Survey and numerical simulation of Lagrangian stability of ideal fluid motion

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Aim

To review

***Lagrangian stability theory due to Arnold**

(including Rouchon's basic result)

To present

***Preliminary results on sectional curvatures by numerics**

1. Introduction

Differential geometric approach
Linear stability
classical mechanics of particles

$$H(\boldsymbol{p},\boldsymbol{x}) = \frac{1}{2}\alpha^{ij}p_i p_j + V(\boldsymbol{x}) = E \ \ \textbf{(const.)}$$

$$\delta \int ds = 0$$

$$ds^2 = g_{ij}dx^i dx^j, \ g_{ij} = (E - V(\boldsymbol{x}))\alpha_{ij}$$

geodesic

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

Jacobi field

$$w_i = \frac{\partial x_i}{\partial \sigma}, \ \ \sigma = \ \textbf{perturbation}$$

geodesic deviation

$$\frac{d^2 w^i}{dt^2} = -R^i_{jkl} \frac{dx^j}{dt} w^k \frac{dx^l}{dt}$$

$$\frac{d^2}{dt^2} \frac{w_i w^i}{2} = \frac{dw_i}{dt} \frac{dw^i}{dt} + w_i \frac{d^2 w^i}{dt^2}$$

$$= \frac{dw_i}{dt} \frac{dw^i}{dt} - R_{ijlk} w^i \frac{dx^j}{dt} w^l \frac{dx^k}{dt}$$

the 2nd term $> 0 \rightarrow |w|$ **exponential growth**

Preliminary observation: vorticity $\sim$ Jacobi field

By

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}, \quad \frac{D^2\boldsymbol{\omega}}{Dt^2} = -\boldsymbol{P} \cdot \boldsymbol{\omega}$$

we have

$$\frac{D^2}{Dt^2}\frac{|\boldsymbol{\omega}|^2}{2} = \left(\frac{D\boldsymbol{\omega}}{Dt}\right)^2 - \boldsymbol{\omega} \cdot \boldsymbol{P} \cdot \boldsymbol{\omega}$$

$$= (\boldsymbol{\omega} \cdot \boldsymbol{S})(\boldsymbol{S} \cdot \boldsymbol{\omega}) - \boldsymbol{\omega} \cdot \boldsymbol{P} \cdot \boldsymbol{\omega}$$

$$\boldsymbol{P} = \nabla\nabla p$$

Pressure hessian $\sim$ sectional curvature

2. Arnold's theory a lá Rouchon

V. Arnold,
Sur la geometrie differentielle des groupes de Lie
de dimension infinie et ses applications a l'hydrodynamique
des fluides parfaits.
Annales de l'institut Fourier, 16 no. 1 (1966), 319--361.
available at http://www.numdam.org

P Rouchon,
Jacobi equation, Riemannian curvature and the motion of a perfect
incompressible fluid.
Eur. J. Mech. B/Fluids, 11(1992)317--336.
http://cas.ensmp.fr/~rouchon/publications/PR1992/GEODESIC.pdf

P Rouchon,
Dynamique des Fluides Parfaits
Principe de Moindre Action Stabilite Lagranienne
http://cas.ensmp.fr/~rouchon/publications/PR1991/onera1.pdf
compressible fluid

Arnold-Khesin
‘‘Topological methods in hydrodynamics’’,
Springer(1998)

Arnold
‘‘Mathematical Methods of Classical Mechanics’’
Springer(1978)

Eq. for Jacobi field: derivation

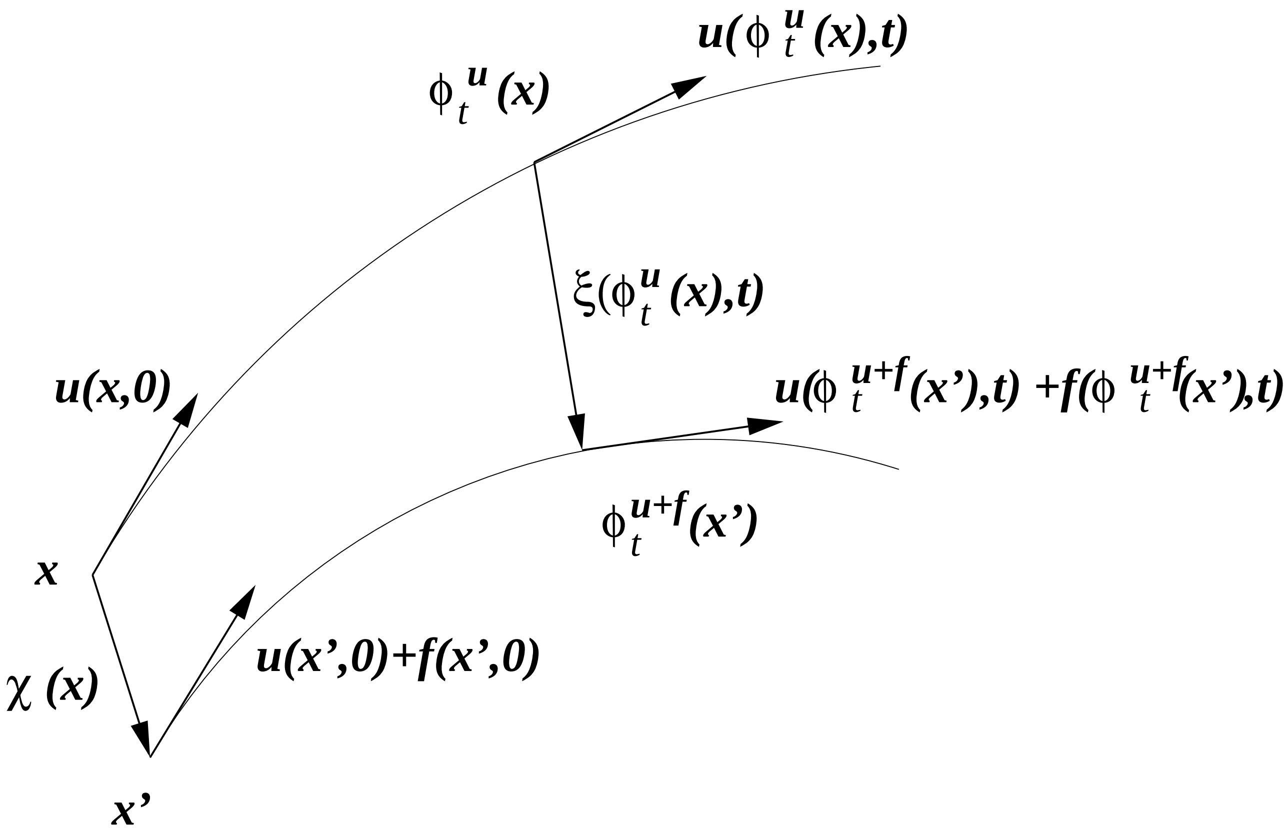
$x(\boldsymbol{a}, t; s)$, s **characterises perturbation**

