Analytical Study of Certain Sub-grid Scale Turbulence Models

Edriss S. Titi

University of California – Irvine
and
Weizmann Institute of Science

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The Navier-Stokes

The Navier-Stokes equations

	$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla} \vec{v}) - \nu \Delta \vec{v} = -\vec{\nabla} p + \vec{f}$	$in\ Q$
Incompressibilitycondition	$\vec{\nabla} \cdot \vec{v} = 0$	$in\ Q$
initial condition	$\vec{v}(0, x) = v_o$	$in\ \Omega$
boundary condition	$\vec{v} = 0$	$on\ [0, T] \times \partial\Omega$

or periodic boundary condition

cylinder: $Q = (0, T) \times \Omega$

unknowns:

$\vec{v} : [0, T] \times \Omega \to \mathbb{R}^n$ - is the velocity field

$p : [0, T] \times \Omega \to \mathbb{R}$ - is the pressure

forcing: $\vec{f} : [0, T] \times \Omega \to \mathbb{R}^n$

ν - is the viscosity

The Navier-Stokes Equations

$$\frac{\partial}{\partial t}\vec{u} - \upsilon\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho_0}\nabla p = \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

Plus Boundary conditions, say periodic in the box

$$\Omega = [0, L]^3$$

Sobolev Spaces

$$H^s(\Omega) = \{\varphi = \sum_{\vec{k} \in \mathrm{Z}^d} \hat{\varphi}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}\frac{2\pi}{L}}$$

such that

$$\sum_{\vec{k} \in \mathrm{Z}^d} \left|\hat{\varphi}_{\vec{k}}\right|^2 (1 + \left|\vec{k}\right|^2)^s < \infty\}$$

By Poincare' inequality

$$\frac{d}{dt}\left\|\vec{u}\right\|_{L^2}^2 + c\,\frac{\upsilon}{L^2}\left\|\vec{u}\right\|_{L^2}^2 \le \frac{cL^2}{\upsilon}\left\|\vec{f}\right\|_{L^2}^2$$

By Gronwall's inequality

$$\left\|\vec{u}(t)\right\|_{L^2}^2 \le e^{-c\upsilon L^{-2}t}\left\|\vec{u}(0)\right\|_{L^2}^2 + \frac{cL^4}{\upsilon^2}\left(1-e^{-c\upsilon L^{-2}t}\right)\left\|\vec{f}\right\|_{L^2}^2 \qquad \forall t \in [0,\mathrm{T}]$$

and

$$\upsilon\int_0^{\mathrm{T}}\left\|\nabla\vec{u}(\tau)\right\|_{L^2}^2 d\tau \le K\left(L,\left\|\vec{u}_0\right\|_{L^2},\left\|\vec{f}\right\|_{L^2},\upsilon,\mathrm{T}\right)$$

Theorem (Leray 1932-34)

For every $\mathrm{T} > 0$ there exists a weak solution (in the sense of distribution) of the Navier-stokes equations, which also satisfies

$$\vec{u} \in C_w([0,\mathrm{T}], L^2(\Omega)) \bigcap L^2([0,\mathrm{T}], H^1(\Omega))$$

The uniqueness of weak solutions in the three dimensional Navier-Stokes equations case is still an open question.

Strong Solutions of Navier-Stokes

$$\vec{u} \in C([0,\mathrm{T}], H^1(\Omega)) \bigcap L^2([0,\mathrm{T}], H^2(\Omega))$$

Enstrophy

$$\left\| \nabla \times \vec{u} \right\|_{L^2}^2 = \left\| \vec{\omega} \right\|_{L^2}^2 = \left\| \nabla \vec{u} \right\|_{L^2}^2$$

Formal Enstrophy Estimates

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla\vec{u}\right\|_{L^2}^2+\upsilon\left\|\Delta\vec{u}\right\|_{L^2}^2+\int(\vec{u}\cdot\nabla)\vec{u}\cdot(-\Delta\vec{u})+\int\nabla p(-\Delta\vec{u})=\int\vec{f}\cdot(-\Delta\vec{u})$$

Observe that $\int\nabla p\cdot(-\Delta\vec{u})dx=0$

By Cauchy-Schwarz $\left|\int\vec{f}\cdot(-\Delta\vec{u})\right|\leq\frac{\left\|\vec{f}\right\|_{L^2}^2}{\upsilon}+\frac{\upsilon}{4}\left\|\Delta\vec{u}\right\|_{L^2}^2$

By Hőlder inequality

$$\left|\int(\vec{u}\cdot\nabla)\vec{u}\cdot(-\Delta\vec{u})\right|\leq\left\|\vec{u}\right\|_{L^4}\left\|\nabla\vec{u}\right\|_{L^4}\left\|\Delta\vec{u}\right\|_{L^2}$$

Calculus/Interpolation (Ladyzhenskaya) Inequatlities

$$\|\varphi\|_{L^4} \le \begin{cases} c\,\|\varphi\|_{L^2}^{1/2}\ \ \|\nabla\varphi\|_{L^2}^{1/2} & 2-D \\ c\,\|\varphi\|_{L^2}^{1/4}\ \ \|\nabla\varphi\|_{L^2}^{3/4} & 3-D \end{cases}$$

Denote by $y = e_0 + \|\nabla\vec{u}\|_{L^2}^2$

The Two-dimensional Case

$$\dot{y} \le c\, y^2 \qquad \& \qquad \int_0^{\mathrm{T}} y(\tau)\,d\tau \le K(\mathrm{T})$$

$$\Rightarrow y(t) \le \tilde{K}(\mathrm{T})$$

Global regularity of strong solutions to the two-dimensional Navier-Stokes equations.

Navier-Stokes Equations

- Two-dimensional Case

  * Global Existence and Uniqueness
    of weak and strong solutions

  * Finite dimension global attractor

One can instead use the following Sobolev inequality

$$\left\|\vec{u}\right\|_{L^6} \leq c\left\|\nabla \vec{u}\right\|_{L^2}$$

Which leads to $\dot{y} \leq c y^3 \quad \& \quad \int_0^T y(\tau)d\tau \leq K$

Theorem (Leray 1932-1934)

There exists $T_*(\left\|\vec{u}_0\right\|_{L^2}, \left\|\vec{f}\right\|_{L^2}, \upsilon, L)$ such that

$y(t) < \infty$ for every $t \in [0, T_*)$.

Navier-Stokes Equations

- The Three-dimensional Case

  * Global existence of the weak solutions

  * Short time existence of the strong solutions

  * Uniqueness of the strong solutions

- Open Problems:

  * Uniqueness of the weak solution

  * Global existence of the strong solution.

Vorticity Formulation

$$\frac{\partial \vec{\omega}}{\partial t} - \nu \Delta \vec{\omega} + (\vec{u} \cdot \nabla)\vec{\omega} - \underline{\underline{(\vec{\omega} \cdot \nabla)\vec{u}}} = \nabla \times \vec{f}$$

Vorticity Stretching Term $(\vec{\omega} \cdot \nabla)\vec{u}$

Two dimensional case $(\vec{\omega} \cdot \nabla)\vec{u} \equiv \vec{0}$

$$\frac{\partial \vec{\omega}}{\partial t} - \nu \Delta \vec{\omega} + (\vec{u} \cdot \nabla)\vec{\omega} = \nabla \times \vec{f}$$

$|\vec{\omega}(x,t)|^2$ Satisfies a maximum principle.

The Three-dimensional Case

$$(\vec{\omega} \cdot \nabla)\vec{u} \not\equiv 0$$

$$\vec{\omega} \sim z$$

$$(\vec{\omega} \cdot \nabla)\vec{u} \sim z^2$$

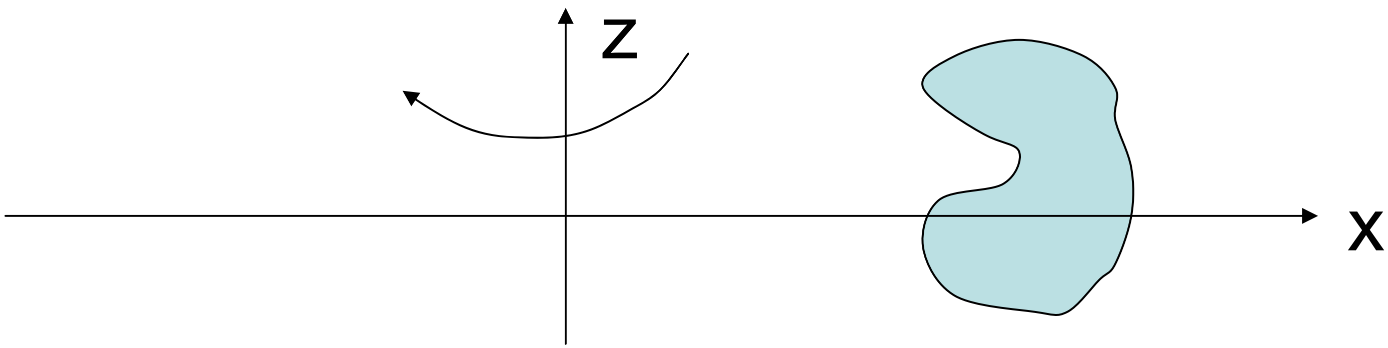
For large initial data $\vec{\omega}_0$ the vorticity balance takes the form

$$\dot{z} \sim z^2 \Rightarrow$$ **Potential "Blow Up"!!**

Special Results of Global Existence for the three-dimensional Navier-Stokes

Theorem (Fujita and Kato)

Let $\|u_0\|_{H^{1/2}}$ be small enough. Then the 3D Navier-Stokes equations are globally well-posed for all time with such initial data. The same result holds if the initial data is small in $L^3(\Omega)$ (Kato, Giga & Miyakawa)



• Ω – Revolution Domain around the z - axis [away from z - axis]

• Let us move to Cylindrical coordinates

Theorem (Ladyzhenskaya) Let

$$\vec{u}_0(x,y,z) = (\varphi_r^0(r,z), \varphi_\theta^0(r,z), \varphi_z^0(r,z))$$

be axi-symmetric initial data. Then the three-dimensional Navier-Stokes equations have globally (in time) strong solution corresponding to such initial data. Moreover, such strong solution remains axi-symmetric.

Theorem (Leiboviz, Mahalov and E.S.T.)

Consider the three-dimensional Navier-Stokes equations in an infinite Pipe. Let

$$\vec{u}_0 = (\varphi_r^0(r, n\theta + \alpha z), \varphi_\theta^0(r, n\theta + \alpha z), \varphi_z^0(r, n\theta + \alpha z))$$

(Helical symmetry). For such initial data we have global existence and uniqueness. Moreover, such a solution remains helically symmetric.

Remarks

- For axi-symmetric and helical flows the vorticity stretching term is nontrivial, and the velocity field is three-dimensional.

- In the inviscid case, i.e. $\upsilon = 0$, the question of global regularity of the three-dimensional helical or axi-symmetrical Euler equations is still open. Except the invariant sub-spaces where the vorticity stretching term is trivial.

- **Theorem [Cannone, Meyer & Planchon] [Bondarevsky] 1996**

Let M *be given, as large as we want. Then there exists* K(M) *such that for every initial data of the form*

$$\vec{u}_0 = \sum_{|\vec{k}| \geq K(M)} \hat{\vec{u}}^{\,0}_{\vec{k}} \, e^{i\vec{k}\cdot\vec{x}\frac{2\pi}{L}}$$

[VERY OSCILLATORY]

the three-dimensional Navier-Stokes equations have global existence of strong solutions.

Remark Such initial data satisfies $\|u_0\|_{H^{1/2}} << 1.$

So, this is a particular case of Kato's Theorem.

The Effect of Rotation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p + \vec{\Omega} \times \vec{u} = 0$$

$$\nabla \cdot \vec{u} = 0$$

- There is $\Omega_0(\mathrm{T}, \vec{u}_0)$ such that if $|\Omega| > \Omega_0$ the solution exists on $[0, \mathrm{T})$.
- That is there exists $\mathrm{T}_0\ (\vec{u}_{0,} |\vec{\Omega}|)$ such that the solution

exists on $[0, \mathrm{T}_0)$. Observe that

$$\mathrm{T}_0 \to \infty \ \text{as} \ |\vec{\Omega}| \to \infty$$

- Babin - Mahalov - Nicolaenko.
- Embid - Majda.
- Chemin, Ghalagher, Granier, Masmoudi,...
- Liu and Tadmor.

An Illustrative Example

Inviscid Burgers Equation

$$u_t + uu_x = 0 \quad \text{in } \mathrm{R}$$

$$u(x,0) = u_0(\mathrm{x})$$

- If $u_0(x)$ is decreasing function on some subinterval of $\mathbb{R}$ then the solution of the above equation develops a singularity (Shock) in finite time.

The solution is given implicitly by the relation:

$$u(x,t) = u_0(x - tu(x,t))$$

The Effect of the Rotation

$$u \in \mathrm{C} \qquad z \in \mathrm{C}$$

$$u_t + uu_z + i\Omega u = 0$$

$$u_0(z) = u(z,0)$$

$$v(z,t) = e^{i\Omega t} u(z,t)$$

$$v_t + e^{-i\Omega t} v v_z = 0$$

$$v(z,t) = v_0\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)$$

$$\frac{\partial}{\partial z} v = \frac{v_0'\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)}{1 + \frac{e^{-i\Omega t} - 1}{-i\Omega} v_0'\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)}$$

If $\Omega >> 1$, (i.e. $\Omega > \Omega_0(u_0)$)

$\frac{\partial}{\partial z} v$ remains finite and the solution remains regular for all $t \geq 0$.

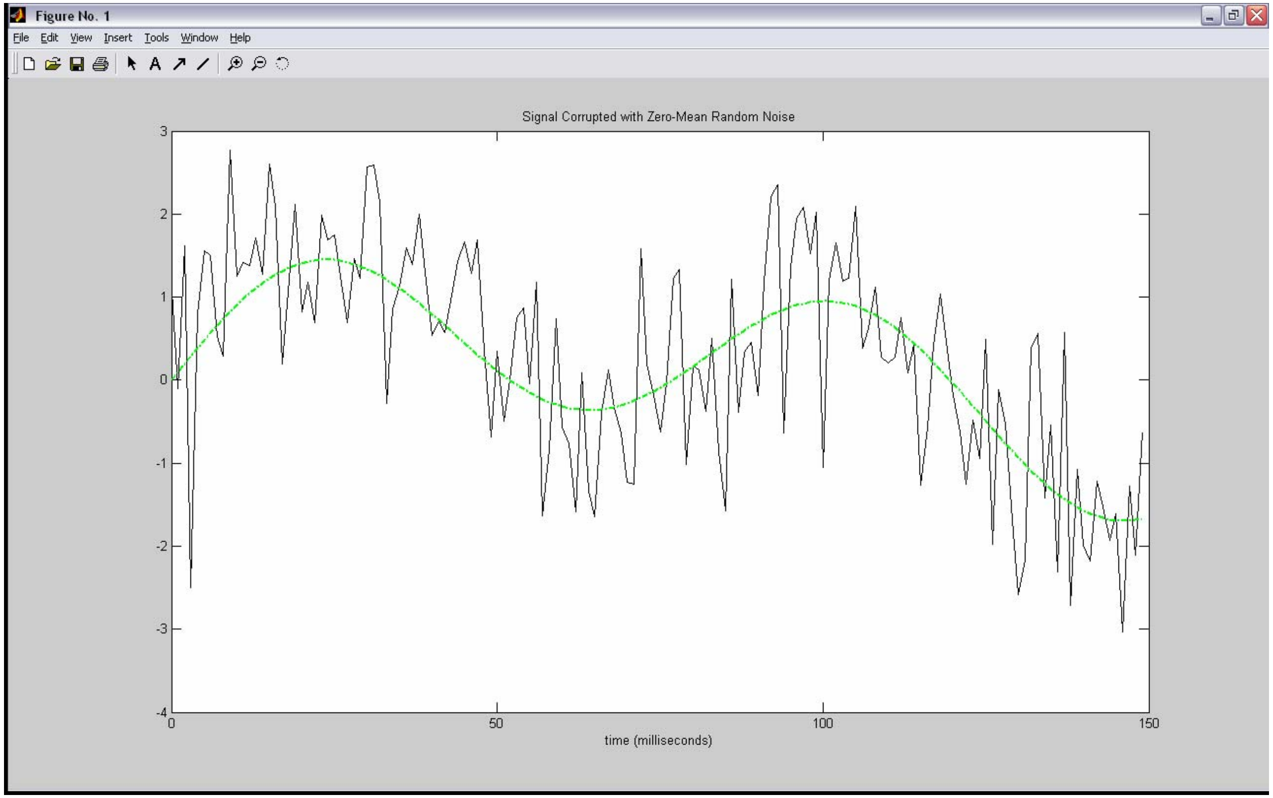
The above complex system is equivalent to 2D Rotating Burgers:

$$u = u_1 + iu_2, \qquad z = x + iy$$

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \Omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{u} = 0$$

Reynolds Equations

$$\phi = \bar{\phi} + \phi'$$



$\bar{\phi}$ - mean

$$\langle \phi \rangle (x) = \bar{\phi}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(x,t)\ dt$$

ϕ' - fluctuations around the mean

Averaged Equations of Motion

$$v = \bar{v} + v'$$ 1

$$p = \bar{p} + p'$$ 2

$$\overline{v'} = \overline{p'} = 0$$ 3

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla} \vec{v}) - \nu \Delta \vec{v} = -\vec{\nabla} p$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

NSE

$$\overline{v'} = 0 \qquad \nabla \cdot v = 0 \qquad \nabla \cdot \bar{v} = \nabla \cdot v' = 0$$ 4

Reynolds averaged Navier-Stokes Equations

$$(\bar{v} \cdot \nabla)\bar{v} + \overline{(v' \cdot \nabla)v'} = \nu \Delta \bar{v} - \nabla \bar{p}$$

$$\nabla \cdot \bar{v} = 0$$

Incompressibility condition

Closure Problem

Reynolds averaged Navier-Stokes Equations

$$(\bar{v} \cdot \nabla)\bar{v} + \overline{\nabla \cdot (v' \otimes v')} = \nu \Delta \bar{v} - \nabla \bar{p}$$

$$\nabla \cdot \bar{v} = 0$$

Incompressibility condition

Fundamental Problem in Turbulence – The Closure Problem

(equations are not closed: more unknowns than equations)

Turbulence Modeling

Reynolds averaged Navier-Stokes Equations

$$(\bar{v} \cdot \nabla)\bar{v} + \overline{\nabla \cdot (v' \otimes v')} = \nu \Delta \bar{v} - \nabla \bar{p}$$

$$\nabla \cdot \bar{v} = 0$$

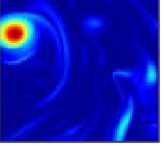
Incompressibility condition

How to model this in terms of $\bar{v}$?

