t')$$

$$+\int d^3jL(\mathbf{k},\mathbf{j})\,\theta(t-t')\int_{t'}^{t}dsH(j;t,s)\,H(k;s,t')\,\theta(t-s)\,Q(|\mathbf{k}-\mathbf{j}|;t,s)$$

$$=\int d^3jL(\mathbf{k},\mathbf{j})\,\theta(t-t')\int_0^{t'}ds\frac{\theta(t-s)\,Q(|\mathbf{k}-\mathbf{j}|;t,s)}{Q(k;t',t')}\times$$

$$\times\{H(k;t',s)\,\theta(t-s)\,Q(j;t,s)-H(j;t,s)\,\theta(t'-s)\,Q(k;t',s)\}$$

DIA response

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)G(k;t,t')$$

$$+\int d^3jL(\mathbf{k},\mathbf{j})\int_{t'}^{t}dsG(j;t,s)\,G(k;s,t')\,Q(|\mathbf{k}-\mathbf{j}|;t,s)$$

$$=\delta(t-t')$$

Properties of the LET response/propagator

Response function transitivity

$$H\left(k;t,t'\right) = H\left(k;t,s\right) H\left(k;s,t'\right)$$

Linking of single-time correlators

$$Q\left(k;t\right) = \theta\left(t-s\right) \widetilde{H}\left(k;t,s\right) Q\left(k;s\right)$$

$$\widetilde{H}\left(k;t,s\right) := H\left(k;t,s\right) H\left(k;t,s\right)$$

i.e. propagates from time ‘s’ to time ‘t’

Behaviour of LET response in the limit of infinite Reynolds number

Compatibility with K41 dimensional results for energy spectra and eddy damping rate in the inertial range.

$$Q(k) = \frac{\alpha \varepsilon^{2/3}}{4\pi} k^{-11/3}, \qquad \omega(k) = \beta \varepsilon^{1/3} k^{2/3}$$

$$\omega(k) = \nu k^2 + \left\{ \int dj \int d\mu \frac{k j^3 (\mu^2 - 1)[\mu(k^2 + j^2) - kj(1 + 2\mu^2)]}{k^2 + j^2 - 2kj\mu} \times \right.$$
$$\left. \times \frac{\alpha \beta^{-1} \varepsilon^{1/3} |\mathbf{k} - \mathbf{j}|^{-11/3} \left[j^{-11/3} - k^{-11/3} \right]}{k^{-11/3} \left[k^{2/3} + j^{2/3} + |\mathbf{k} - \mathbf{j}|^{2/3} \right]} \right\}$$

LET is only 2-t Eulerian closure that is K41 compatible in this sense.

Single-time LET

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) Q\left(k;t\right)$$
$$= 2\int d^3 j L\left(\mathbf{k},\mathbf{j}\right) \int_0^t ds \mathcal{H}\left(k;t,s\right) \mathcal{H}\left(j;t,s\right) \mathcal{H}\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right) \times$$
$$\left[Q\left(j;s\right) Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right) - Q\left(k;s\right) Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)\right]$$

$$\theta(t-t')\left[\frac{\partial}{\partial t} + \nu k^2 + \eta(k;t,t')\right] \mathcal{H}\left(k;t,t'\right) = 0$$

$$\eta(k;t,t')$$
$$= \theta\left(t-t'\right) \int d^3 j L\left(\mathbf{k},\mathbf{j}\right) \int_{t'}^t ds \left\{\mathcal{H}\left(k;s,t'\right) \times\right.$$
$$\left. \times \mathcal{H}\left(j;t,s\right) \mathcal{H}\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right) Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)\right\}$$
$$-\theta\left(t-t'\right) \int d^3 j L\left(\mathbf{k},\mathbf{j}\right) \int_0^{t'} ds \left\{ \mathcal{H}\left(k;t',s\right) \mathcal{H}\left(j;t,s\right) \mathcal{H}\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right) \times\right.$$
$$\left. \times \frac{Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)}{Q\left(k;t'\right)} \left[Q\left(j;s\right) - Q\left(k;s\right)\right]\right\}$$