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p, \\ \dfrac{D\boldsymbol{x}(\boldsymbol{a}, t)}{Dt} = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a}, t), t) \end{cases}$$

variation $\boldsymbol{f} \equiv \dfrac{\partial}{\partial s}\boldsymbol{u}\Big|_{s=0} = \delta \boldsymbol{u}$

variation $\boldsymbol{\xi} \equiv \dfrac{\partial}{\partial s}\boldsymbol{x}(t, s)\Big|_{s=0}$

$$\frac{D}{Dt} \underbrace{\frac{\partial x_i}{\partial s}}_{=\xi_i} = \underbrace{\frac{\partial u_i}{\partial s}}_{=f_i} + \frac{\partial u_i}{\partial x_j} \underbrace{\frac{\partial x_j}{\partial s}}_{=\xi_j}$$



$$\frac{d\phi}{dt} = u(\phi, t),\ \phi(0) = x, \quad \phi = \phi_t^{u}(x)$$

Jacobi eqs.

$$\begin{cases} \dfrac{D\boldsymbol{\xi}}{Dt} = (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{u} + \boldsymbol{f}, \;\; \nabla\cdot\boldsymbol{\xi} = 0 \\ \dfrac{D\boldsymbol{f}}{Dt} = -(\boldsymbol{f}\cdot\nabla)\boldsymbol{u} - \nabla q, \;\; \nabla\cdot\boldsymbol{f} = 0 \end{cases}$$

Rouchon (1991)

Covariant derivative (Moreau)

$$\frac{\delta\boldsymbol{\xi}}{\delta t} = \frac{D\boldsymbol{\xi}}{Dt} + \nabla\alpha_{\boldsymbol{\xi}} \text{ such that } \quad \nabla\cdot\boldsymbol{\xi} = 0$$

$$\nabla\cdot((\boldsymbol{\xi}\cdot\nabla)\boldsymbol{u}) + \triangle\alpha_{\boldsymbol{\xi}} = 0$$

$$\frac{\delta^2\boldsymbol{\xi}}{\delta t^2} = -\boldsymbol{P}\cdot\boldsymbol{\xi} + (\boldsymbol{u}\cdot\nabla)\nabla\alpha_{\boldsymbol{\xi}} - \nabla\gamma$$

$$\frac{D\boldsymbol{\xi}}{Dt} = (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{u} + \boldsymbol{f}, \quad \frac{D\boldsymbol{f}}{Dt} = -(\boldsymbol{f}\cdot\nabla)\boldsymbol{u} - \nabla q$$

$$\frac{D^2\boldsymbol{\xi}}{Dt^2} = -\boldsymbol{P}\cdot\boldsymbol{\xi} - \nabla q \tag{1}$$

Auxiliary formula

$$\left[\frac{D}{Dt}, \boldsymbol{\xi}\cdot\nabla\right] = \boldsymbol{f}\cdot\nabla \text{ or } \frac{D}{Dt}(\boldsymbol{\xi}\cdot\nabla) = \boldsymbol{f}\cdot\nabla$$

Derivation of (1)

$$\frac{D^2\boldsymbol{\xi}}{Dt^2} = (\boldsymbol{\xi}\cdot\nabla)\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{f}\cdot\nabla\boldsymbol{u} + \frac{D\boldsymbol{f}}{Dt}$$

$$= (\boldsymbol{\xi}\cdot\nabla)(-\nabla p) + (\boldsymbol{f}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{f}\cdot\nabla)\boldsymbol{u} - \nabla q$$

$$= -\boldsymbol{P}\cdot\boldsymbol{\xi} - \nabla q$$

Derivation

$$\begin{aligned}
\frac{\delta^2 \boldsymbol{\xi}}{\delta t^2} &= \frac{\delta}{\delta t}\frac{\delta \boldsymbol{\xi}}{\delta t} \\
&= \frac{D}{Dt}\frac{\delta \boldsymbol{\xi}}{\delta t} + \nabla \beta \\
&= \frac{D^2 \boldsymbol{\xi}}{Dt^2} + \frac{D}{Dt}\nabla \alpha_{\boldsymbol{\xi}} + \nabla \beta \\
&= -\boldsymbol{P} \cdot \boldsymbol{\xi} - \nabla q + \frac{\partial}{\partial t}\nabla \alpha_{\boldsymbol{\xi}} + (\boldsymbol{u} \cdot \nabla)\nabla \alpha_{\boldsymbol{\xi}} + \nabla \beta \\
&= -\boldsymbol{P} \cdot \boldsymbol{\xi} + (\boldsymbol{u} \cdot \nabla)\nabla \alpha_{\boldsymbol{\xi}} - \nabla \gamma,
\end{aligned}$$

where

$$\gamma = q - \beta - \frac{\partial}{\partial t}\nabla \alpha_{\boldsymbol{\xi}}$$

$$\frac{\delta^2 \boldsymbol{\xi}}{\delta t^2} + \boldsymbol{A}\boldsymbol{u}(\boldsymbol{\xi}) = 0.$$

$$\boldsymbol{A}\boldsymbol{u}(\boldsymbol{\xi}) \equiv \boldsymbol{P} \cdot \boldsymbol{\xi} - (\boldsymbol{u} \cdot \nabla)\nabla \alpha_{\boldsymbol{\xi}} + \nabla \gamma$$

$$\left\langle \frac{\delta \boldsymbol{\xi}_1}{\delta t} \cdot \nabla \alpha_{\boldsymbol{\xi}_2} \right\rangle = 0$$

leads to symmetry

$$\langle \boldsymbol{\xi}_1, \boldsymbol{A}\boldsymbol{u}(\boldsymbol{\xi}_2) \rangle = \langle \boldsymbol{\xi}_2, \boldsymbol{A}\boldsymbol{u}(\boldsymbol{\xi}_1) \rangle$$

$$= \int \left(\boldsymbol{\xi}_1 \cdot \boldsymbol{P} \cdot \boldsymbol{\xi}_2 - \nabla \alpha_{\boldsymbol{\xi}_1} \cdot \nabla \alpha_{\boldsymbol{\xi}_2} \right) \boldsymbol{dx}$$