How to close the Reynolds averaged system?

$$\tau_{ij}^R = ((v - \bar{v}) \otimes (v - \bar{v}))_{ij}$$

$$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$



Large Eddy Simulations

- Spatial Filtering
- Large Eddy Simulations
- Sub-grid Scale Model

Let ϕ be nice/smooth spatial filter/kernel

$$\bar{v} = \int \phi(x-y)v(y)$$

$$\frac{\partial \bar{v}}{\partial_t} - \nu \Delta \bar{v} + (\bar{v} \cdot \nabla)\bar{v} = -\nabla \cdot (\tau^R + \bar{p}I)$$

$$\nabla \cdot \bar{v} = 0$$

Here again the problem is to model:

$$div \ \ \tau^R$$

and close the system in terms of $\bar{v}$

$$\tau^R_{ij} = ((v-\bar{v}) \otimes (v-\bar{v}))_{ij}$$

$$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

Smogarinsky Model

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)$$

$$|\bar{S}|^2 = 2\sum_{i,j}(\bar{S}_{ij})^2$$

$$\tau_{ij}^R \approx -2\nu_T \bar{S}_{ij}$$

$$\nu_T = l_S^2 |\bar{S}|$$

$$\tau_{ij}^R = ((v - \bar{v}) \otimes (v - \bar{v}))_{ij}$$

$$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

$$\partial_t \bar{v} - \nu \Delta \bar{v} + (\bar{v} \cdot \nabla)\bar{v} = -\nabla \bar{p} + \nu_1 \nabla \cdot (|S| S(\bar{v})) + \bar{f}$$

Navier-Stokes-α

- S. Chen
- C. Foias
- D. Holm
- E. Olson
- S. Wynne
- E. S. Titi

Camassa-Holm Water Wave Equation

Camassa-Holm Water Wave Equation

Hamiltonian

$$\int \left(|u|^2 + \alpha^2 |u_x|^2\right)\, dx$$

Inviscid Equations

Euler equations

Hamiltonian

$$\int |u(x,t)|^2 \, dx$$

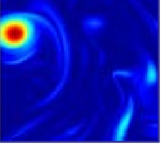
$$\nabla \cdot u = 0 \text{ constraint}$$

Euler-α equations (Holm-Ratiu-Marsden)

Hamiltonian

$$\int |u|^2 + \alpha^2 |\nabla u|^2 \, dx$$

$$\nabla \cdot u = 0 \text{ constraint}$$



Euler-α (Inviscid Second-Grade Fluid)

$$\frac{\partial v}{\partial t} + (u \cdot \nabla)v - \sum_{j=1}^{3} v_j \nabla u_j + \nabla \pi = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta)u$$

Or Equivalently

$$\frac{\partial v}{\partial t} - u \times (\nabla \times v) + \nabla p = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta)u$$

Euler-α (inviscid second grade fluid)

—º Δu

$$\frac{\partial v}{\partial t} + (u \cdot \nabla) v - \sum_{j=1}^{3} v_j \nabla u_j + \nabla \pi = 0$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta) u$$

3D (no global well-posedness)
Euler equations when α=0

or Equivalently

$$\frac{\partial v}{\partial t} - u \times (\nabla \times v) + \nabla p = 0$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta) u$$

Navier-Stokes-α (The viscous Camassa-Holm equations)

$$\frac{\partial v}{\partial t} - \nu \Delta v + (u \cdot \nabla) v - \sum_{j=1}^{3} v_j \nabla u_j + \nabla \pi = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta) u$$

$$\frac{\partial v}{\partial t} - \nu \Delta v - u \times (\nabla \times v) + \nabla p = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta) u$$

The Navier-Stokes-α as a closure model

Reynolds averaged Navier-Stokes Equations

$$(\bar{v} \cdot \nabla)\bar{v} = \nabla \cdot \tau$$

$$\tau = \nu(\nabla \bar{v} + \nabla \bar{v}^T) - \bar{p}I - \overline{v' \otimes v'}$$

Chen, Foias, Holm, Olson, Titi and Wynne, Physics of Fluids 1999

The steady state Navier-Stokes-alpha analytic subgrid scale model of turbulence

$$(u \cdot \nabla)u = \nabla \cdot \tau_\alpha$$

$$\tau_\alpha = 2\nu(1 - \alpha^2 \Delta)D - pI + \alpha^2 \dot{D}$$

where

$$D = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$\Omega = \frac{1}{2}(\nabla u - \nabla u^T)$$

$$\dot{D} = u \cdot \nabla D + D\Omega - \Omega D$$

Vorticity Formulation

NSE $\omega = \nabla \times u$

$$\frac{\partial \omega}{\partial t} - \nu \Delta \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = \nabla \times f$$

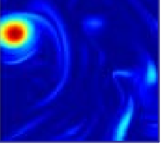
$$\nabla \cdot u = 0$$

VCHE $q = \nabla \times v$ $v = u - \alpha^2 \Delta u$

$$\frac{\partial q}{\partial t} - \nu \Delta q + (u \cdot \nabla)q - (q \cdot \nabla)u = \nabla \times f$$

$$\nabla \cdot u = 0$$

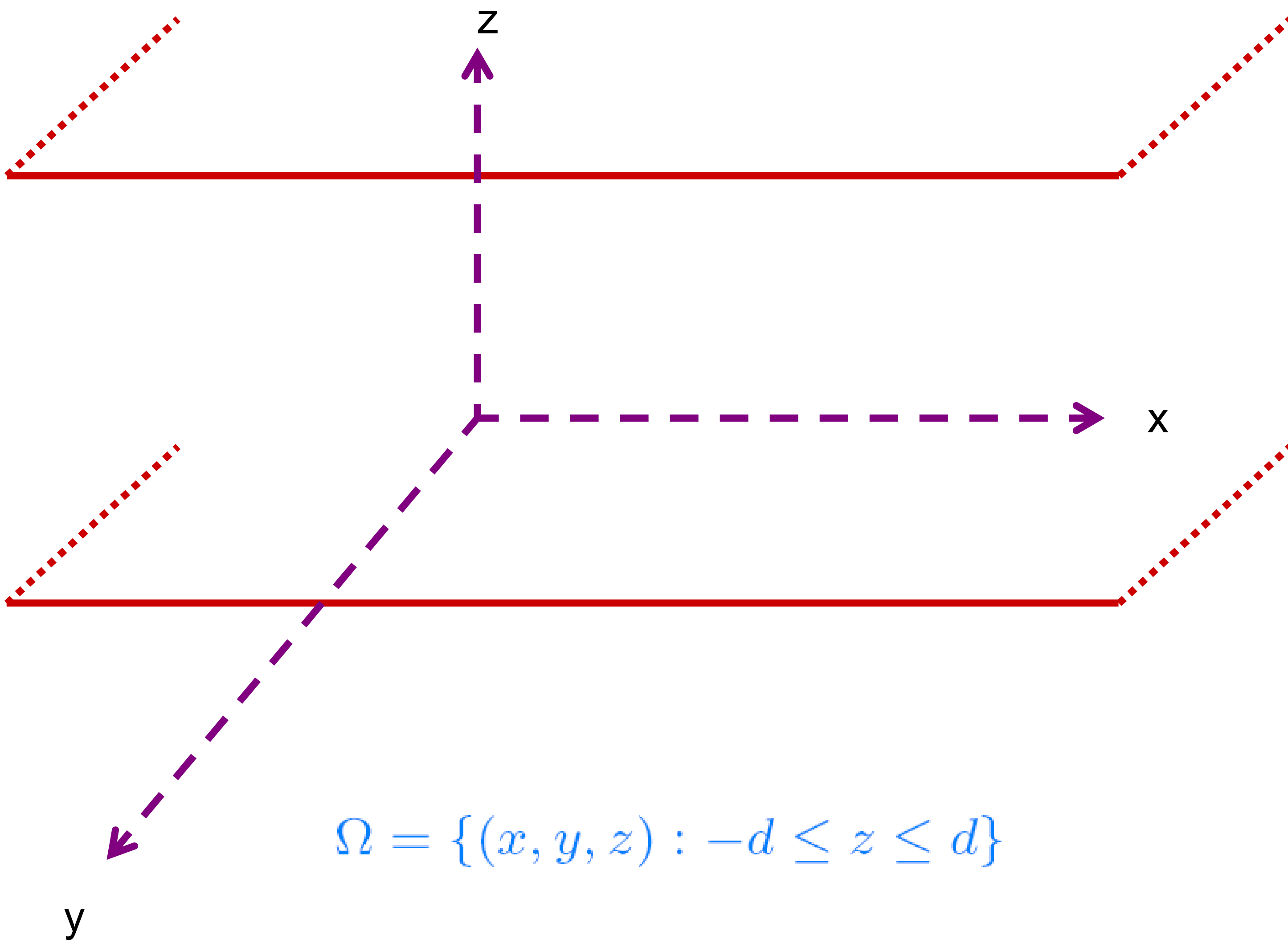
$$u \cdot \nabla q - q \cdot \nabla u \quad \Longleftrightarrow \quad v \cdot \nabla q - q \cdot \nabla v$$



Dimension of Global Attractor (NS-α)

$$d(\mathcal{A}) \leq c\left(\frac{L}{\alpha}\right)^{3/2}\left(\frac{L}{l_d}\right)^{3}$$

Turbulent Channel Flow



$$\Omega = \{(x, y, z) : -d \leq z \leq d\}$$

Reynolds Averaged Equations

$$-\nu\Delta\langle u\rangle = \langle(u\cdot\nabla u)\rangle + \nabla\langle p\rangle = 0$$

$$\nabla\cdot\langle u\rangle = 0$$

Facts:

(i)

$$\langle u\rangle = \begin{pmatrix} \bar{U}(z) \\ 0 \\ 0 \end{pmatrix}$$

(ii)

$$\bar{U}(z) = \bar{U}(-z)$$

(iii)

$$u = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \langle u\rangle = \begin{pmatrix} u+\bar{U} \\ v \\ w \end{pmatrix}$$

Reynolds Stresses

The Reynold stresses

$$\langle u^2 \rangle, \langle uv \rangle, \langle uw \rangle, \langle v^2 \rangle, \langle vw \rangle, \langle w^2 \rangle$$

are functions of z alone.

Reynolds Equations

$$-\nu \bar{U}'' + \partial_z \langle wu \rangle = -\partial_x \bar{P}$$

$$\partial_z \langle wv \rangle = -\partial_y \bar{P}$$

$$\partial_z \langle w^2 \rangle = -\partial_z \bar{P}$$

Steady VCHE

ansatz $$u = \begin{pmatrix} U(z) \\ 0 \\ 0 \end{pmatrix}$$

Steady VCHE	Reynolds equations
$-\nu U'' + \nu\alpha^2 U'''' = -\partial_x p$	$-\nu\bar{U}'' + \partial_z \langle wu \rangle = -\partial_x \bar{P}$
$0 = -\partial_y p$	$\partial_z \langle wv \rangle = -\partial_y \bar{P}$
$0 = -\partial_z (p - \alpha^2 (U')^2)$	$\partial_z \langle w^2 \rangle = -\partial_z \bar{P}$

Identifying Terms in VCHE & Reynolds equations

$$\text{(i)}\quad \bar{U} = U$$

$$\text{(ii)}\quad \partial_z \langle wu \rangle = \nu\alpha^2 U'''' + p_0$$

$$\text{(iii)}\quad \partial_z \langle wv \rangle = 0$$

$$\text{(iv)}\quad \nabla(\bar{P} + \langle w^2 \rangle) = \nabla(p - p_0 x - \alpha^2 (U')^2)$$

The General Solution of VCHE

$$U(z) = a\left(1 - \frac{\cosh(z/\alpha)}{\cosh(d/\alpha)}\right) + b\left(1 - \frac{z^2}{d^2}\right)$$

a, b constants

Physical Parameters

- Boundary Stress

$$\pm\tau_0 = -\langle\tau_{13}\rangle\,|_{z=\pm d} = \nu\bar{U}'(z) + \langle wu\rangle\,|_{z=\pm d}$$

$$\tau_0 = -\nu\bar{U}'(z=-d)$$

- Averaged Streamwise Velocity Across the Channel

$$\bar{u} = \frac{1}{2d}\int_{-d}^{d} U(z)dz$$

- Reynolds Numbers

$$R = \frac{\bar{u}d}{\nu} \qquad R_0 = \frac{\tau_0^{1/2}d}{\nu}$$

- Length Scales

$$d, \qquad \alpha, \qquad l_* = \frac{\nu}{\tau_0^{1/2}}$$ wall unit

Normalized quantities

Let $$\eta = \frac{z+d}{l_*}$$ normalized distance from the wall

$$\phi(\eta) = \frac{U(\eta l_* - d)}{\tau_0^{1/2}}$$

normalized velocity profile

Drag Law

The drag law for the wall friction

$$D = \frac{2\tau_0}{\bar{u}^2} = \frac{2R_0^2}{R^2}$$

$$\sqrt{\frac{2}{D}} = \frac{R}{R_0}$$

Profile

The Profile ϕ depends on:

$$\text{(i)}\quad c,\, R,\, R_0 \ \text{or}$$

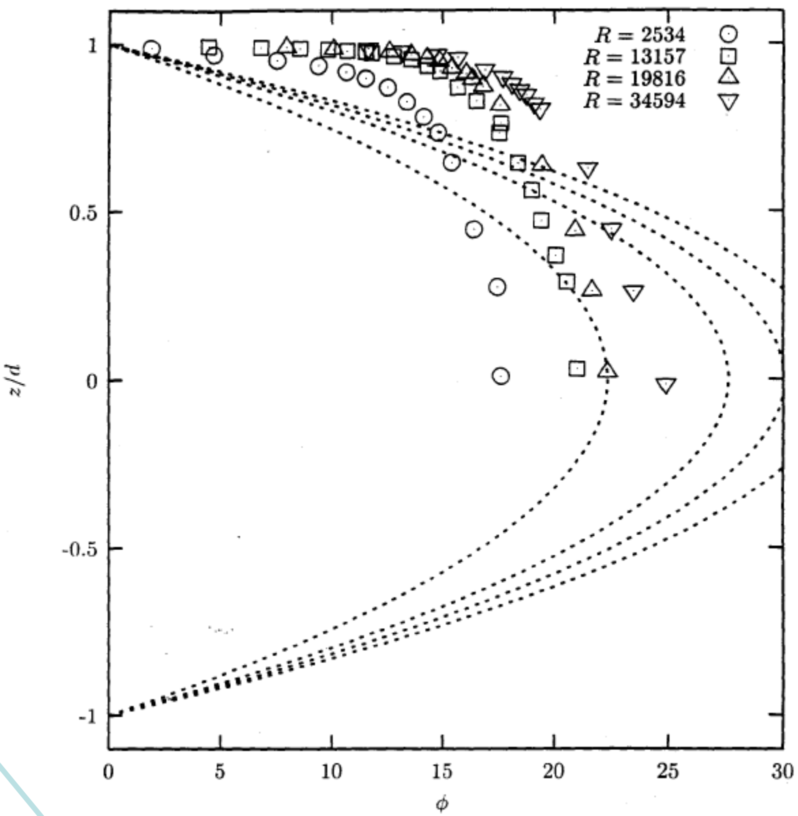
$$\text{(ii)}\quad c,\, R,\, D$$

Blasius drag law

$$D = \lambda R^{-1/4}$$

$$\lambda = \text{constant}$$

Figure 1 and Figure 2



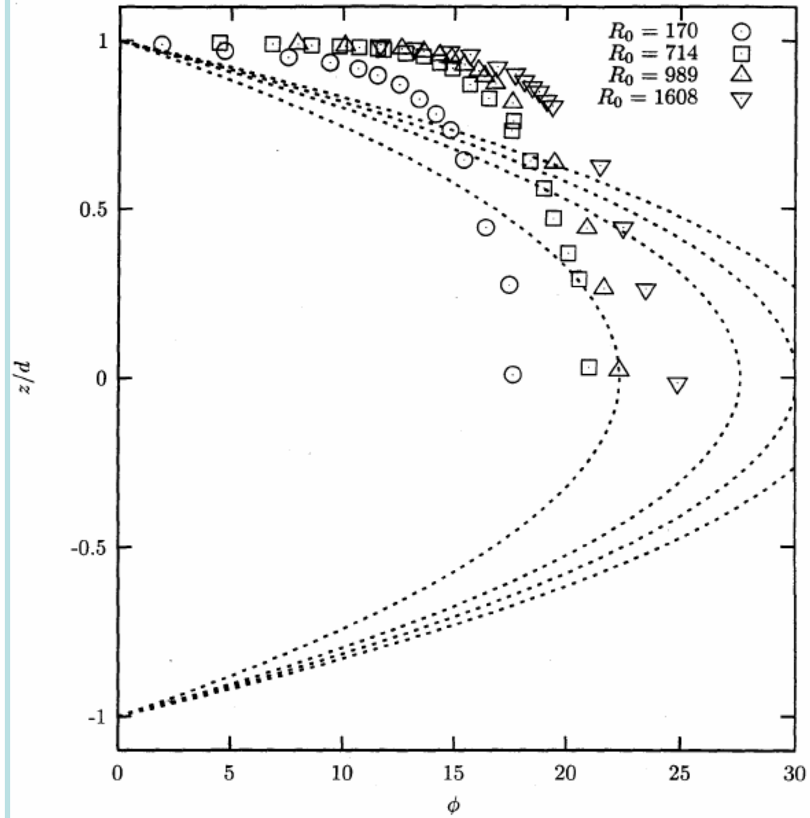
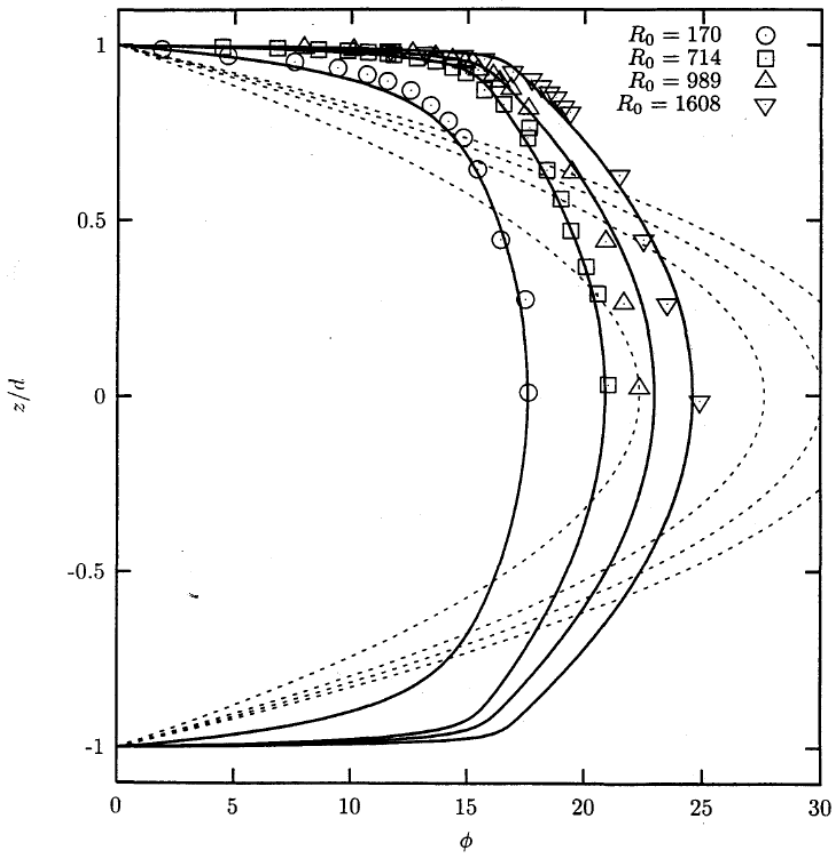