Langevin equation
form for propagator

Markovian assumption

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right) \quad = \quad 2\int d^3 j L\left(\mathbf{k},\mathbf{j}\right)D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)\times$$
$$\times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)\left[Q\left(j;t\right)-Q\left(k;t\right)\right]$$

Memory time

$$D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)=\int_0^t ds\mathcal{H}\left(k;t,s\right)\mathcal{H}\left(j;t,s\right)\mathcal{H}\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)$$

Using Langevin propagator form

$$\frac{\partial}{\partial t}D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t) \quad = \quad 1-\left[\left(\nu k^2+\nu j^2+\nu\left|\mathbf{k}-\mathbf{j}\right|^2\right)+\eta(k;t)+\eta(j;t)\right.$$
$$\left.+\eta\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)\right]D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)$$

Markovianized single-time LET eqns

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q(k;t) = 2\int d^3j L(\mathbf{k},\mathbf{j})\, D(k,j,|\mathbf{k}-\mathbf{j}|;t)\times \times Q(|\mathbf{k}-\mathbf{j}|;t)\left[Q(j;t)-Q(k;t)\right] = -2\eta(k;t)Q(k;t)$$

$$\eta(k;t) = -\int d^3j L(\mathbf{k},\mathbf{j})\, D(k,j,|\mathbf{k}-\mathbf{j}|;t)\frac{Q(|\mathbf{k}-\mathbf{j}|;t)}{Q(k;t)}\left[Q(j;t)-Q(k;t)\right]$$

$$\frac{\partial}{\partial t}D(k,j,|\mathbf{k}-\mathbf{j}|;t) = 1-\left[\left(\nu k^2+\nu j^2+\nu|\mathbf{k}-\mathbf{j}|^2\right)+\eta(k;t)+\eta(j;t)+\eta(|\mathbf{k}-\mathbf{j}|;t)\right]D(k,j,|\mathbf{k}-\mathbf{j}|;t)$$

No adjustable parameters for memory time function

Eddy Damped Quasi-normal Markovian

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) Q\left(k;t\right)$$
$$= 2\int d^3 j L\left(\mathbf{k},\mathbf{j}\right)\theta\left(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t\right) \times$$
$$\times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)\left[Q\left(j;t\right) - Q\left(k;t\right)\right]$$

$$\theta\left(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t\right) = \frac{1-\exp\left\{-\left[\omega(k)+\omega(j)+\omega\left(\left|\mathbf{k}-\mathbf{j}\right|\right)\right]t\right\}}{\omega(k)+\omega(j)+\omega\left(\left|\mathbf{k}-\mathbf{j}\right|\right)},$$

$$\omega(k) = \nu k^2 + \beta\epsilon^{\frac{1}{3}} k^{\frac{2}{3}}$$

Test Field Model

Single-time LET equations +

$$\eta(k;t) = \int d^3 j L(\mathbf{k},\mathbf{j}) D(k,j,|\mathbf{k}-\mathbf{j}|;t) Q(|\mathbf{k}-\mathbf{j}|;t)$$

$$\eta^S(k;t) = \pi g^2 \int d^3 j L(\mathbf{k},\mathbf{j}) D^G(k,j,|\mathbf{k}-\mathbf{j}|;t) Q(|\mathbf{k}-\mathbf{j}|;t)$$

$$\eta^C(k;t) = 2\pi g^2 \int d^3 j L(\mathbf{k},\mathbf{j}) D^G(k,j,|\mathbf{k}-\mathbf{j}|;t) Q(|\mathbf{k}-\mathbf{j}|;t)$$

+ eqn for D^G + 2 eqns for G^S and G^C

Summary and Conclusions

- All of this work is in the following papers:

  W. D. McComb and K. Kiyani, Phys. Rev. E **72**, 016309 (2005)

  K. Kiyani and W. D. McComb, Phys. Rev. E **70**, 066303 (2004)

- Numerical results for LET:

  W. D. McComb and A. P. Quinn, Physica A, **317**: 487-508 (2003)

- Need to compute the single time Markovian theory and compare to TFM, EDQNM and DIA, LET etc
- Further analysis needed to see if the closure is realizable in the presence of waves (Bowman, Krommes Phys. Plasma 1997) -- application to real flows -- MHD, geophysical flows etc.

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