$\boldsymbol{\xi}$ $(\parallel u)$ **will not contribute**

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{\perp} + \boldsymbol{\xi}^{\parallel},\ \boldsymbol{\xi}^{\parallel} = \frac{\langle \boldsymbol{\xi}, \boldsymbol{u} \rangle}{\|\boldsymbol{u}\|^2}\boldsymbol{u}$$

$$\begin{cases} \dfrac{\delta^2 \boldsymbol{\xi}^{\parallel}}{\delta t^2} = 0, \quad \because \alpha \boldsymbol{u} = p,\ \gamma = 0, \\ \dfrac{\delta^2 \boldsymbol{\xi}^{\perp}}{\delta t^2} + \boldsymbol{A}_{\boldsymbol{u}}(\boldsymbol{\xi}^{\perp}) = 0 \end{cases}$$

$$U_{\boldsymbol{u}}(\boldsymbol{\xi}^{\perp}) = \frac{1}{2}\left\langle \boldsymbol{A}_{\boldsymbol{u}}(\boldsymbol{\xi}^{\perp}), \boldsymbol{\xi}^{\perp} \right\rangle$$

$$\frac{\delta^2 \boldsymbol{\xi}^{\perp}}{\delta t^2} = -\frac{\delta U_{\boldsymbol{u}}(\boldsymbol{\xi}^{\perp})}{\delta \boldsymbol{\xi}^{\perp}},\ \ K(\boldsymbol{u}, \boldsymbol{\xi}) = \frac{2U_{\boldsymbol{u}}(\boldsymbol{\xi}^{\perp})}{\|\boldsymbol{u}\|^2 \|\boldsymbol{\xi}^{\perp}\|^2}$$

Rouchon's basic result

$$\boldsymbol{M} \equiv \boldsymbol{P} - (\nabla \boldsymbol{u})^T \nabla \boldsymbol{u},$$

$$\lambda_{\text{min}} = \textbf{minimum eigenvalue of } M$$

$[M] =$ **time**$^{-2}$**, time scale of particle dispersion**

$$2U_{\boldsymbol{u}}(\boldsymbol{\xi}^{\perp}) \underbrace{\geq}_{*} \int \boldsymbol{\xi}^{\perp} \cdot \boldsymbol{M} \cdot \boldsymbol{\xi}^{\perp} \boldsymbol{dx}$$

$$\frac{1}{3} \min_{\boldsymbol{x}} \text{tr}(\boldsymbol{M}(\boldsymbol{x},t)) \geq \min_{\|\boldsymbol{\xi}\|=1} 2U_{\boldsymbol{u}}(\boldsymbol{\xi}) \geq \min_{\boldsymbol{x}} \lambda_{\text{min}}(\boldsymbol{x},t)$$

$$\text{tr}(\boldsymbol{M}(\boldsymbol{x},t)) = \triangle p - (\partial_i u_j)(\partial_i u_j) = -\boldsymbol{S} : \boldsymbol{S} \leq 0$$

Unless $S : S \equiv 0$**, we have negative sectional curvature**

$$\mathrm{tr}(\boldsymbol{M}(\boldsymbol{x},t)) = \triangle p - (\partial_i u_j)(\partial_i u_j) = -\boldsymbol{S} : \boldsymbol{S} \leq 0$$

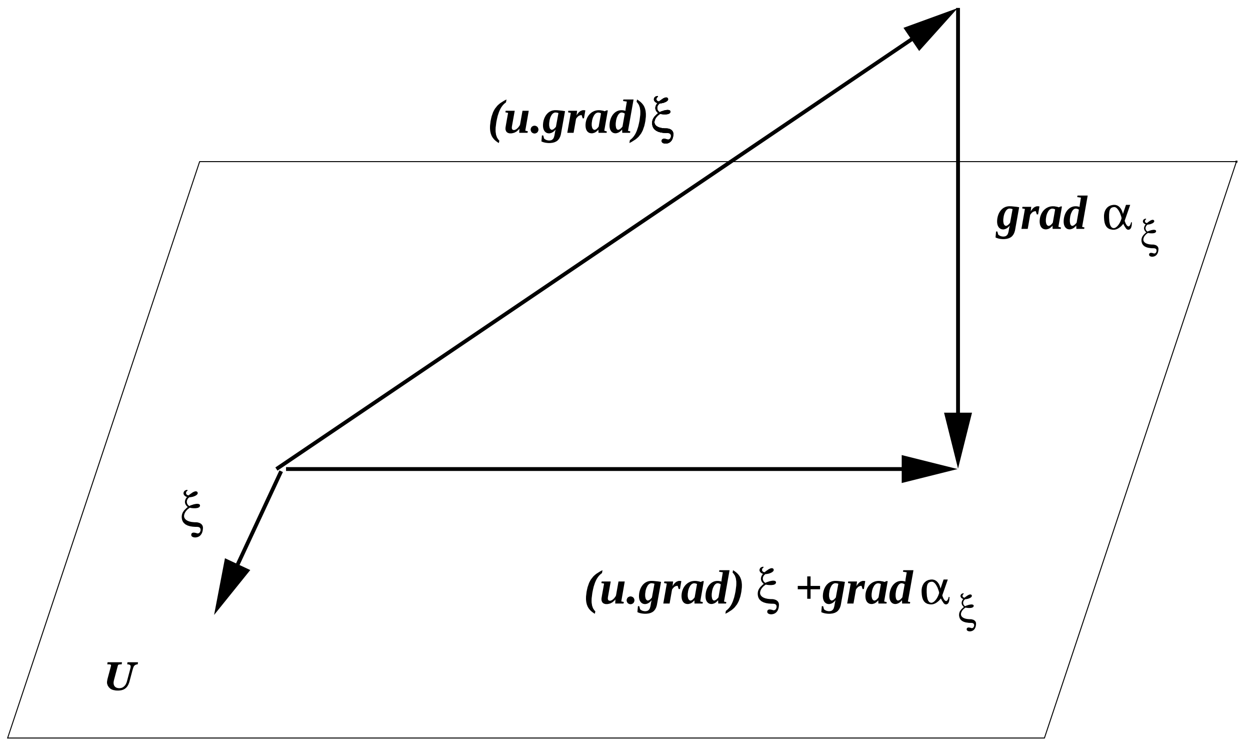
$$\because \begin{cases} \triangle p = -(\partial_i u_j)(\partial_j u_i) = |\boldsymbol{\omega}|^2 - \frac{1}{2}\boldsymbol{S} : \boldsymbol{S}, \\ (\partial_i u_j)(\partial_i u_j) = |\boldsymbol{\omega}|^2 + \frac{1}{2}\boldsymbol{S} : \boldsymbol{S} \end{cases}$$