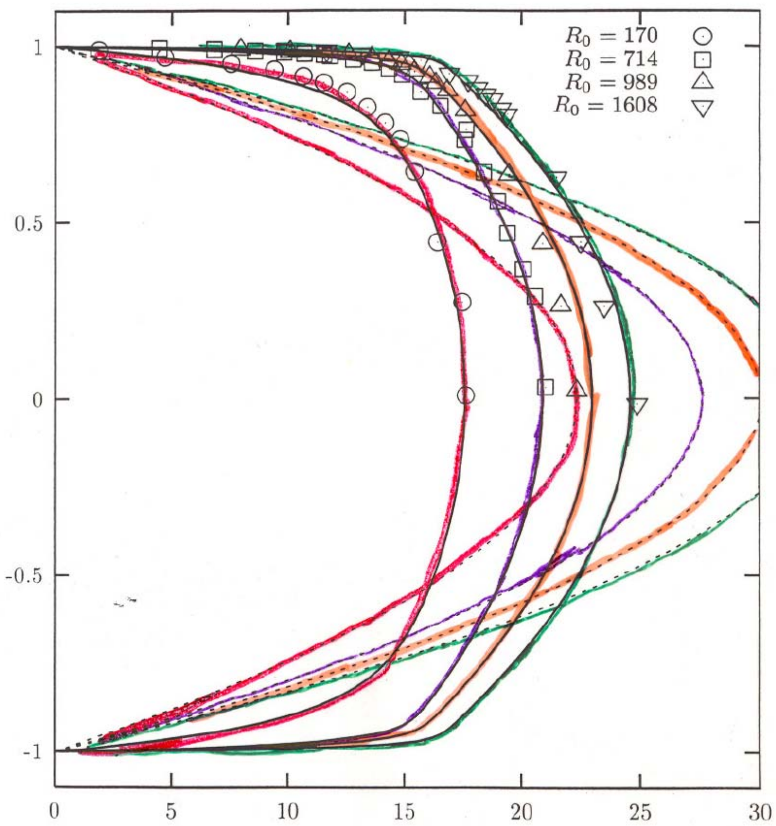
Figure 3



R_c	R_0	ϕ_m	$b/2a$	d/α
2970	170	17.6	1.2	12.850378
14914	714	20.9	1.1	48.782079
22776	989	23	.9	65.777777
39582	1608	24.6	.9	100.569105

$$\phi = \frac{a}{u_*}\left(1 - \frac{\cosh(z/\alpha)}{\cosh(d/\alpha)}\right) + \frac{b}{u_*}\left(1 - \frac{z^2}{d^2}\right)$$

Figure 4

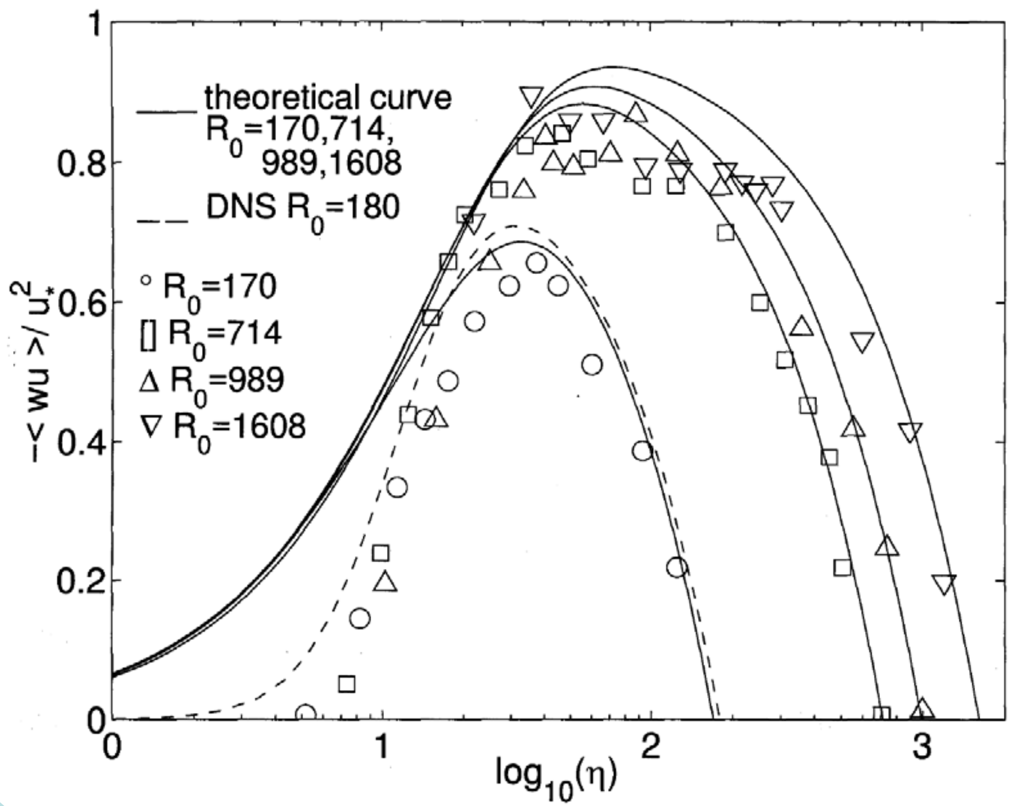


R_c	R_0	ϕ_m	$b/2a$	d/α
2970	170	17.6	1.2	12.850378
14914	714	20.9	1.1	48.782079
22776	989	23	.9	65.777777
39582	1608	24.6	.9	100.569105

$$\phi = \frac{a}{u_*}\left(1 - \frac{\cosh(z/\alpha)}{\cosh(d/\alpha)}\right) + \frac{b}{u_*}\left(1 - \frac{z^2}{d^2}\right)$$

- Experimental data from:
  T. Wei and W.W. Willmarth
- Having blasius drag law
  $\lambda = 0.06$

Figure 5



- Experimental data from:
  T. Wei and W.W. Willmarth
- DNS Kim, Moin & Moser
- Having blasius drag law
  λ = 0.06

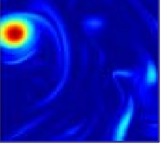
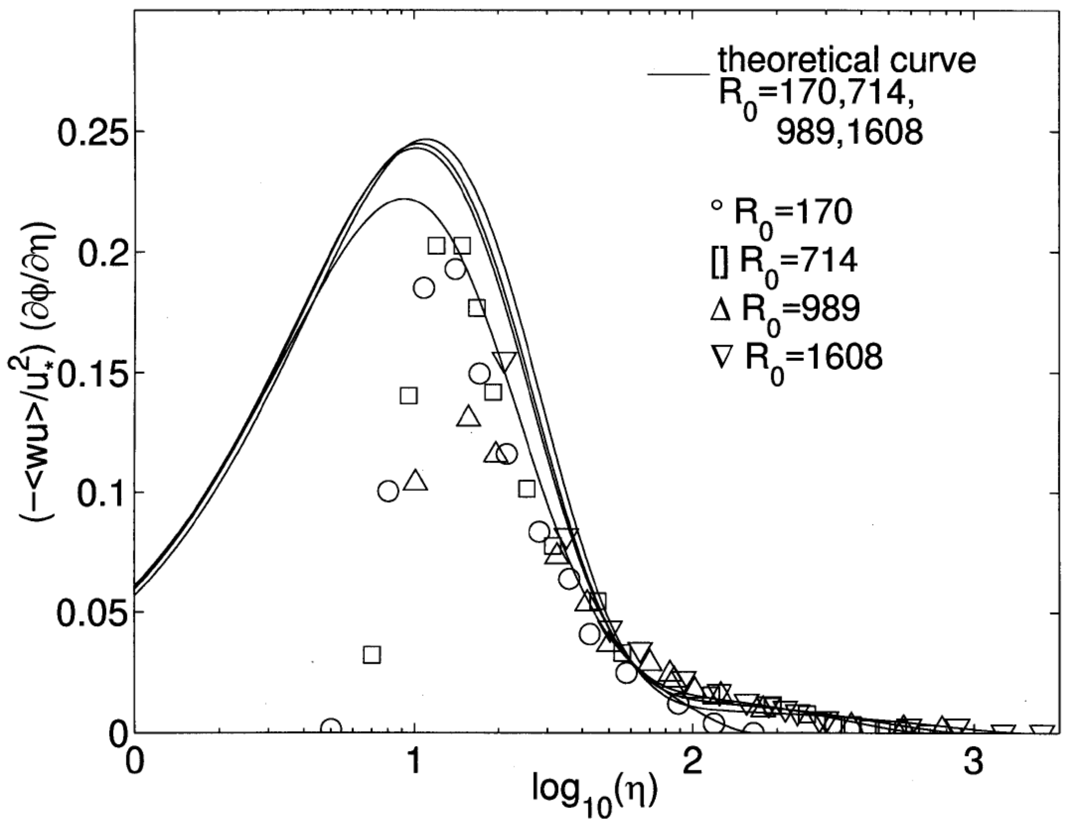
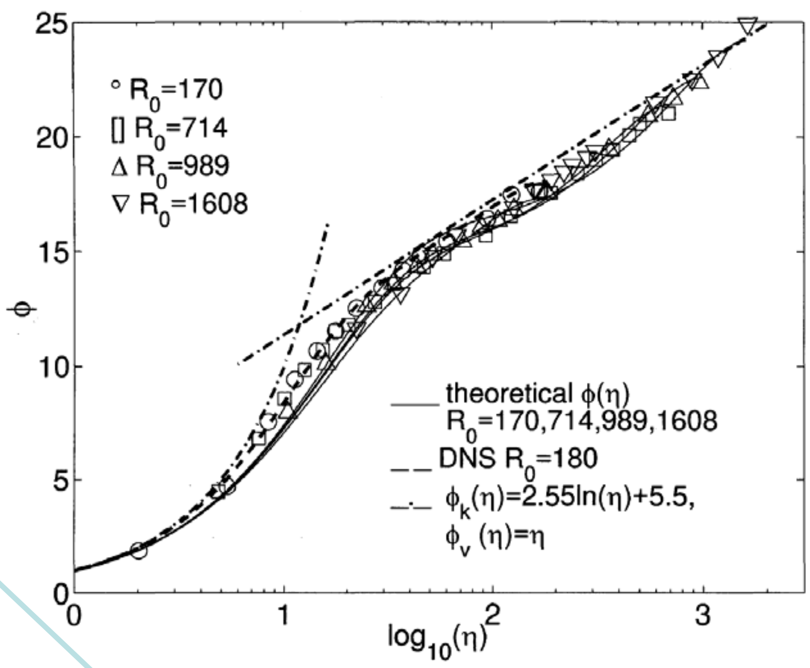


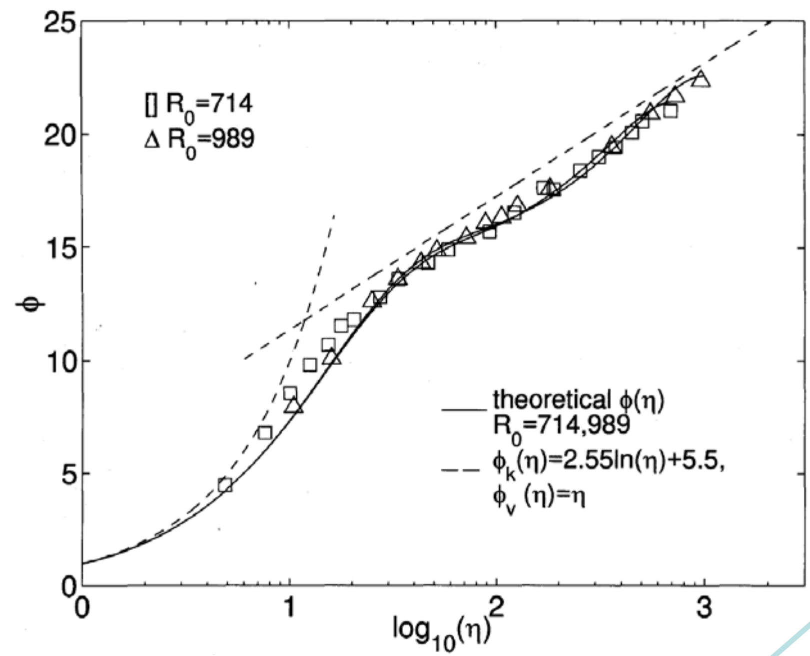
Figure 6



- Experimental data from:
  T. Wei and W.W. Willmarth
- Having blasius drag law
  $\lambda = 0.06$

Figure 7 and Figure 8





- Experimental data from:
  T. Wei and W.W. Willmarth
- DNS Kim, Moin & Moser
- Having blasius drag law
  λ = 0.06

Room for Improvement

α -- constant away from the boundary

Near the boundary

α -- is a function of the distance from the boundary

First Attempt:

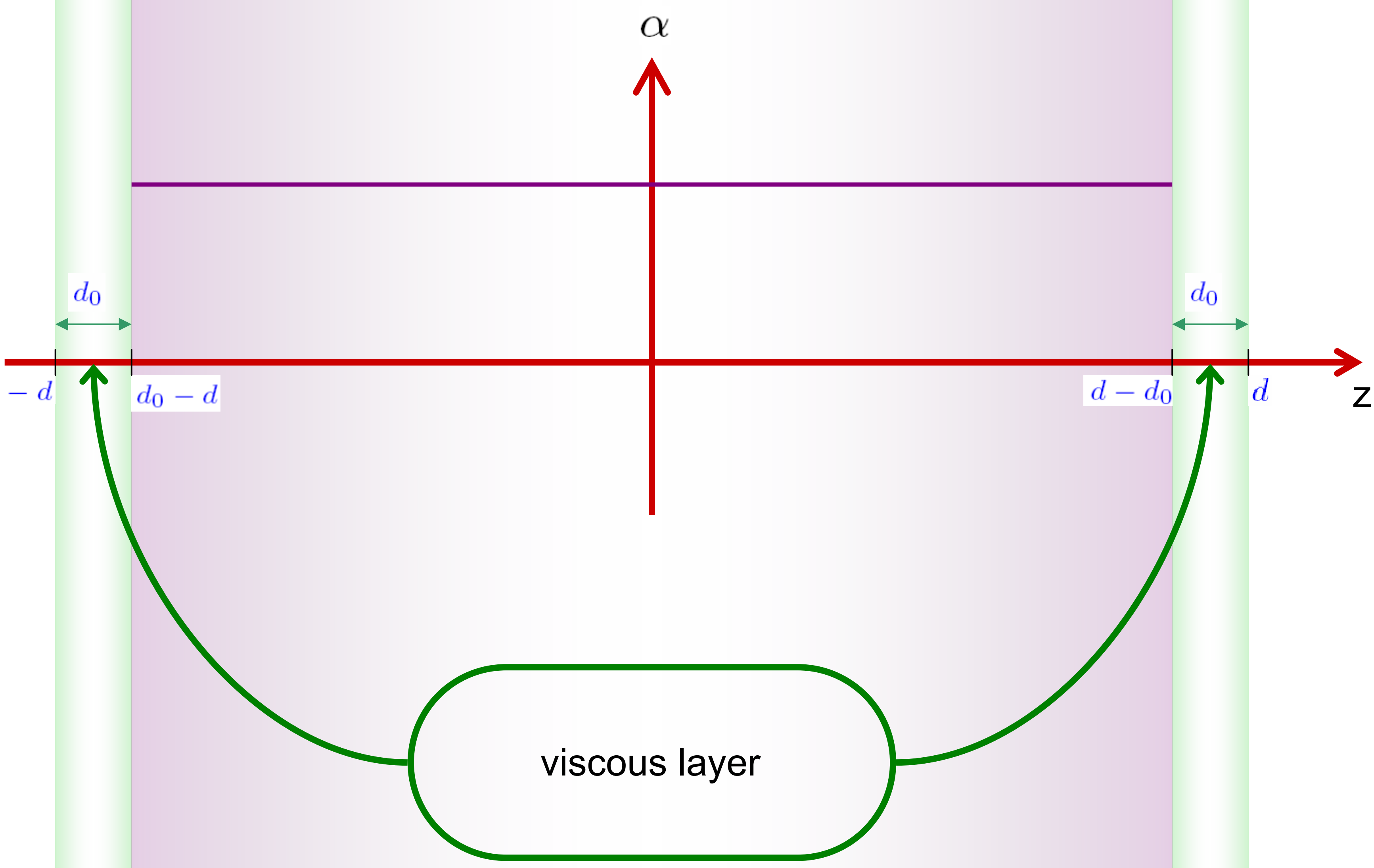
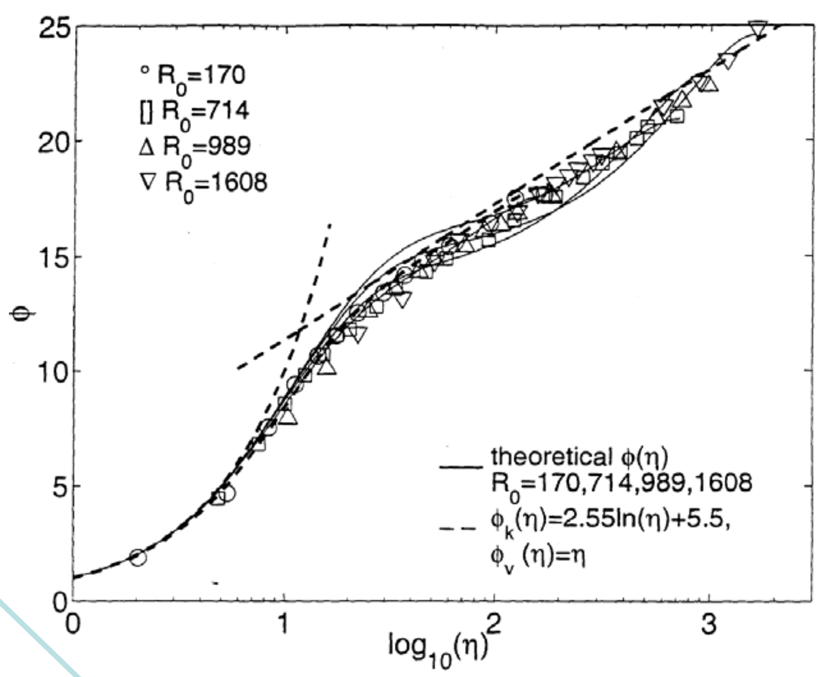
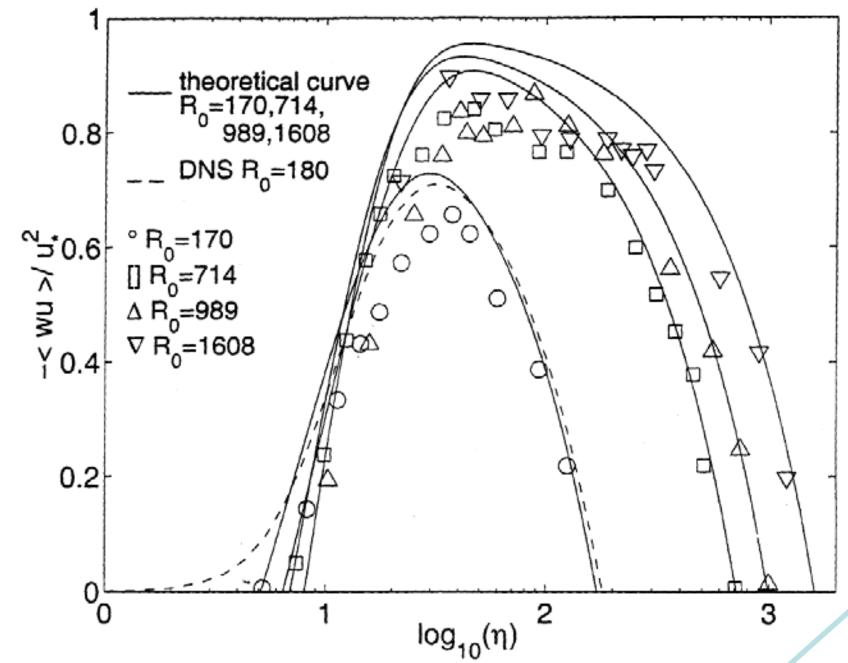


Figure 9 and Figure 10



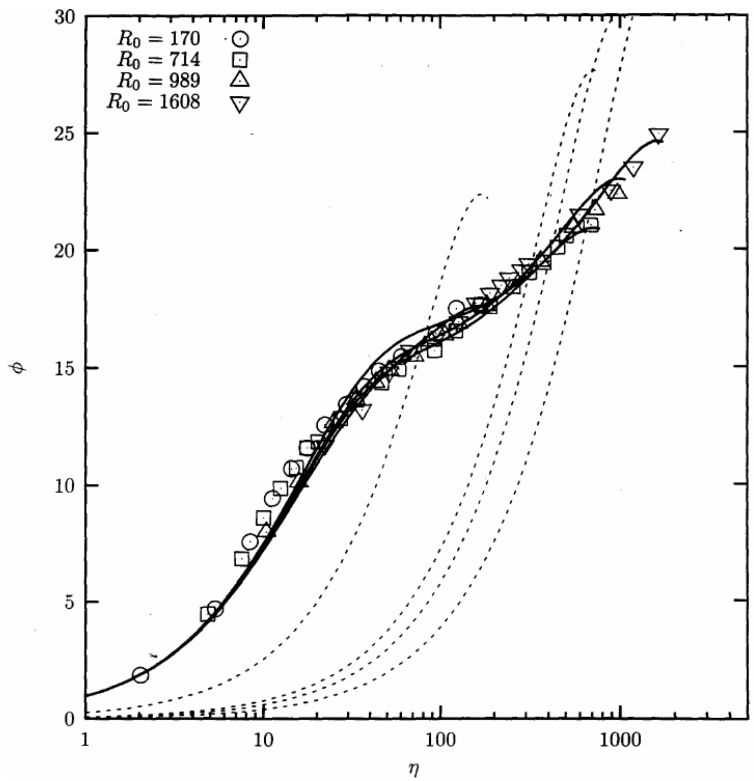


$$\frac{d - d_0}{d} = [:97; :991; :993; :997]$$

Blasius Drag Law $D = :06R^{-1=4}$

Using $Á(R_0) = Á_{max}$ as input

Figure 11 and Figure 12



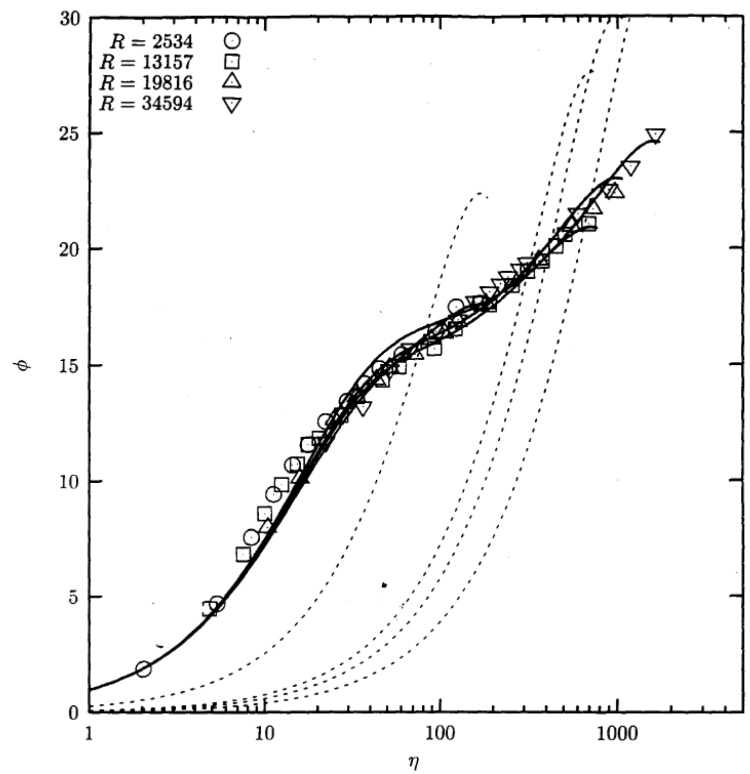
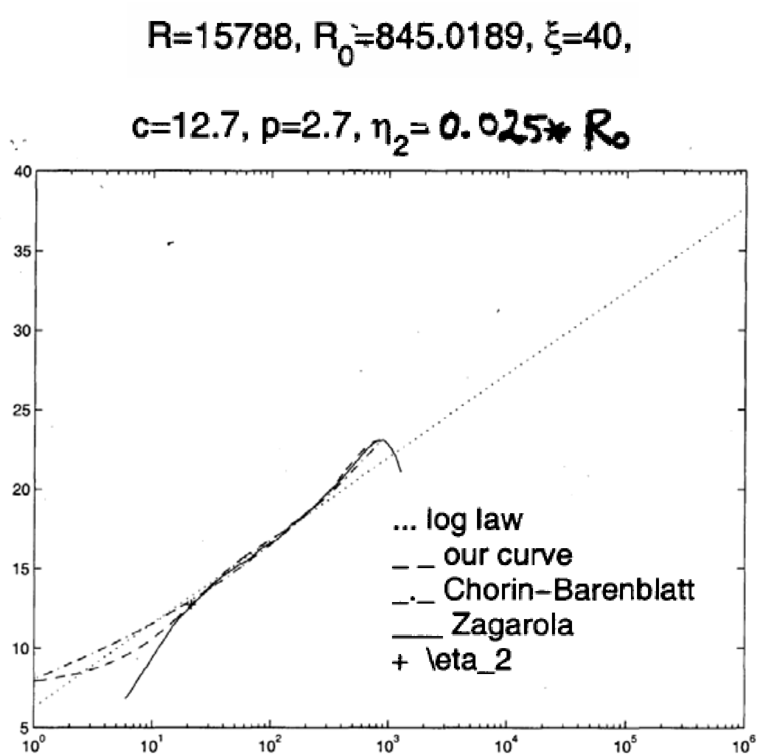


Figure 15 and Figure 16



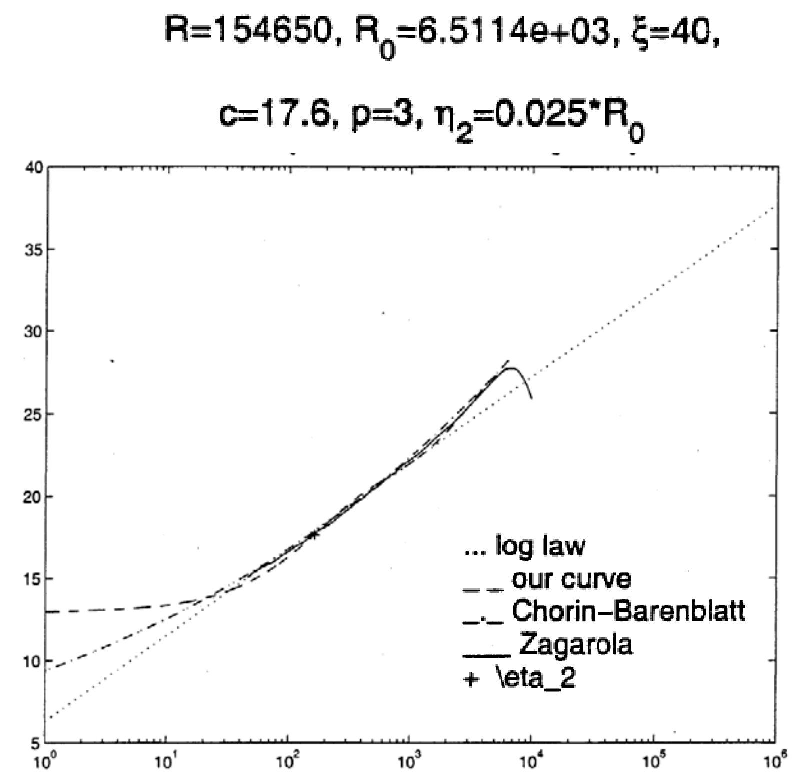
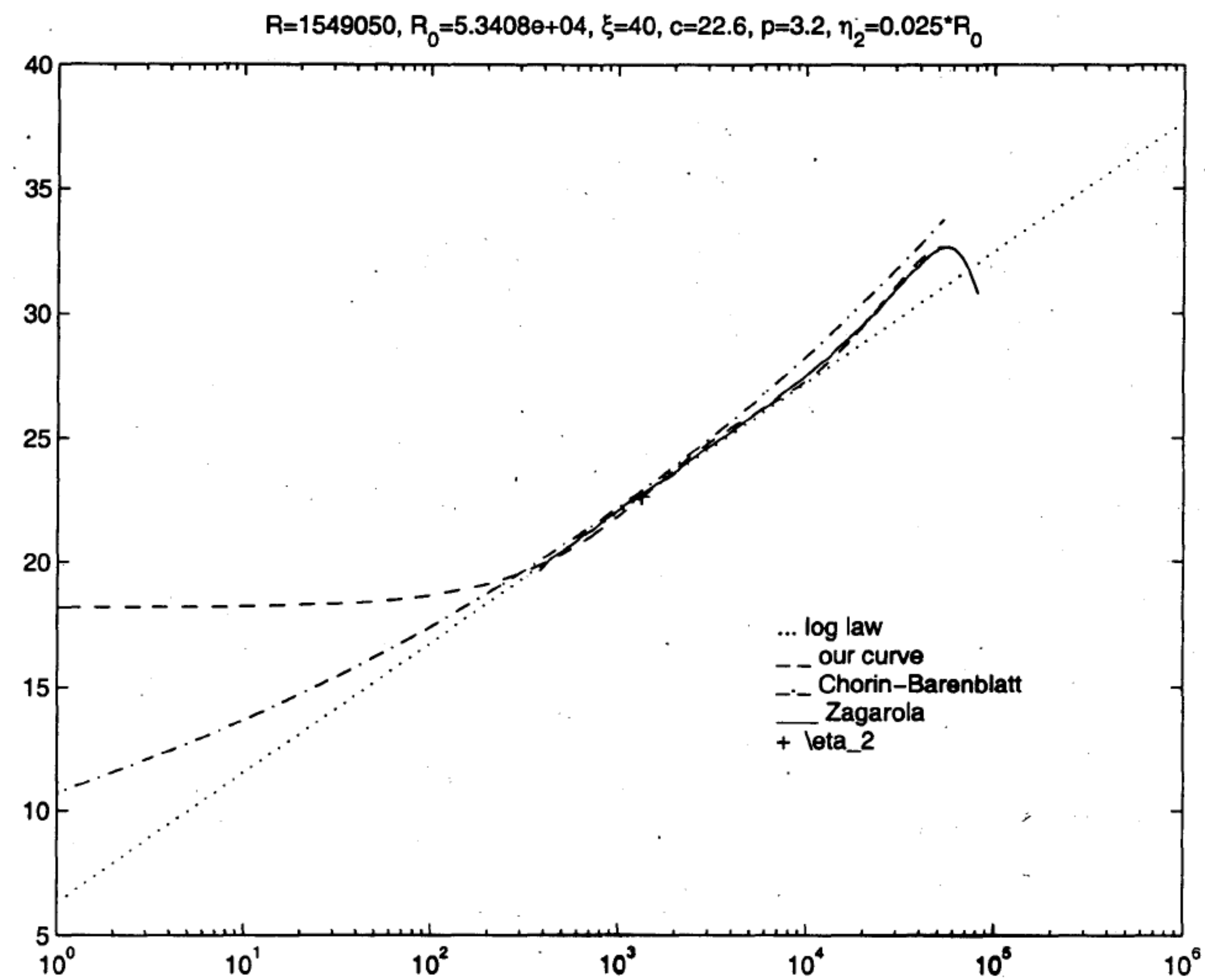
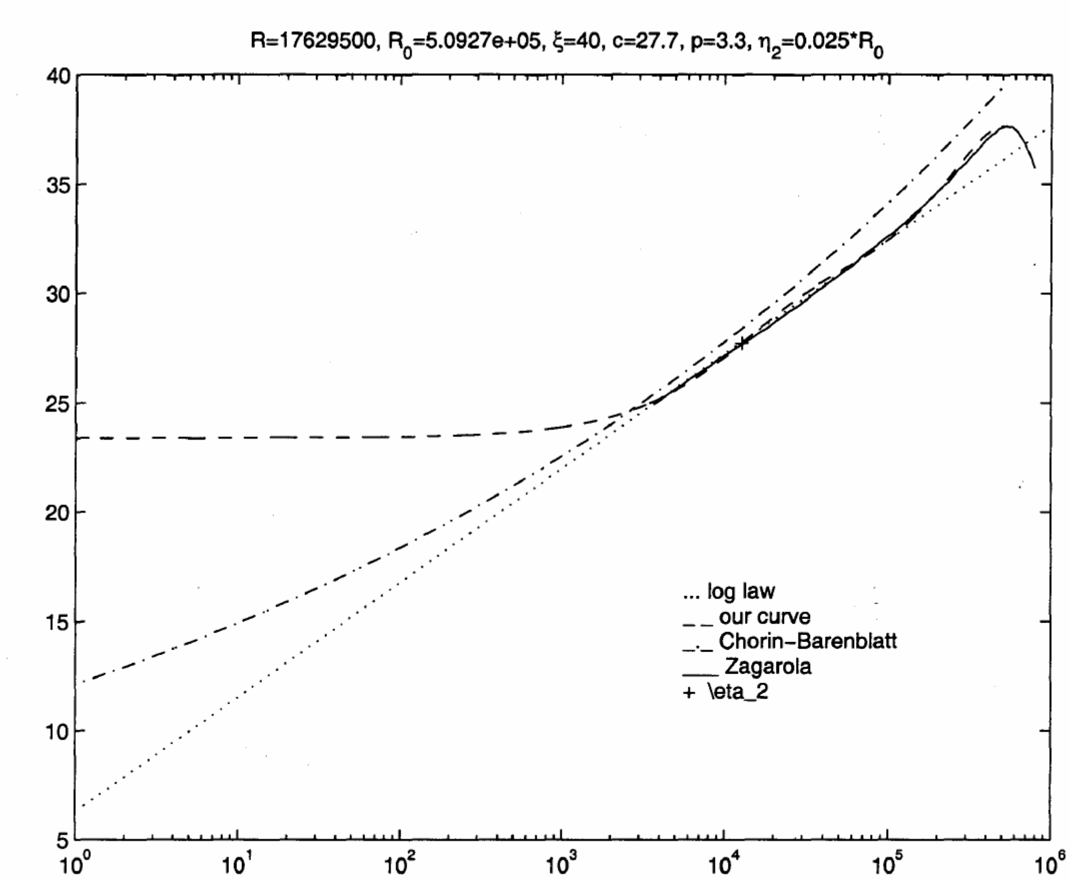