Proof of (*) $2U\boldsymbol{u}(\boldsymbol{\xi}^{\perp}) = \int \left(\boldsymbol{\xi}^{\perp} \cdot \boldsymbol{P} \cdot \boldsymbol{\xi}^{\perp} - |\nabla \alpha_{\boldsymbol{\xi}^{\perp}}|^2\right) \boldsymbol{dx}$

$$\geq \int \left(\boldsymbol{\xi}^{\perp} \cdot \boldsymbol{P} \cdot \boldsymbol{\xi}^{\perp} - |(\boldsymbol{\xi}^{\perp} \cdot \nabla)\boldsymbol{u}|^2\right) \boldsymbol{dx}$$

$$= \int \boldsymbol{\xi}^{\perp} \cdot \boldsymbol{M} \cdot \boldsymbol{\xi}^{\perp} \quad (\because \; M_{ij} \equiv P_{ij} - (\partial_i u_k)(\partial_j u_k) \;)$$

$$\because \; \|\nabla \alpha_{\boldsymbol{\xi}^{\perp}}\|^2 \leq \|(\boldsymbol{\xi}^{\perp} \cdot \nabla)\boldsymbol{u}\|^2$$



3. Numerical results
No previous works
ABC flow

$$\boldsymbol{u} = \begin{pmatrix} a\sin z + c\cos y \\ b\sin x + a\cos z \\ c\sin y + b\cos x \end{pmatrix}, \ \boldsymbol{\xi} = \begin{pmatrix} A\sin z + C\cos y \\ B\sin x + A\cos z \\ C\sin y + B\cos x \end{pmatrix}$$

$$\nabla \cdot ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) + \triangle p = 0,$$

$$\nabla \cdot ((\boldsymbol{\xi}\cdot\nabla)\boldsymbol{u}) + \triangle \alpha_{\boldsymbol{\xi}} = 0$$

$$2U_{\boldsymbol{u}}(\boldsymbol{\xi}) = \int \left(\boldsymbol{\xi}\cdot\boldsymbol{P}\cdot\boldsymbol{\xi} - |\nabla\alpha_{\boldsymbol{\xi}}|^2\right)\boldsymbol{dx}$$

$$= -\frac{1}{8}\left((aB-bA)^2 + (bC-cB)^2 + (cA-aC)^2\right) \le 0$$

Nakamura *et al.* **(1992): Fourier series**

Taylor-Green vortex

$$\boldsymbol{u} = \begin{pmatrix} A \cos x \sin y \sin z \\ B \sin x \cos y \sin z \\ C \sin x \sin y \cos z \end{pmatrix}, A + B + C = 0$$

Case.1 velocity

$$\boldsymbol{\xi} = \begin{pmatrix} a \cos x \sin y \sin z \\ b \sin x \cos y \sin z \\ c \sin x \sin y \cos z \end{pmatrix}, a + b + c = 0$$

$$2U\boldsymbol{u}(\boldsymbol{\xi}) = 0 \ \textbf{(new)}$$

Case.2 vorticity

$$\boldsymbol{\xi} = \begin{pmatrix} (c-b)\sin x \cos y \cos z \\ (a-c)\cos x \sin y \cos z \\ (b-a)\cos x \cos y \sin z \end{pmatrix}, a+b+c=0$$

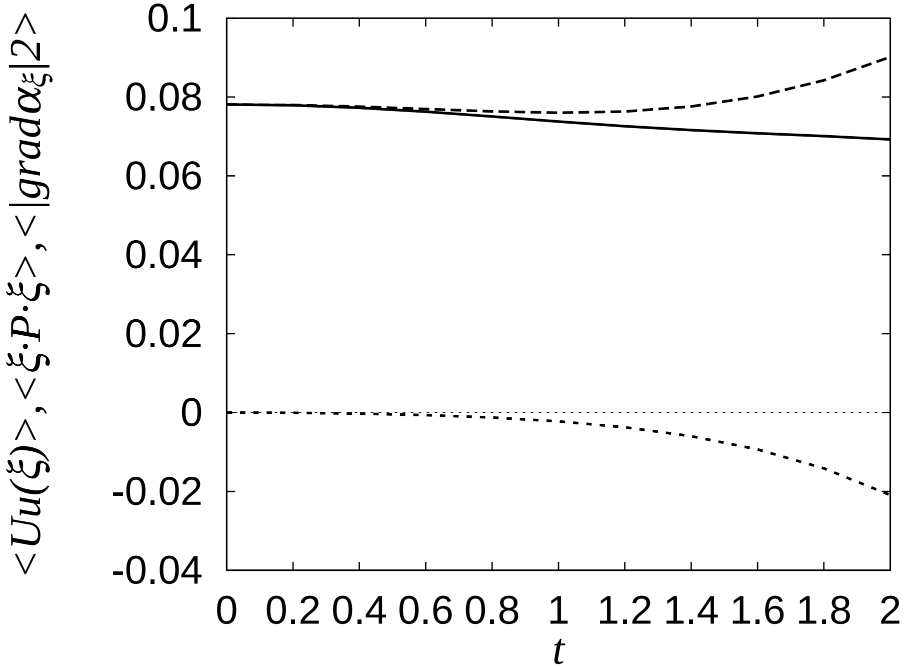
$$2U\boldsymbol{u}(\boldsymbol{\xi}) = -\frac{3}{128}\Big(a^2B^2 + b^2A^2 + 2(bA-aB)^2$$

$$+(A+B)^2(a^2+b^2) + (a+b)^2(A^2+B^2)\Big) \leq 0 \ \textbf{(new)}$$

Case.1 Numerics $A = -B = 1, C = 0$

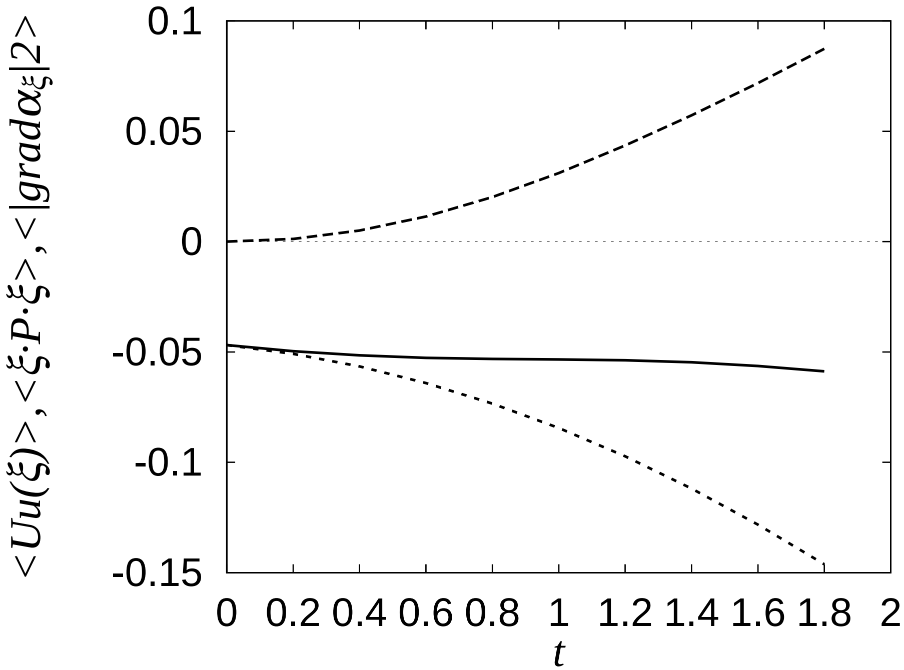
$\left\langle |\nabla \alpha_{\boldsymbol{\xi}}|^2 \right\rangle$, $\langle \boldsymbol{\xi} \cdot \boldsymbol{P} \cdot \boldsymbol{\xi} \rangle$, $\langle U\boldsymbol{u}(\boldsymbol{\xi}) \rangle$

sectional curvature becomes negative (new)