Figure 17



R=1549050, R_0=5.3408e+04, ξ=40, c=22.6, p=3.2, η_2=0.025*R_0
40
35
30
25
20
15
10
5
10^0
10^1
10^2
10^3
10^4
10^5
10^6
... log law
_ _ our curve
_._ Chorin-Barenblatt
____ Zagarola
+ \eta_2

Figure 18



R=17629500, R0=5.0927e+05, ξ=40, c=27.7, p=3.3, η2=0.025*R0
40
35
30
25
20
15
10
5
10^0
10^1
10^2
10^3
10^4
10^5
10^6
... log law
_ _ our curve
_._ Chorin−Barenblatt
___ Zagarola
+ \eta_2

Energy Spectrum

C. Foias et al. / Physica D 152–153 (2001) 505–519

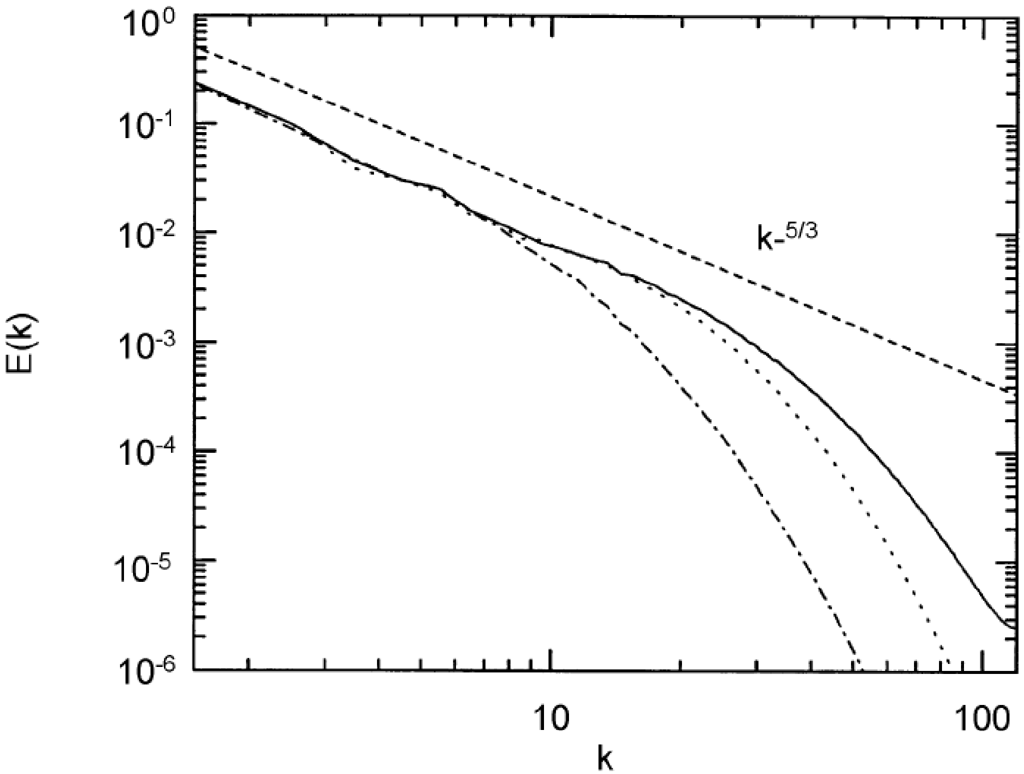
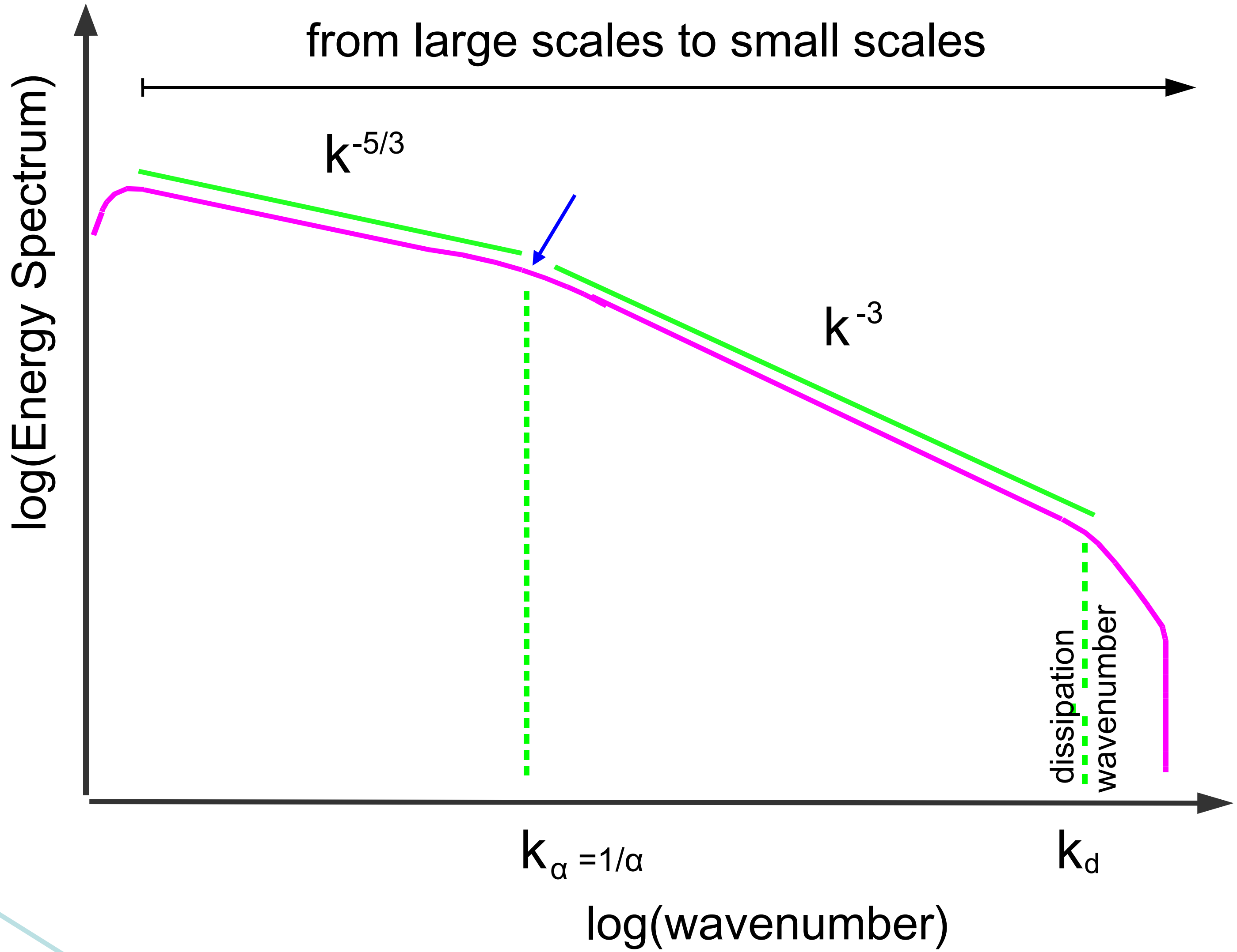


Fig. 1. The DNS energy spectrum, $E(k) = E_\alpha(k)$, versus the wavenumber k for three cases with the same viscosities, same forcings and mesh sizes of 256^3 for $\alpha = 0$ (solid line), $\frac{1}{32}$ (dotted line) and $\frac{1}{8}$ (dotted-dash line). In the inertial range ($k < 20$), a power spectrum with $k^{-5/3}$ can be identified. For finite α, this behavior is seen to roll off to a steeper spectrum for $k \geq 1/\alpha$.

Energy Spectrum (NS-α)



Foias, Holm, Titi (J. Dyn. Diff. Eqns. 2001)

The Navier-Stokes-alpha subgrid scale model of turbulence

$$\frac{\partial}{\partial_t} v - \nu \Delta v + (u \cdot \nabla) v + \sum_{j=1}^{3} v_j \nabla u_j + \nabla p = f$$

$$\nabla \cdot u = \nabla \cdot v = 0,$$

$$v = u - \alpha^2 \Delta u$$

Inviscid equation – introduced by

Holm, Marsden and Ratiu

(Phys. Rev. Let. 1998), called

Lagrangian-Averaged Euler - α

(No global well-posedness.)

original velocity

Making the nonlinearity milder

Turns into a complete gradient

under the channel and pipe symmetry

The smallest eddy scale

still participating actively

in the evolution of the flow

Lagrangian-**A**veraged

Navier-**S**tokes- α (alpha) model

(LANS-α) or

viscous **C**amassa-**H**olm equations (VCHE)

as $\alpha \to 0$ we recover NSE

Leray-α Model

NS-α

$$\frac{\partial v}{\partial t} - \nu \Delta v + (u \cdot \nabla) v - \sum_{j=1}^{3} v_j \nabla u_j + \nabla \pi = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta) u$$

Cheskidov, Holm, Olson, Titi (Royal Soc. A, MPES 2005)

The Leray-alpha analytic subgrid scale model of turbulence

$$\frac{\partial}{\partial_t} v - \nu \Delta v + (u \cdot \nabla) v + \nabla p = f$$

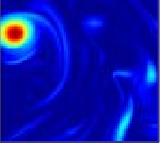
$$\nabla \cdot u = \nabla \cdot v = 0,$$

$$v = u - \alpha^2 \Delta u$$

Aside: Leray Acta Math. 1934 – Regularized NSE

$$u = \phi_\alpha * v$$

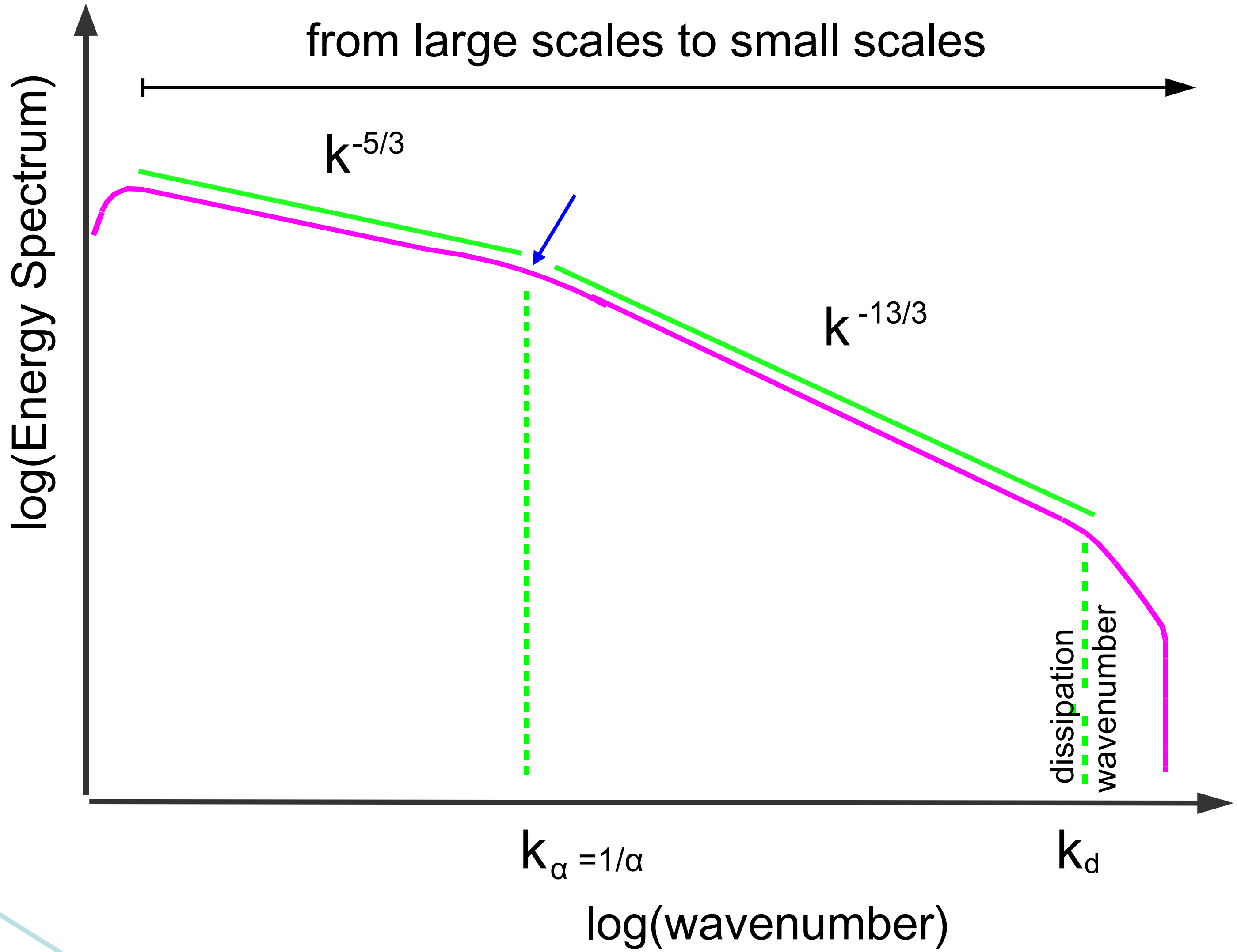
ϕ_α - the Green's function associated with $(1 - \alpha^2 \Delta)$

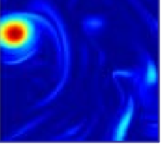


Dimension of Global Attractor (Leray-α)

$$d(\mathcal{A}) \leq C \left(\frac{L}{l_d} \right)^{12/7} \left(1 + \frac{L}{\alpha} \right)^{9/14}$$

Energy Spectrum (Leray-α)





Clark-α Model

C. Cao, D. Holm and E.S.T., *Jour. Of Turbulence,* ***6*** (2005)

The Clark-alpha subgrid scale model of turbulence

$$\partial_t v - \nu \Delta v + (u \cdot \nabla) v + (v \cdot \nabla) u - (u \cdot \nabla) u - \alpha^2 \nabla \cdot (\nabla u \cdot \nabla u^T) + \nabla q = g,$$

$$\nabla \cdot u = \nabla \cdot v = 0$$

Global Existence and Uniqueness

Attractors dimension and Energy Sepctrum like Navier-Stokes-alpha

ML-α Model

A. Ilyin, E. Lunasin and E.S.T., *Journ. Nonlinear Science,* **19,** (2006)

The Modified-Leray-alpha subgrid scale model of turbulence

$$\frac{\partial}{\partial_t} v - \nu \Delta v + (v \cdot \nabla) u + \nabla p = f$$

$$\nabla \cdot v = 0$$

$$v = u - \alpha^2 \Delta u$$

Global Existence and Uniqueness

Attractor’s dimension and Energy Spectrum like Navier-Stokes-alpha

Simplified Bardina Model

Y. Cao, E. Lunasin, and E.S.T, *Comm. Math Sci.* ***4,*** *(2006)*

Simplified Bardina turbulence model

$$\partial_t v - \nu \Delta v + (u \cdot \nabla) u = -\nabla p + f,$$

$$\nabla \cdot u = \nabla \cdot v = 0,$$

$$v = u - \alpha^2 \Delta u,$$

Y. Cao, E. Lunasin, E.S. Titi (CMS 2006)

Simplified Bardina turbulence model

$$\partial_t v - \nu \Delta v + (u \cdot \nabla) u = -\nabla p + f,$$
$$\nabla \cdot u = \nabla \cdot v = 0,$$
$$v = u - \alpha^2 \Delta u,$$

1980 Bardina $\quad \mathcal{R}(v,v) \approx \overline{\bar{v} \otimes \bar{v}} - \bar{\bar{v}} \otimes \bar{\bar{v}}$

2003 Layton, Lewandowski $\quad \mathcal{R}(v,v) \approx \overline{\bar{v} \otimes \bar{v}} - \bar{v} \otimes \bar{v}$

(no global well-posedness)

The Navier-Stokes equations

$$\partial_t v - \nu \Delta v + \nabla \cdot (v \otimes v) = -\nabla p + f,$$
$$\nabla \cdot v = 0,$$
$$v(x,0) = v^{in}(x),$$

Reynolds Average Navier-Stokes

$$\partial_t \bar{v} - \nu \Delta \bar{v} + \nabla \cdot (\overline{v \otimes v}) = -\nabla \bar{p} + \bar{f},$$
$$\nabla \cdot \bar{v} = 0,$$

$$\nabla \cdot (\overline{v \otimes v}) = \nabla \cdot (\bar{v} \otimes \bar{v}) + \nabla \cdot \mathcal{R}(v,v),$$
$$\mathcal{R}(v,v) = \overline{v \otimes v} - \bar{v} \otimes \bar{v}$$

Simplified Bardina Model

Improvement from Layton and Lewandowski (2003)

initial data: $f \in L^2,\ u(0) = u^{in} \in H^1$

weak solution: $u \in C([0,T]; H^1) \cap L^2([0,T]; H^2)$

$$\frac{du}{dt} \in L^2\left([0,T); L^2\right)$$

$$d_H(\mathcal{A}) \leq d_F(\mathcal{A}) \leq c \left(\frac{L}{\alpha}\right)^{12/5} \left(\frac{L}{l_d}\right)^{12/5}$$

Y. Cao, E. Lunasin, E.S. Titi (CMS 2006)

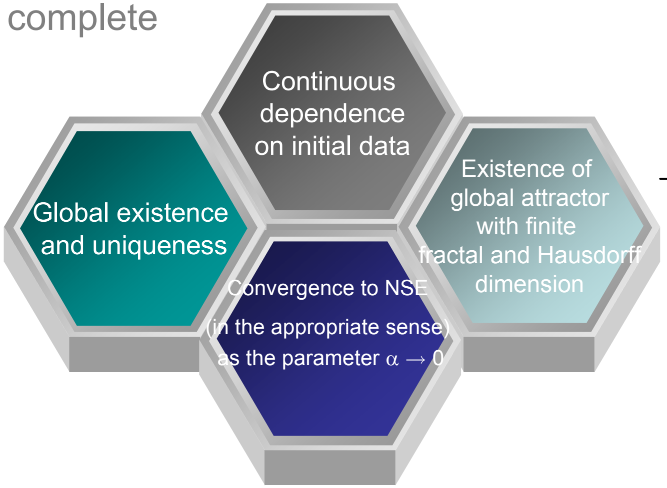
Simplified Bardina turbulence model

$$\partial_t v - \nu \Delta v + (u \cdot \nabla) u = -\nabla p + f,$$

$$\nabla \cdot u = \nabla \cdot v = 0,$$

$$v = u - \alpha^2 \Delta u,$$

The mathematical theory of simplified Bardina is complete



Excellent match with experimental data

Energy spectra

Inviscid Simplified Bardina Model

Y. Cao, E. Lunasin, E.S.T., *Communications in Math. Sciences,* **4** (2006)

$$\partial_t v - \nu\Delta v + (u \cdot \nabla)u = -\nabla p + f,$$
$$\nabla \cdot u = \nabla \cdot v = 0,$$
$$v = u - \alpha^2 \Delta u,$$

$$-\alpha^2 \Delta \frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = f,$$
$$\nabla \cdot u = 0,$$
$$u(x,0) = u^{in}$$

This result has important application in computational fluid dynamics when the inviscid model is considered as a regularizing model of the 3D Euler equations.