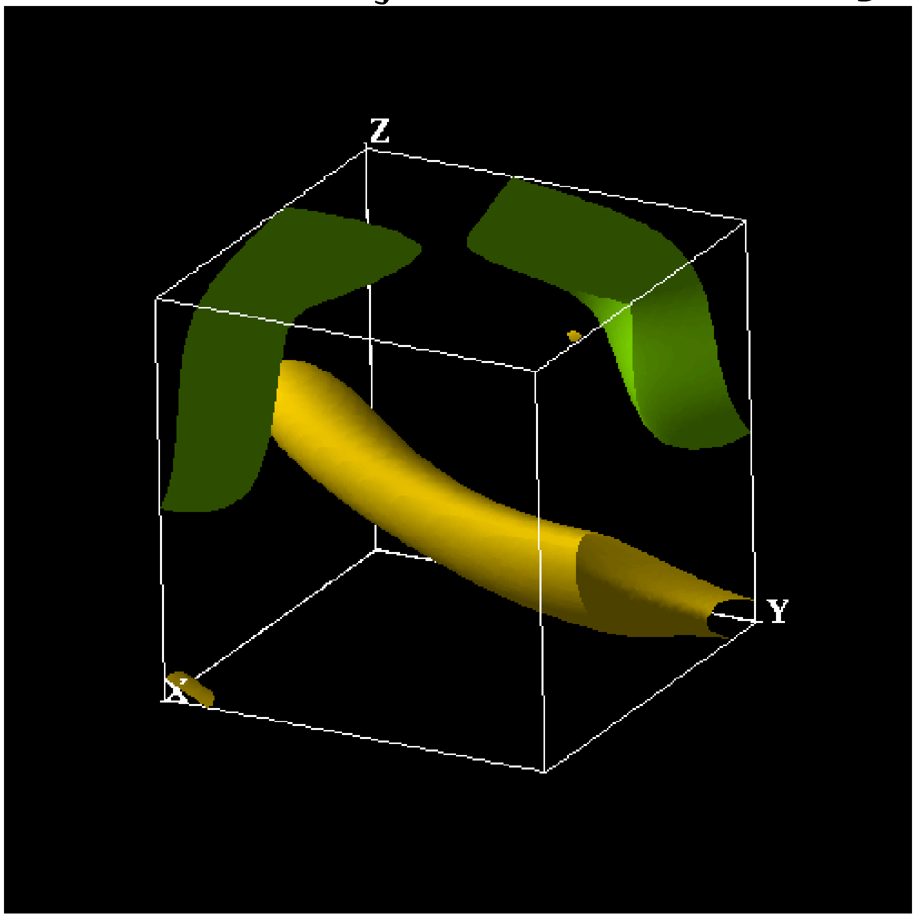


Case.2 Numerics $A = -B = 1, C = 0$

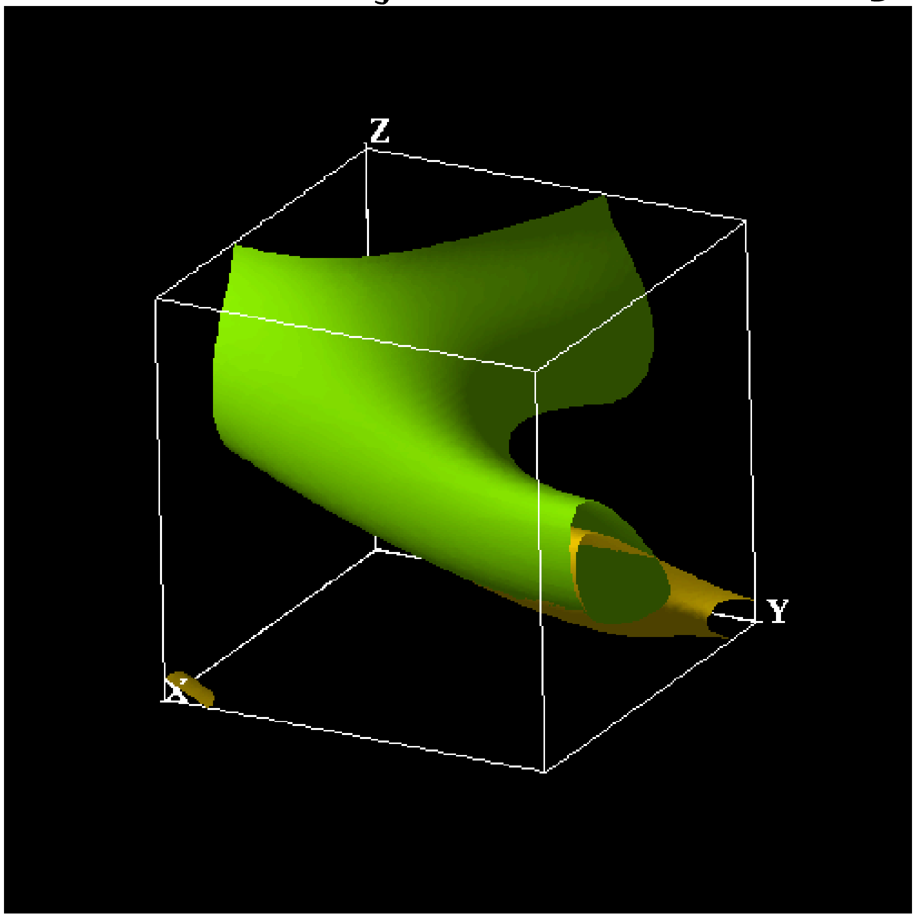
$\left\langle |\nabla \alpha_{\boldsymbol{\xi}}|^2 \right\rangle$, $\langle \boldsymbol{\xi} \cdot \boldsymbol{P} \cdot \boldsymbol{\xi} \rangle$, $\langle U\boldsymbol{u}(\boldsymbol{\xi}) \rangle$



Jacobi field ξ, Case 1 velocity



Jacobi field ξ, Case 2 vorticity



Preston (2004)

Jacobi field does not always grow exponentially in time even though sectional curvatures are negative

Example: plane parallel Couette flow, Orr (1907)

Negative sectional curvature arises for such a peculiar case, which is stable but has only continuous spectrum

Case (1960)

$$\boldsymbol{U} = (U(y), 0)$$

$$\boldsymbol{u} = (u(x,y,t), v(x,y,t))$$

$$v_k(y,t) = \int_{-\infty}^{\infty} v(x,y,t) e^{-ikx} dx$$

$$v_p(y) = \int_0^{\infty} e^{-pt} v_k(y,t) dt$$

$$(p + ikU) \underbrace{\left(\frac{d^2}{dy^2} - k^2 - \frac{ikU''}{p + ikU} \right)}_{=L_p} v_p = \left(\frac{d^2}{dy^2} - k^2 \right) v_k(y,0)$$

$$U(y) = y$$

normal mode equation

$$(p + ikU)L_p v_p = 0$$

(a)discrete spectrum

$L_p v_p = 0, \;\; v_p(0) = v_p(1) = 0$ **empty**

(b) continuous spectrum

$$p_{y_0} + ikU(y_0) = 0$$

$$L_{p_{y_0}} v_p = \delta(y - y_0)$$

$|v| = O(t^{-1})$ **Orr(1907), Case(1960)**

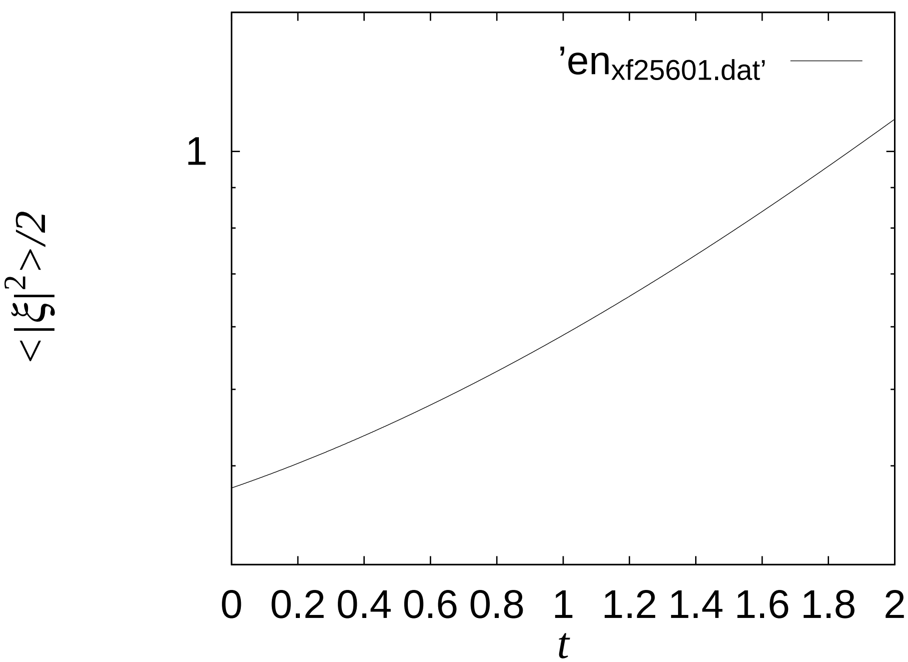
Jabobi field $= O(t)$ **Preston (2004)**

Relationship between Eulerian and Lagrangian instabilities **u=stationary solution 2D Euler eq. with no stagnation points**

$$\|\boldsymbol{\xi}\|(t) \leq \sqrt{3+2At^2}\frac{\sup|\boldsymbol{u}|}{\inf|\boldsymbol{u}|}\int_0^t \|\boldsymbol{f}\|(t')dt'$$

Preston (2004)

exponential growth (rather than algebraic)



5. Summary and outlook

- **Review of Arnold's theory**
- **Numerical evaluation of sectional curvature**
- **Remark on Couette flow**

Application to dispersion of fluid particle

- **Structure of Jacobi fields**
- **Eigenvalue problem of M**

Extention to Viscous case (Rouchon 1992)