Also note that the inviscid simplified Bardina model is a globally well-posed model approximating the Euler equations without adding hyperviscous regularizing term.

The Navier-Stokes-Voight Model

$$-\alpha^2 \Delta \partial_t u + \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f$$

$$\nabla \cdot u = 0$$

This is a global regularization of the three-dimensional Navier-Stokes.

This regularization works also in the case of no-slip Dirichlet Boundary conditions.

Inspired by the inviscid simplified Bardina model, we propose

Navier-Stokes-Voight equations

$$-\alpha^2 \Delta \frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f,$$

$$\nabla \cdot u = 0,$$

$$u(x,0) = u^{in}$$

Introduced by Oskolkov (1973) as a model of motion of linear,viscoelastic fluids.
Models dynamics of Kelvin-Voight viscoelastic incompressible fluids.

Global attractors, estimates of the number of determining modes
by V. Kalantarov and E.S.Titi (preprint)

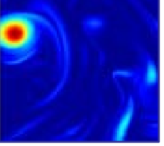
Inviscid Regularization of the Surface Quasi-Geostrophic

B. Khouider and E.S. Titi, *Communications Pure Applied Math.* (2007)

$$-\alpha^2 \Delta \theta_t + \theta_t + u \cdot \nabla \theta = 0$$

$$u = \nabla^{\perp} \left(-\Delta \right)^{-1/2} \theta$$

This inviscid regularization retains the maximum principle.



Energy Spectra for Navier-Stokes

S. Kurien
E. M. Lunasin
M. Taylor
E. S. Titi

Observation:

In 2d NS-α the conserved (° $= 0$ and f $= 0$)
"energy" and "enstrophy" are as follows

Recall that we have two kinds of velocity

NS-alpha

$$\partial_t v - u \times (\nabla \times v) + \nabla \tilde{p} = \nu \Delta v + f$$

$$v = (1 - \alpha^2 \Delta) u$$

Don't forget

v un-smoothed velocity field
u smoothed velocity field

$$\frac{1}{2}\frac{d}{dt}\langle v, u\rangle = -\nu(|\nabla u|^2 + \alpha^2 |\Delta u|^2) + \langle f, u\rangle$$

$$\langle u \times \nabla \times v, u\rangle = 0$$

energy
conserved $:= \frac{1}{2}(|u|^2 + \alpha^2 |\nabla u|^2)$

NS-alpha vorticity formulation

$$\partial_t q + (u \cdot \nabla) q = \nu \Delta q + \nabla \times f$$

vorticity $q = \nabla \times v$

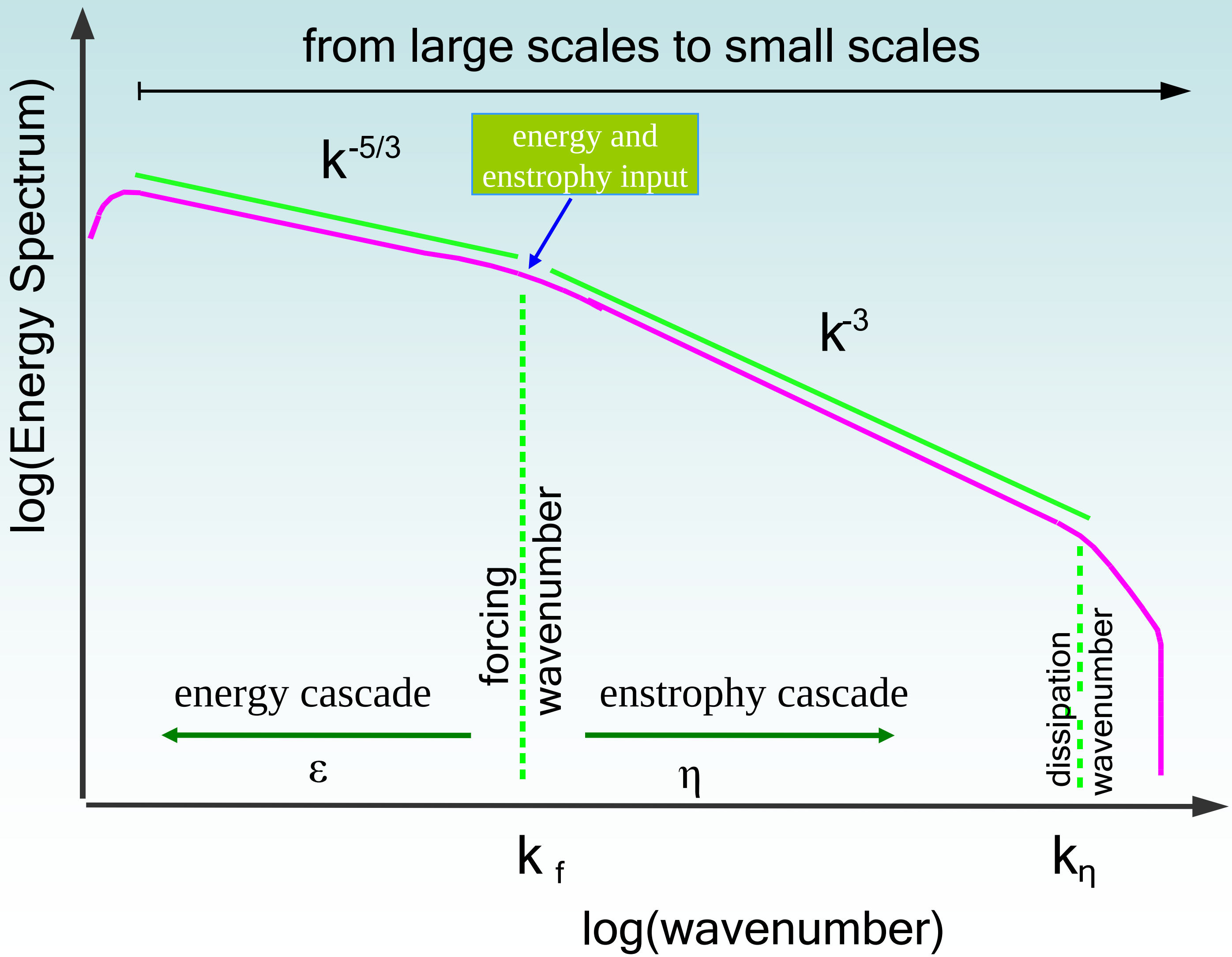
$$(q \cdot \nabla) u = \vec{0}$$

$$\frac{1}{2}\frac{d}{dt}|q|^2 = -\nu |\nabla q|^2 + \langle \nabla \times f, q\rangle$$

$$\langle u \cdot \nabla q, q\rangle = \vec{0}$$

enstrophy
conserved $:= \frac{1}{2}|q|^2$

Energy Spectrum of Two-Dimensional Navier-Stokes equations



Analytical Result 2: Power laws for the 2D NS-α

Proof: LKTT (2007, JOT):

a. Split the flow into 3 wavenumber ranges : [1,k), [k,2k), [2k,∞)

Assume $k_f < k$

$$u = u_k^< + u_k + u_k^>$$
$$v = v_k^< + v_k + v_k^>$$
$$q = q_k^< + q_k + q_k^> .$$

b. Define the energy of an eddy of size 1/k as:

$$E_\alpha(k) = (1 + \alpha^2 |k|^2) \sum_{|j|=k} |\hat{u}_j|^2$$

c. Enstrophy balance for eddy of size 1/k:

$$\frac{1}{2}\frac{d}{dt}(q_k, q_k) + \nu(-\Delta q_k, q_k) = Z_k - Z_{2k}$$

where Z_k represents the net amount of enstrophy per unit time transferred into wavenumbers larger than k.

$$Z_k := -b(u_k^<, q_k^<, q_k + q_k^>) + b(u_k + u_k^>, q_k + q_k^>, q_k^<)$$

d. Candidates for averaged velocity:

Don't forget

$$U_k^0 = \left\langle \frac{1}{L^3} \int_\Omega |v_k|^2 dx \right\rangle^{1/2} \sim$$
$$U_k^1 = \left\langle \frac{1}{L^3} \int_\Omega u_k \cdot v_k dx \right\rangle^{1/2}$$
$$U_k^2 = \left\langle \frac{1}{L^3} \int_\Omega |u_k|^2 dx \right\rangle^{1/2} \sim$$

Therefore we get the following 3 characteristic timescales:

$$\tau_k^n := \frac{1}{kU_k^n} = \frac{(1+\alpha^2k^2)^{(n-1)/2}}{k^{3/2}(E_\alpha(k))^{1/2}} \quad (n = 0, 1, 2)$$

Dissipation rate:

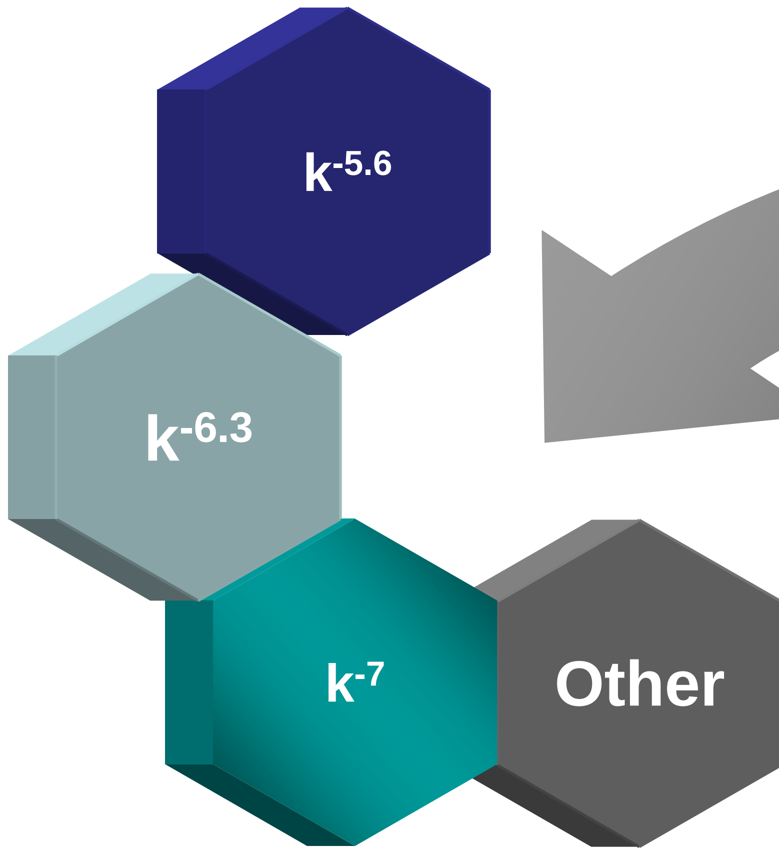
$$\eta \sim \frac{1}{\tau_k^n}\int_k^{2k}(1+\alpha^2k^2)k^2E_\alpha(k)dk \sim \frac{k^{9/2}\left(E_\alpha(k)\right)^{3/2}}{(1+\alpha^2k^2)^{(n-3)/2}}$$

Hence,

$$E_\alpha(k) \sim \frac{\eta^{2/3}(1+\alpha^2k^2)^{(n-3)/3}}{k^3}$$

Main Result: The kinetic energy spectrum for the variable u is:

$$E^u(k) \equiv \frac{E_\alpha(k)}{1+\alpha^2k^2} \sim \begin{cases} \dfrac{\eta_\alpha^{2/3}}{k^3}, & \text{when } k\alpha \ll 1\,, \\ \dfrac{\eta_\alpha^{2/3}}{\alpha^{2(6-n)/3}k^{(21-2n)/3}}, & \text{when } k\alpha \gg 1\,. \end{cases}$$

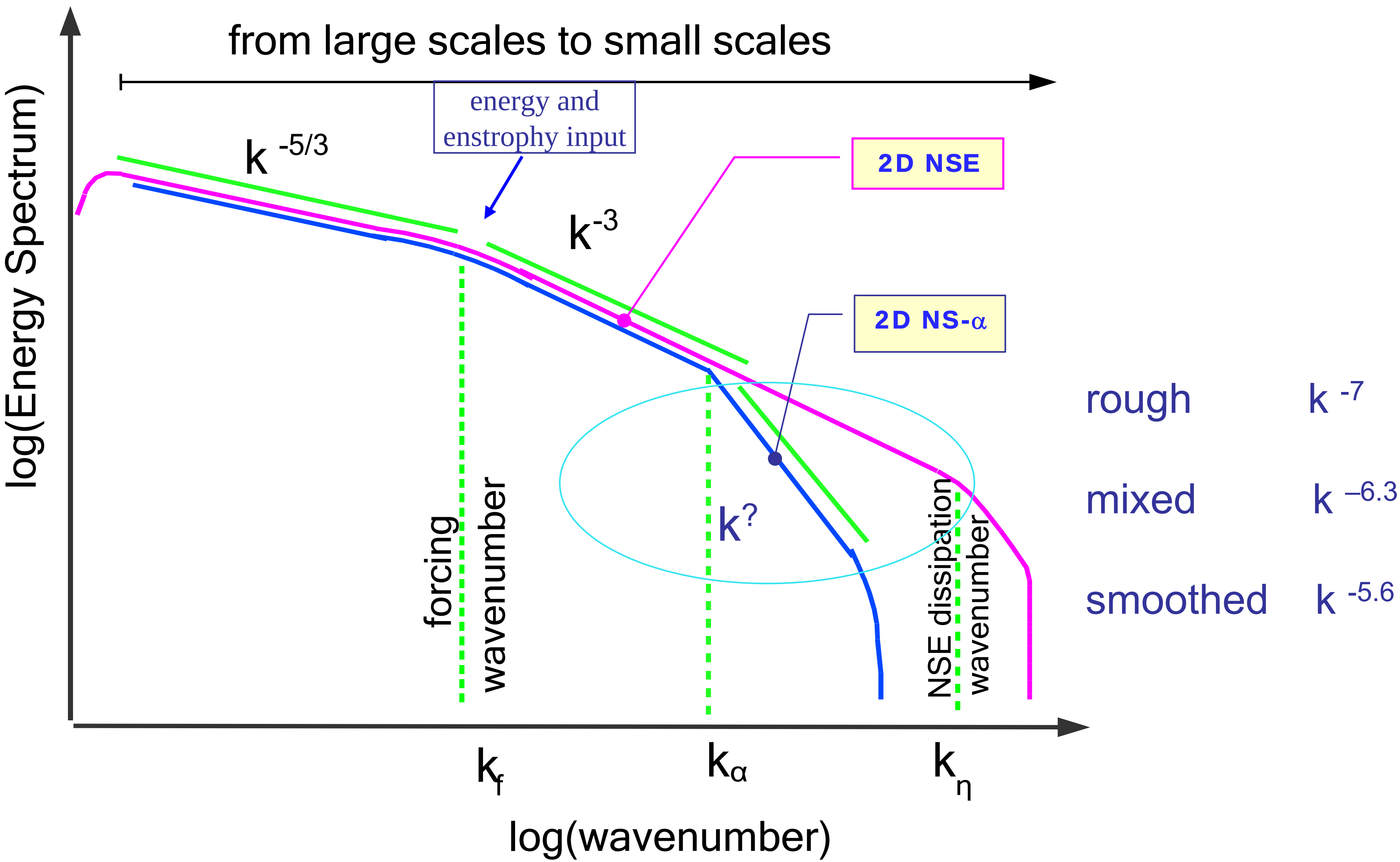


Need to check numerically

$$U_k^0 = \left\langle \frac{1}{L^3} \int_\Omega |v_k|^2 dx \right\rangle^{1/2} \sim$$

$$U_k^1 = \left\langle \frac{1}{L^3} \int_\Omega u_k \cdot v_k dx \right\rangle^{1/2}$$

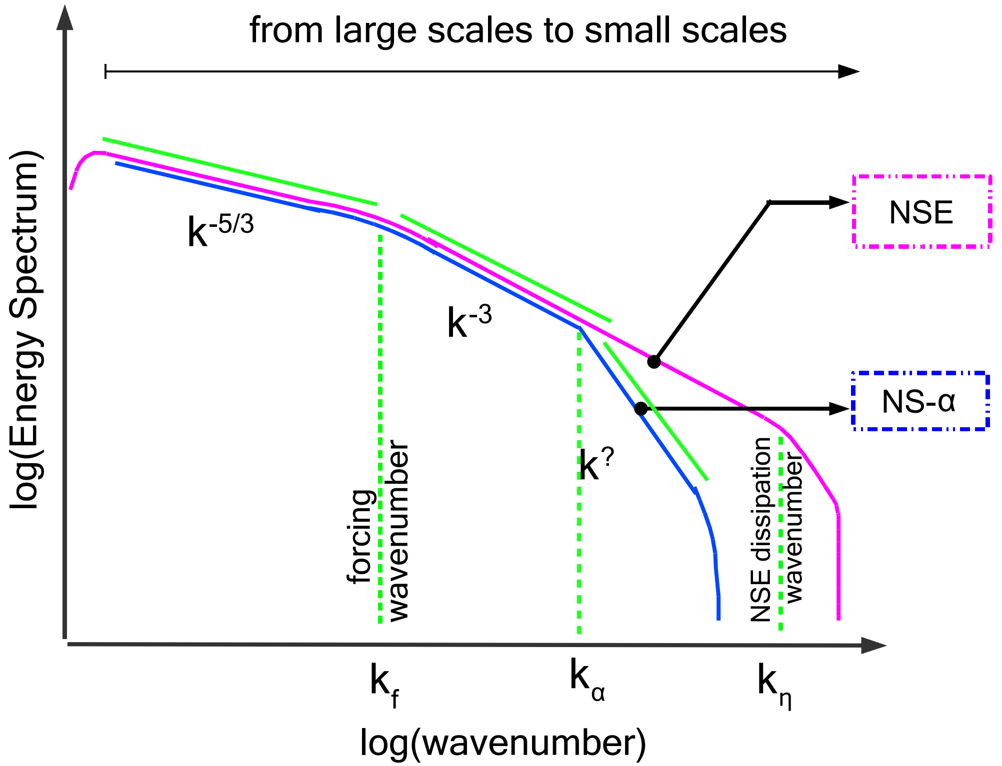
$$U_k^2 = \left\langle \frac{1}{L^3} \int_\Omega |u_k|^2 dx \right\rangle^{1/2} \sim$$



log(Energy Spectrum)
from large scales to small scales
energy and enstrophy input
k -5/3
k-3
2D NSE
2D NS-α
k?
forcing wavenumber
NSE dissipation wavenumber
rough k -7
mixed k –6.3
smoothed k -5.6
kf
kα
kη
log(wavenumber)

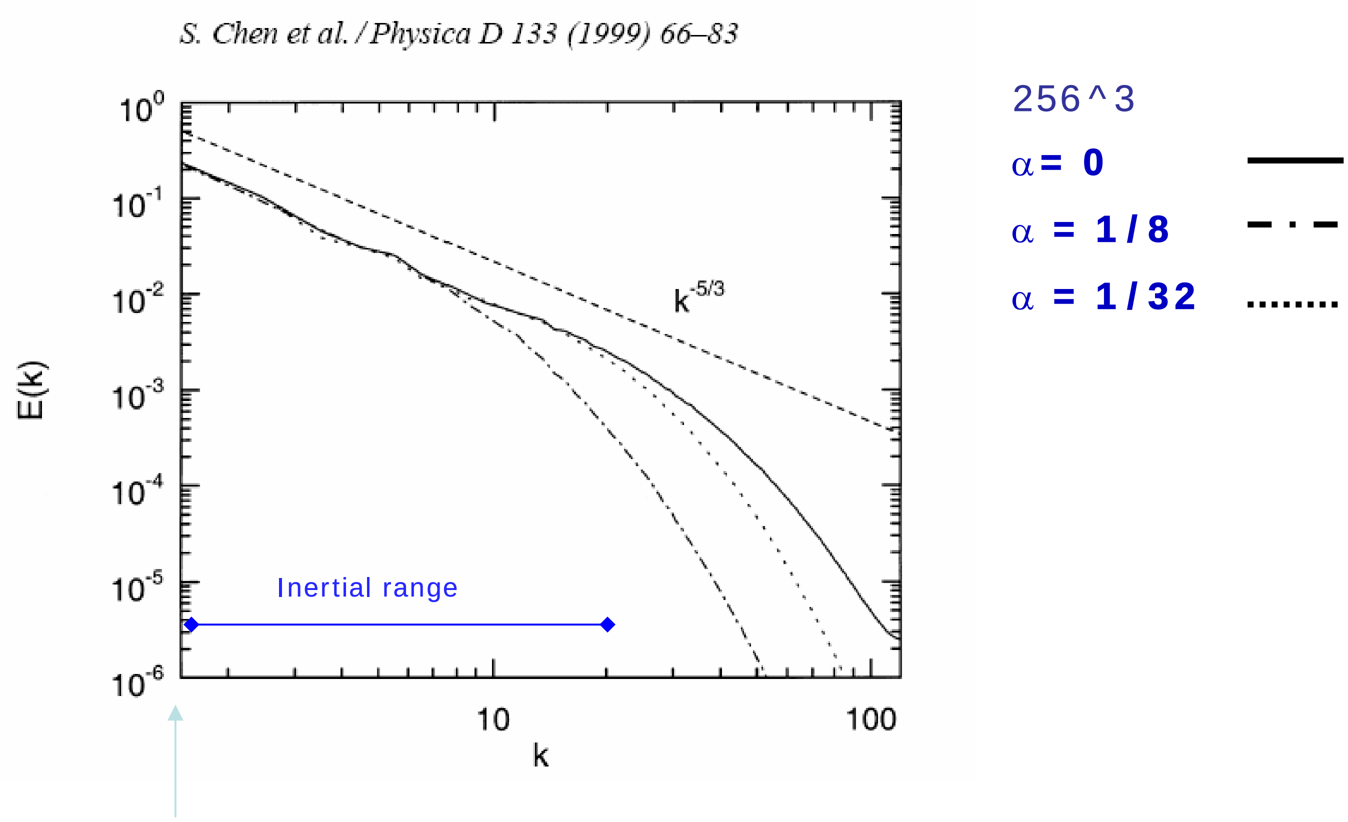
Establish two power laws in the enstrophy inertial subrange range numerically.

Verify the semi-rigorous arguments.



What has been done in 3D NS-α?

Large scale dynamics of the flow is captured by the NS-α eqautions.



Also by Mohseni, Kosovic, Shkoller and J. Marsden (2003 Phys. Fluids)

What has been done in 2D NS-α?

B. Nadiga and S. Shkoller (2001 Phys. Fluids) –
inverse energy inertial range.

Power law prediction for $k > k_\alpha$ in the forward enstrophy cascade regime $\rightarrow$ $k^{-5.6}$ (not enough resolution to verify).

Figure 1. Energy spectra for a 256^2 simulation with fixed viscosity and varying hypoviscosity coefficient μ.

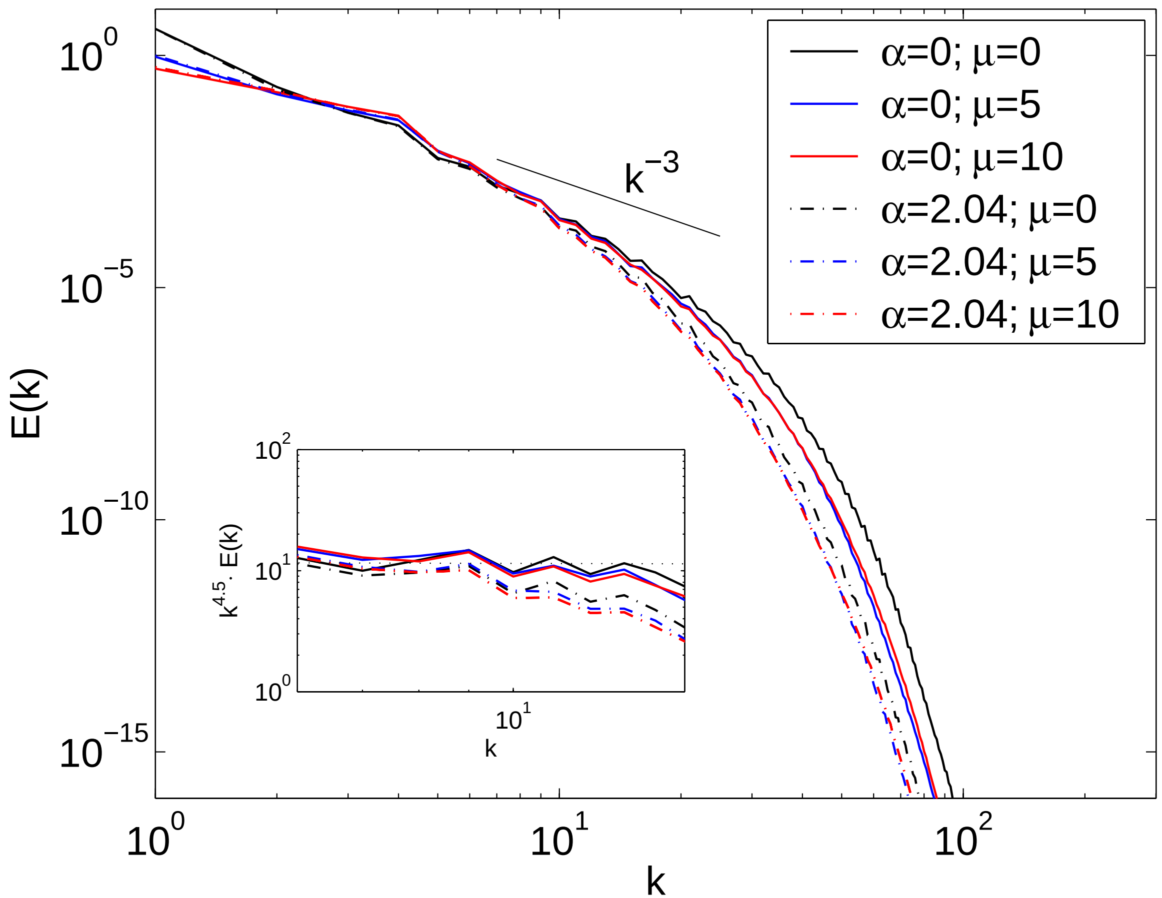
The wavenumber k is in multiples of 2 π. The solid lines are the DNS α=0 calculations of E(k).

The dotted lines are the NS-α model calculations of $E^u(k)$ for small α.

The behaviour of the spectra is largely independent of the magnitude of the hypoviscosity in the enstrophy cascade subrange (6 < k < 15).

The inset shows the spectra compensated by $k^{4.5}$.

The resolution of this simulation is far to small to observe the expected scaling exponent.



Scale (to prevent trivial dynamics)

$$\partial_t v - \nu \Delta v - u \times \nabla \times v = -\left(\alpha/L\right)^2 \nabla p + \left(\alpha/L\right)^2 f$$

$$\nabla \cdot u = \nabla \cdot v = 0$$

$$v = u - \alpha^2 \Delta u$$

Take the limit $\alpha \rightarrow \infty$

$$\partial_t v - \nu \Delta v - u \times \nabla \times v = -\nabla p + f$$

$$\nabla \cdot u = \nabla \cdot v = 0$$

$$v = -L^2 \Delta u$$

1024^2 simulation: Why NS-∞ equations?

Figure 2. Energy spectra for 1024^2 simulation.

The black curve is the DNS ($\alpha = 0$) which shows close to k^{-3} scaling in the enstrophy cascade range $6 < k < 20$.

The solid red curve is the $E^u(k)$ spectrum as $\alpha \to \infty$ which scales close to k^{-7} in the enstrophy cascade range $6 < k < 25$.

The energy spectra for intermediate values of α tend to the $\alpha \to \infty$ limit as α increases.

The inset shows the DNS energy spectrum (black) compensated by $k^{3.7}$ and the $\alpha \to \infty$ energy spectrum (red) compensated by $k^{7.4}$

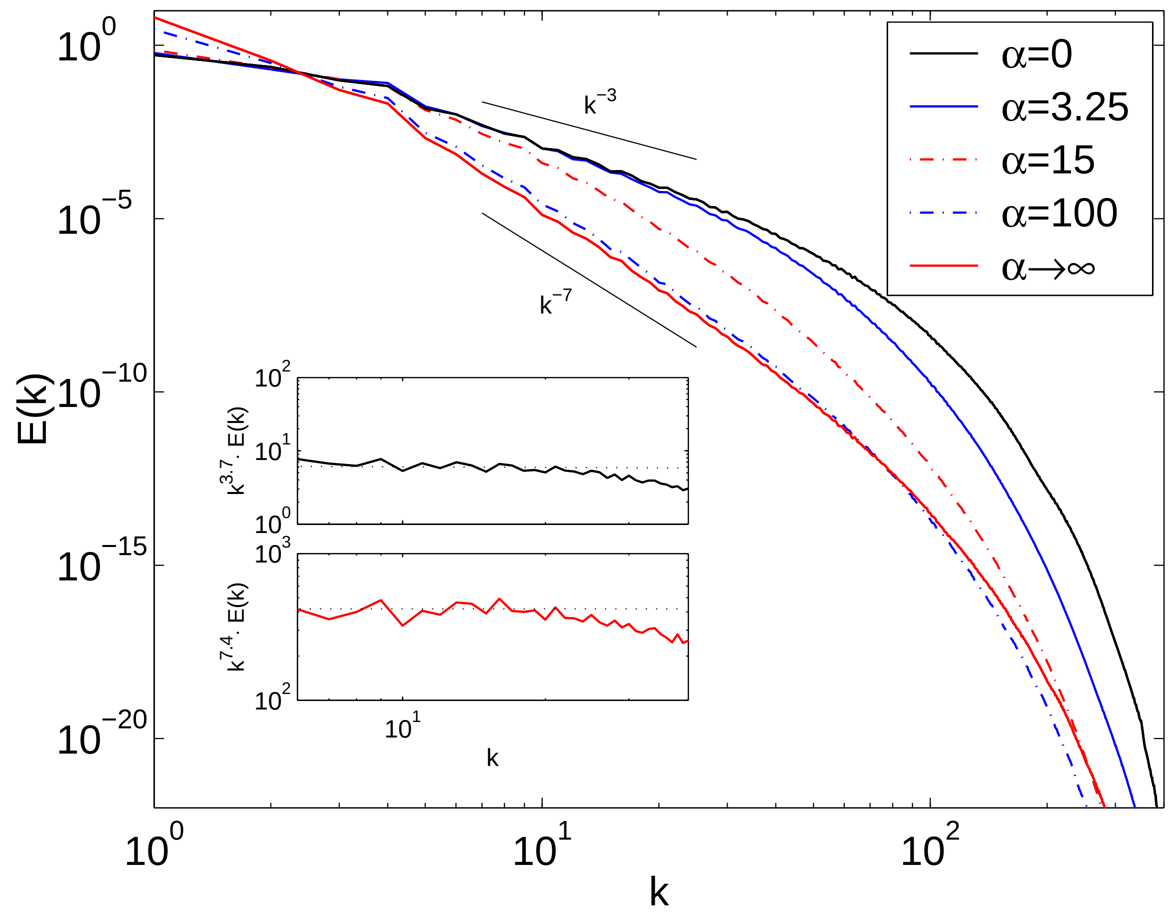


Figure 3. Energy spectra for 2048^2 simulation.

The wavenumber is in multiples of 2π.

The black curve is the energy spectrum of the DNS which shows close to k^{-3} scaling in the enstrophy cascade range 6<k<35.

The solid red curve is the $E^u(k)$ spectrum as $\alpha \to \infty$ which scales approximately as $k^{-7.1}$ in the wavenumber region 6<k<25.

The inset shows the DNS energy spectrum (black) compensated by $k^{3.5}$ and the $\alpha \to \infty$ energy spectrum (red) compensated by $k^{7.1}$

2048^2

Comparing energy spectra for different values of α

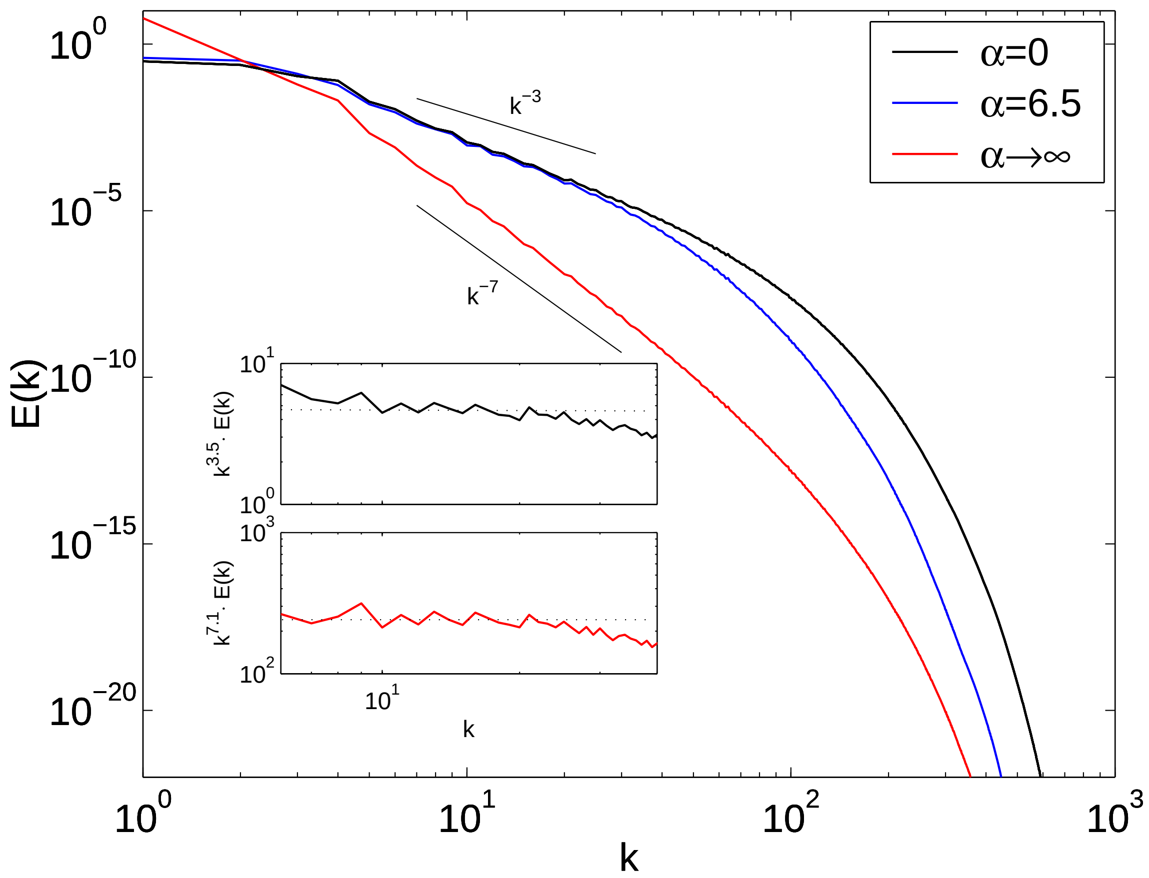


Figure 4. Energy spectra for 4096^2 simulation.