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u}, \\ \dfrac{D\boldsymbol{x}(\boldsymbol{a},t)}{Dt} = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a},t),t) \end{cases}$$

$$\begin{cases} \dfrac{D\boldsymbol{\xi}}{Dt} = (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{u} + \boldsymbol{f}, \;\; \nabla \cdot \boldsymbol{\xi} = 0 \\ \dfrac{D\boldsymbol{f}}{Dt} = -(\boldsymbol{f} \cdot \nabla)\boldsymbol{u} + \nu \triangle \boldsymbol{f} - \nabla q, \;\; \nabla \cdot \boldsymbol{f} = 0 \end{cases}$$

$$\boldsymbol{A}_{\boldsymbol{v},\nu}(\boldsymbol{\xi}) = \boldsymbol{P}\cdot\boldsymbol{\xi} - (\boldsymbol{u}\cdot\nabla)\nabla\alpha_{\boldsymbol{\xi}} - \nu(\boldsymbol{\xi}\cdot\nabla)\triangle\boldsymbol{u} + \nu\triangle\left[(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{u}\right] + \nabla\gamma$$

$$\frac{\delta^2\boldsymbol{\xi}}{\delta t^2} = -\boldsymbol{A}_{\boldsymbol{v},\nu}(\boldsymbol{\xi}) + \nu\triangle\left(\frac{\delta\boldsymbol{\xi}}{\delta t}\right)$$

In

$$\frac{1}{2}\frac{\delta}{\delta t}\int\left(\frac{\delta\boldsymbol{\xi}}{\delta t}\right)^2 \boldsymbol{dx}$$

contribution from the last term is always negative

$$\nu\int\left(\frac{\delta\boldsymbol{\xi}}{\delta t}\right)\cdot\triangle\left(\frac{\delta\boldsymbol{\xi}}{\delta t}\right)\boldsymbol{dx} = -\nu\int\left|\nabla\left(\frac{\delta\boldsymbol{\xi}}{\delta t}\right)\right|^2\boldsymbol{dx} \leq 0$$

F Nakamura, Y Hattori, T Kambe,
Geodesics and curvature of a group of diffeomorphisms
and motion of an
ideal fluid
1992 J. Phys. A: Math. Gen. 25 L45-L50

S.C. Preston,
For Ideal Fluids, Eulerian and Lagrangian Instabilities are Equivalen
Geometric and Functional Analysis, 14(2004)1044--1062.

G. Misiolek,
Stability of flows of ideal fluids and the geometry of the group of
diffeomorphisms. Indiana Univ. Math. J. 42 (1993), no. 1, 215--235.

T Kambe,
Geometrical theory of dynamical systems and fluid flows,
World Scientific, 2004, New Jersey,
Advanced Series in Nonlinear Dynamics 23

Classical Mechanics

Curvature statistics of some few-body Debye-Huckel
and Lennard-Jones systems
JFC van Velsen, J. Phys. A: Math. Gen. 13 833-854

The average Riemann curvature of conservative systems
in classical mechanics

JFC van Velsen 1981 J. Phys. A: Math. Gen. 14 1621-1627

On the Riemann curvature of conservative systems
in classical mechanics
JFC van Velsen - Physics Letters A, 1978 67A 325-327

Hamiltonian system
"Hamiltonian description of the ideal fluid"
P. J. Morrison, Rev. Mod. Phys. (1998)
http://prola.aps.org/abstract/RMP/v70/i2/p467_1

♠ **Lemma**

For M, N**=**3×3 **real symmetric matrix,** $\mathrm{tr}(N) = 0$ **There exists a unit vector** w **s.t.**

$$\boldsymbol{w}^T \boldsymbol{M} \boldsymbol{w} \leq \frac{1}{3}\mathrm{tr}(\boldsymbol{M}), \;\; \boldsymbol{w}^T \boldsymbol{N} \boldsymbol{w} = 0$$

Proof N **can be diagonalized**

$$\boldsymbol{N} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

$$\boldsymbol{w} = (s_1, s_2, s_3)^T, \;\; s_i = \pm\frac{1}{\sqrt{3}}$$

$$\boldsymbol{w}^T \boldsymbol{N} \boldsymbol{w} = s_1^2 A + s_2^2 B + s_3^2 C = 0$$

Let

$$M = \begin{pmatrix} a & p & r \\ p & b & q \\ r & q & c \end{pmatrix}$$

in this principal frame,

$$\boldsymbol{w}^T \boldsymbol{M} \boldsymbol{w} = \frac{1}{3}(a+b+c) + 2(ps_1s_2 + qs_2s_3 + rs_3s_1)$$

$$\sum_{\textbf{all } w} \boldsymbol{w}^T \boldsymbol{M} \boldsymbol{w} = \frac{8}{3}(a+b+c)$$

If

$$\forall \boldsymbol{w}, \boldsymbol{w}^T \boldsymbol{M} \boldsymbol{w} > \frac{\mathrm{tr}(\boldsymbol{M})}{3} \Rightarrow \textbf{contradiction}$$

$\therefore$ **for some** w

$$\boldsymbol{w}^T \boldsymbol{M} \boldsymbol{w} \leq \frac{\mathrm{tr}(\boldsymbol{M})}{3}.$$

♠ **proof (**)**

Let us show

$$\min_{\|\boldsymbol{\xi}^{\perp}\|=1} 2U\boldsymbol{u}(\boldsymbol{\xi}^{\perp}) \leq \frac{1}{3}\mathrm{tr}(\boldsymbol{M}(\overline{\boldsymbol{x}}, t)), \forall \overline{\boldsymbol{x}}$$

Set in the lemma $\boldsymbol{M} = \boldsymbol{M}(\overline{\boldsymbol{x}}, t)$, $\boldsymbol{N} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T$

$\exists$ **ortho-normal frame** $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \in \mathbb{R}^3$

$$\boldsymbol{e}_1 \cdot \boldsymbol{M}(\overline{\boldsymbol{x}}, t) \cdot \boldsymbol{e}_1 \leq \frac{\mathrm{tr}(\boldsymbol{M}(\overline{\boldsymbol{x}}, t))}{3},$$

$$\boldsymbol{e}_1 \cdot \nabla \boldsymbol{u}(\overline{\boldsymbol{x}}, t) \cdot \boldsymbol{e}_1 = 0.$$

Assume

$$\boldsymbol{e}_1 \cdot \nabla \boldsymbol{u}(\overline{\boldsymbol{x}}, t) = a\boldsymbol{e}_2 \ \ (a \geq 0)$$

$$\xi_\epsilon(x_1, x_2, x_3) = \psi\left(\left(\frac{x_1}{\epsilon}\right)^2 + \left(\frac{x_2}{\epsilon^2}\right)^2 + \left(\frac{x_3}{\epsilon}\right)^2\right)\begin{pmatrix} -\frac{x_2}{\epsilon} \\ \epsilon x_1 \\ 0 \end{pmatrix},$$

where

$$\psi(s) = \begin{cases} \exp(1/(s-1)), & \textbf{for } 0 \le s < 1 \\ 0, & \textbf{for } s \ge 1 \end{cases}$$

$$\xi_\epsilon(x_1, x_2, x_3) \parallel \boldsymbol{e}_1, \; (\epsilon \to 0)$$

$$\nabla \cdot \boldsymbol{\xi}_\epsilon \approx 0$$

Define $\alpha_\epsilon, \beta_\epsilon$

$$\nabla \cdot (\nabla \alpha_\epsilon + (\boldsymbol{\xi}_\epsilon \cdot \nabla)\boldsymbol{u}) = 0,$$

$$\beta_\epsilon \equiv a\frac{\epsilon^3}{2}\Psi\left(\left(\frac{x_1}{\epsilon}\right)^2 + \left(\frac{x_2}{\epsilon^2}\right)^2 + \left(\frac{x_3}{\epsilon}\right)^2\right),$$

$$\Psi = \int^s \psi(s')ds'$$

$\beta_\epsilon \to \alpha_\epsilon, \; (\epsilon \to 0)$

By $e_1 \cdot \nabla u = a e_2$

$$\underbrace{e_1 \frac{-x_2}{\epsilon} \psi(s)}_{\approx \boldsymbol{\xi}_\epsilon} \cdot \nabla \boldsymbol{u} = -\underbrace{\frac{\epsilon^3}{2} a \psi(s) \frac{2x_2}{\epsilon^4} \boldsymbol{e}_2}_{\approx \nabla \beta_\epsilon},$$

therefore

$$(\boldsymbol{\xi}_\epsilon \cdot \nabla)\boldsymbol{u} + \nabla \beta_\epsilon = O(\epsilon^2).$$

$$(\boldsymbol{\xi}_\epsilon \cdot \nabla)\boldsymbol{u} + \nabla\beta_\epsilon = O(\epsilon^2)$$

$$O(\epsilon^8) = \left\langle |(\boldsymbol{\xi}_\epsilon \cdot \nabla)\boldsymbol{u} + \nabla\beta_\epsilon|^2 \right\rangle$$

$$= \left\langle |\nabla\alpha_\epsilon - \nabla\beta_\epsilon|^2 \right\rangle + \left\langle |(\boldsymbol{\xi}_\epsilon \cdot \nabla)\boldsymbol{u} + \nabla\alpha_\epsilon|^2 \right\rangle$$

$$\geq \left\langle |\nabla\alpha_\epsilon - \nabla\beta_\epsilon|^2 \right\rangle$$

$$\left\langle |\nabla\alpha_\epsilon - \nabla\beta_\epsilon|^2 \right\rangle = O(\epsilon^8)$$

$$\boldsymbol{\xi}_\epsilon \cdot \boldsymbol{M} \cdot \boldsymbol{\xi}_\epsilon = \boldsymbol{\xi}_\epsilon \cdot \boldsymbol{P} \cdot \boldsymbol{\xi}_\epsilon - (\boldsymbol{\xi}_\epsilon \cdot \nabla)\boldsymbol{u} \cdot (\boldsymbol{\xi}_\epsilon \cdot \nabla)\boldsymbol{u}$$

$$= \boldsymbol{\xi}_\epsilon \cdot \boldsymbol{P} \cdot \boldsymbol{\xi}_\epsilon - |\nabla\beta_\epsilon|^2 + O(\epsilon^3)$$

$$\boldsymbol{\xi}_\epsilon \cdot \boldsymbol{M} \cdot \boldsymbol{\xi}_\epsilon = \boldsymbol{\xi}_\epsilon \cdot \boldsymbol{P} \cdot \boldsymbol{\xi}_\epsilon - |\nabla \beta_\epsilon|^2 + O(\epsilon^3)$$

$$\textbf{LHS} = \left(\frac{x_2}{\epsilon}\right)^2 \psi(s)^2 \boldsymbol{e}_1 \cdot \boldsymbol{M}(\overline{\boldsymbol{x}}, t) \cdot \boldsymbol{e}_1 + O(\epsilon^3)$$

therefore

$$\langle \boldsymbol{A}_{\boldsymbol{u}}(\boldsymbol{\xi}_\epsilon), \boldsymbol{\xi}_\epsilon \rangle = \boldsymbol{e}_1 \cdot \boldsymbol{M}(\overline{\boldsymbol{x}}, t) \cdot \boldsymbol{e}_1 k^2 \epsilon^6 + O(\epsilon^7),$$

where

$$k^2 \epsilon^6 = \int \left(\frac{x_2}{\epsilon}\right)^2 \psi^2 \left(\left(\frac{x_1}{\epsilon}\right)^2 + \left(\frac{x_2}{\epsilon^2}\right)^2 + \left(\frac{x_3}{\epsilon}\right)^2 \right) \boldsymbol{dx}$$

$$= \|\boldsymbol{\xi}_\epsilon\|^2 + O(\epsilon^7)$$

$$\boldsymbol{\xi}_\epsilon^\perp = \boldsymbol{\xi}_\epsilon - \frac{\langle \boldsymbol{\xi}_\epsilon, \boldsymbol{u} \rangle}{\|\boldsymbol{u}\|^2}\boldsymbol{u}, \quad \langle \boldsymbol{\xi}_\epsilon, \boldsymbol{u} \rangle = O(\epsilon^5)$$

$$\|\boldsymbol{\xi}_\epsilon\|^2 = \|\boldsymbol{\xi}_\epsilon^\perp\|^2 + O(\epsilon^{10})$$

therefore

$$\frac{2U\boldsymbol{u}(\boldsymbol{\xi}_\epsilon^\perp)}{\|\boldsymbol{\xi}_\epsilon^\perp\|^2} = \underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{M}(\overline{\boldsymbol{x}}, t) \cdot \boldsymbol{e}_1}_{\leq \frac{\mathrm{tr}(\boldsymbol{M}(\overline{\boldsymbol{x}}, t))}{3}} + O(\epsilon)$$