The black curve is the spectrum for the DNS, the red curve is the spectrum for $\alpha \rightarrow \infty$.

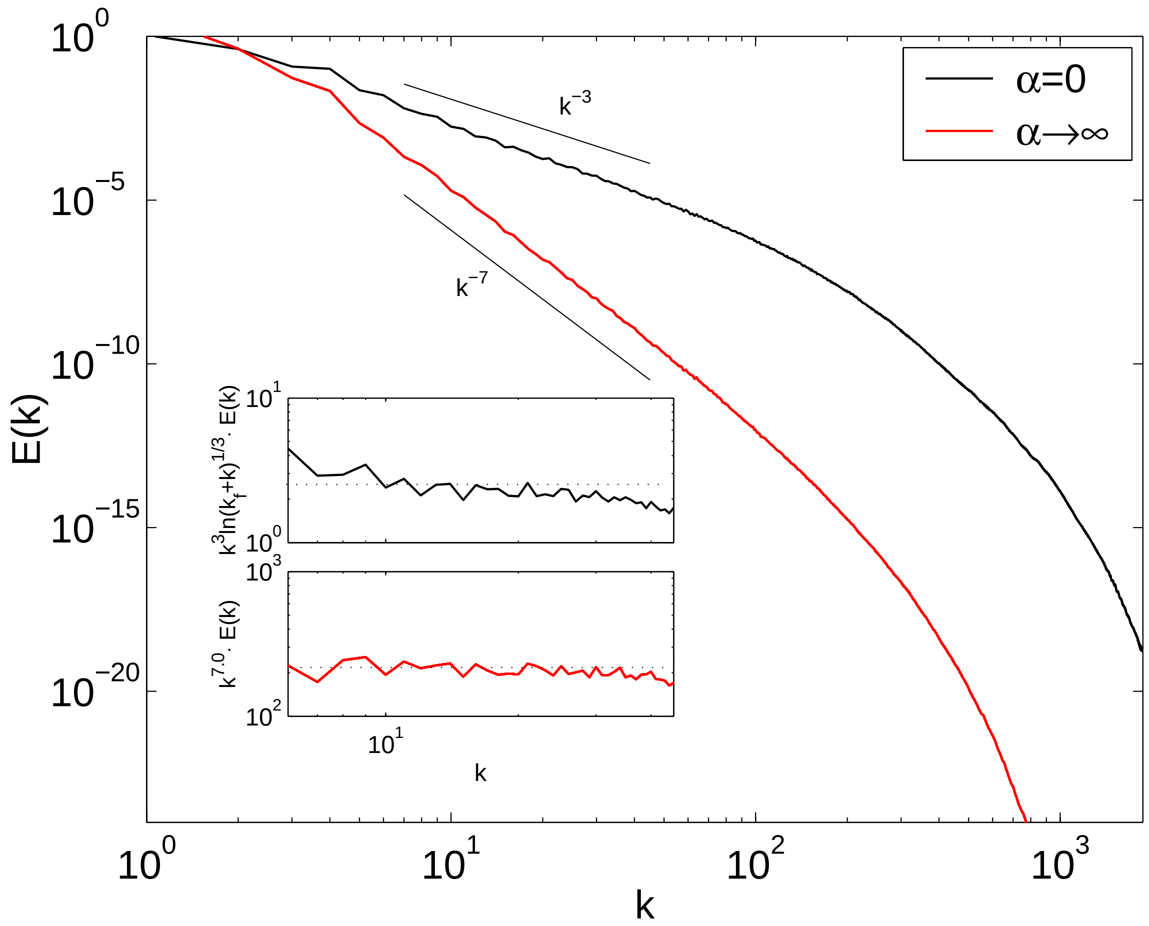
The black curve in the inset corresponds to the NSE energy spectrum compensated by $k^3 \ln(k_f+k)^{1/3}$.

The red curve in the inset is the energy spectrum $E^u(k)$ for NS-∞ compensated by k^7.

The region 6 < k < 40 is flat indicating the nominal range over which the k^{-7} scaling holds.

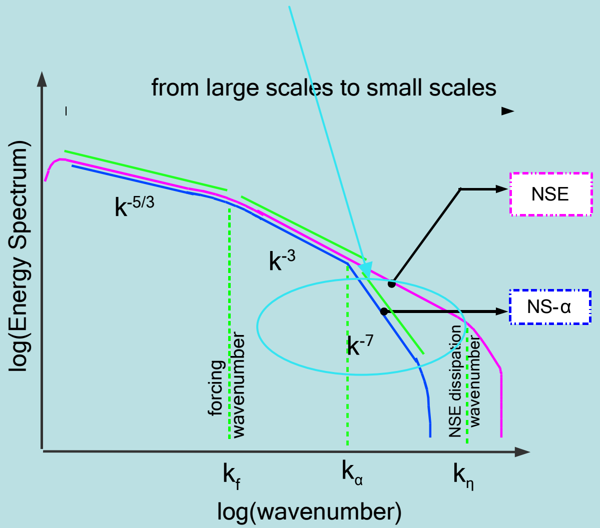
4096^2

Power law for NS-∞



Conclusion:

k^{-7} power law



4096^2

Power law for NS-∞

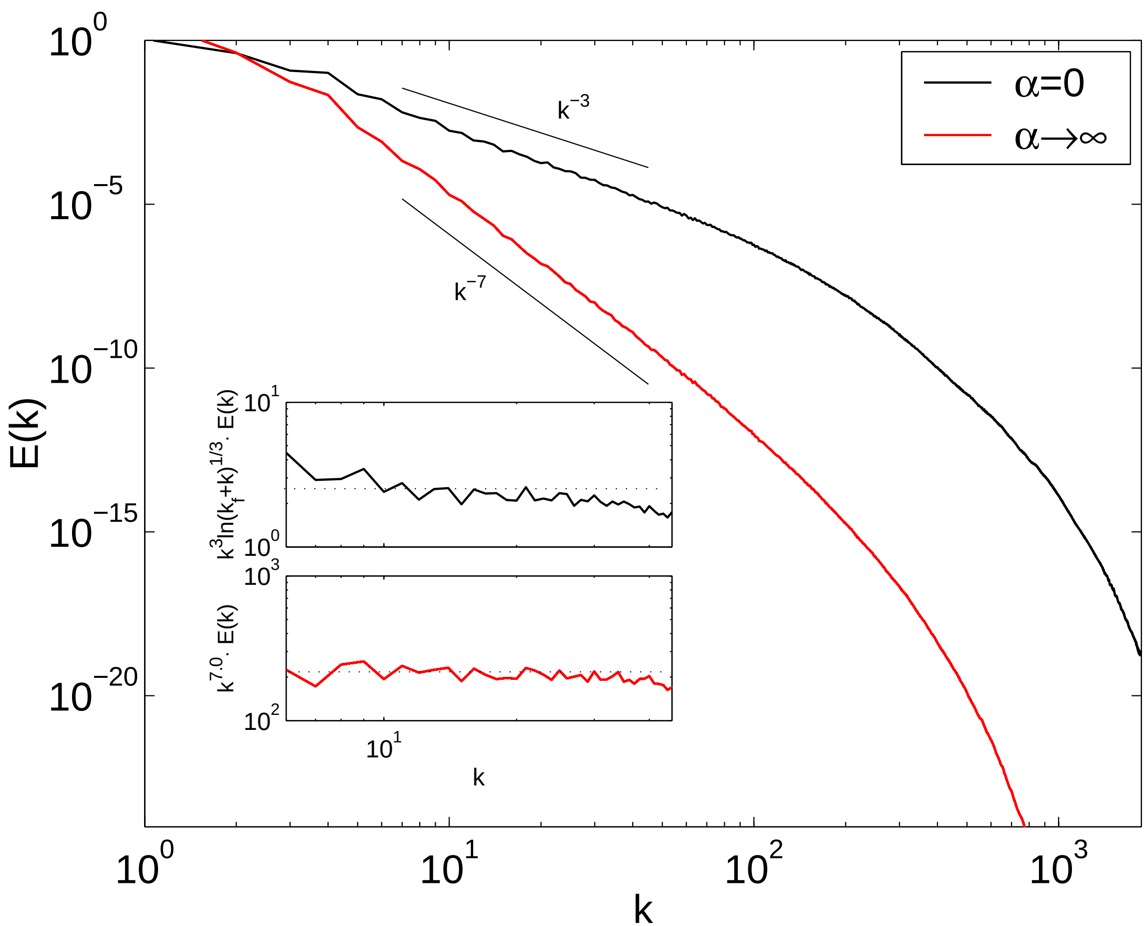
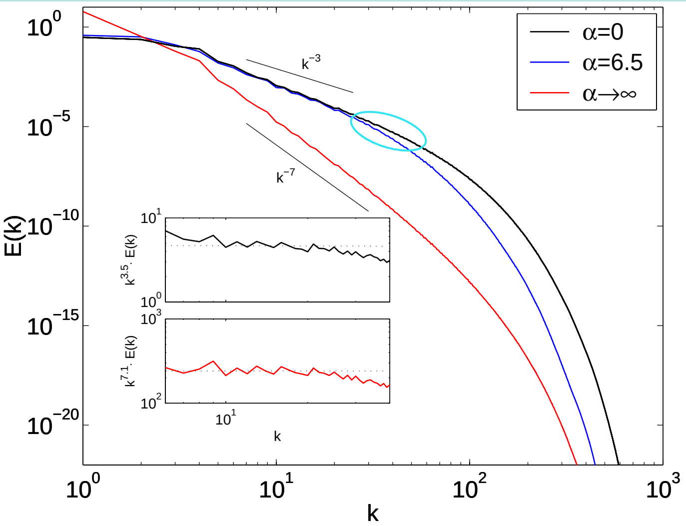


Figure 5. Compensated energy spectra for 2048^2 simulation for α =6.5 (k_α=39.75; vertical dashed line).

The energy spectrum is compensated by k^7, $k^{19/3}$, and, $k^{17/3}$ respectively.

The region 39 < k < 70 in the first subplot follows a flat regime which indicates the nominal range over which the k^{-7} scaling holds.



2048^2

Power law for finite $\alpha = 6.5$

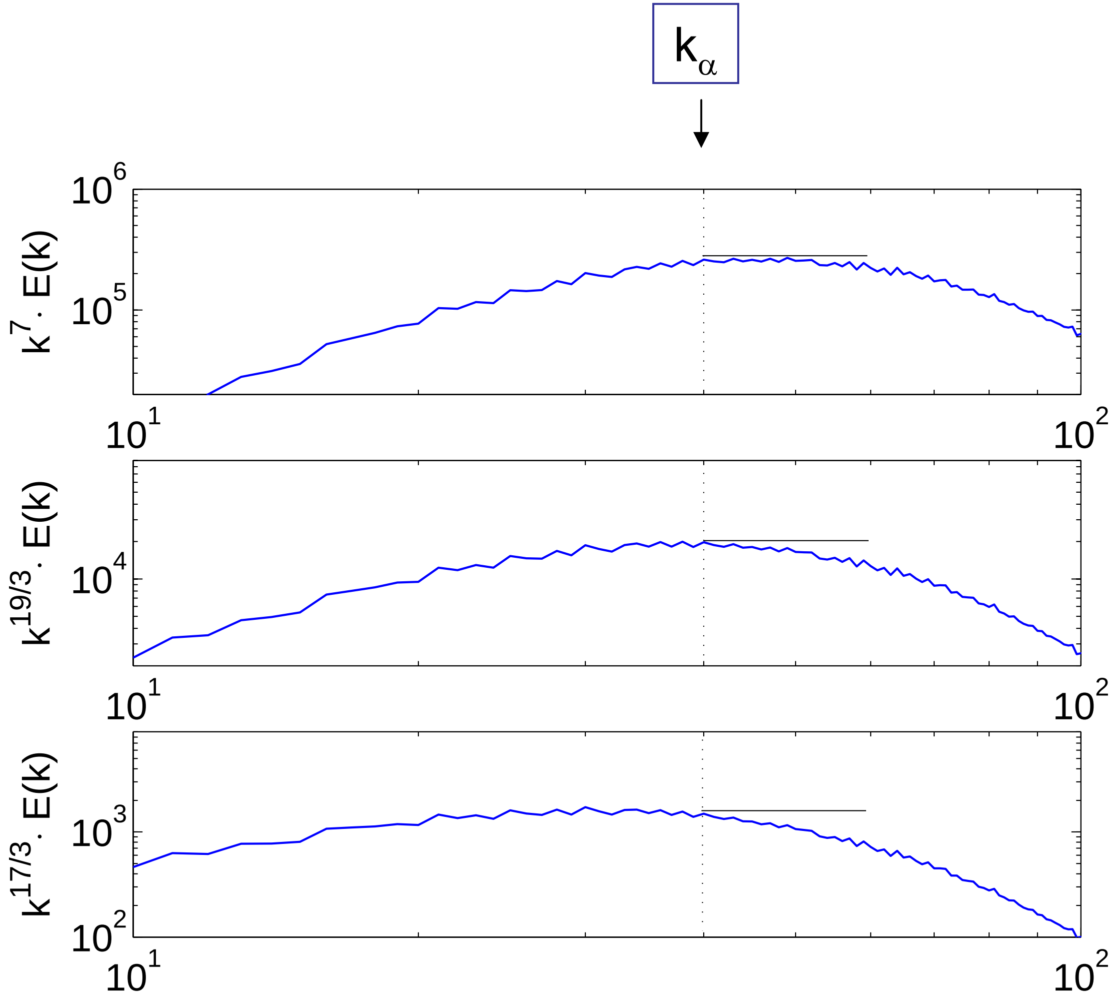


Figure 6. Isosurfaces of vorticity $\nabla \times$ v for the 1024^2 simulation. α = 0, 3.25, 15, 100, ∞, reading each row of figures from left to right. The vorticity field exhibits increasingly fine structures as α is increased.

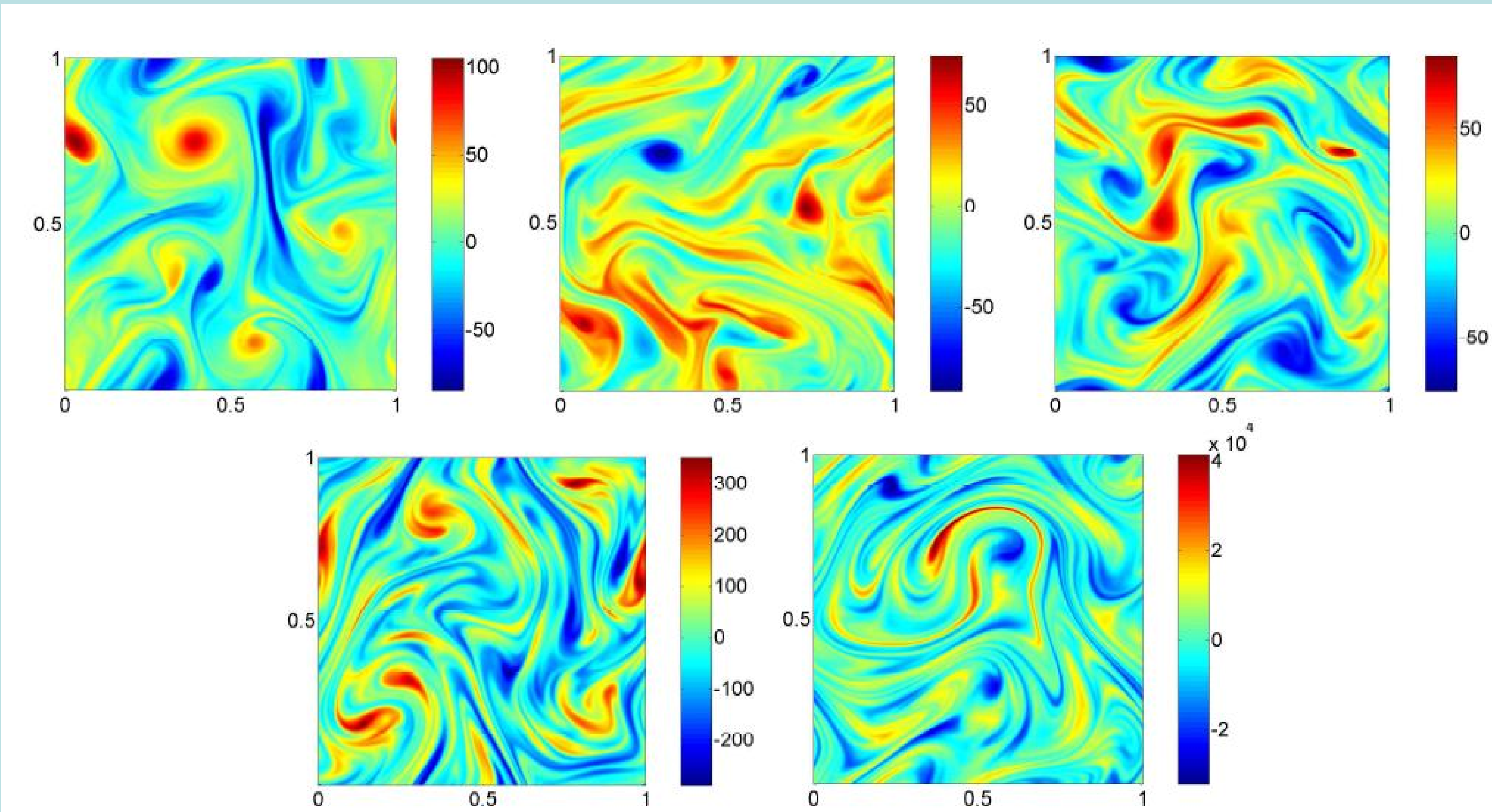
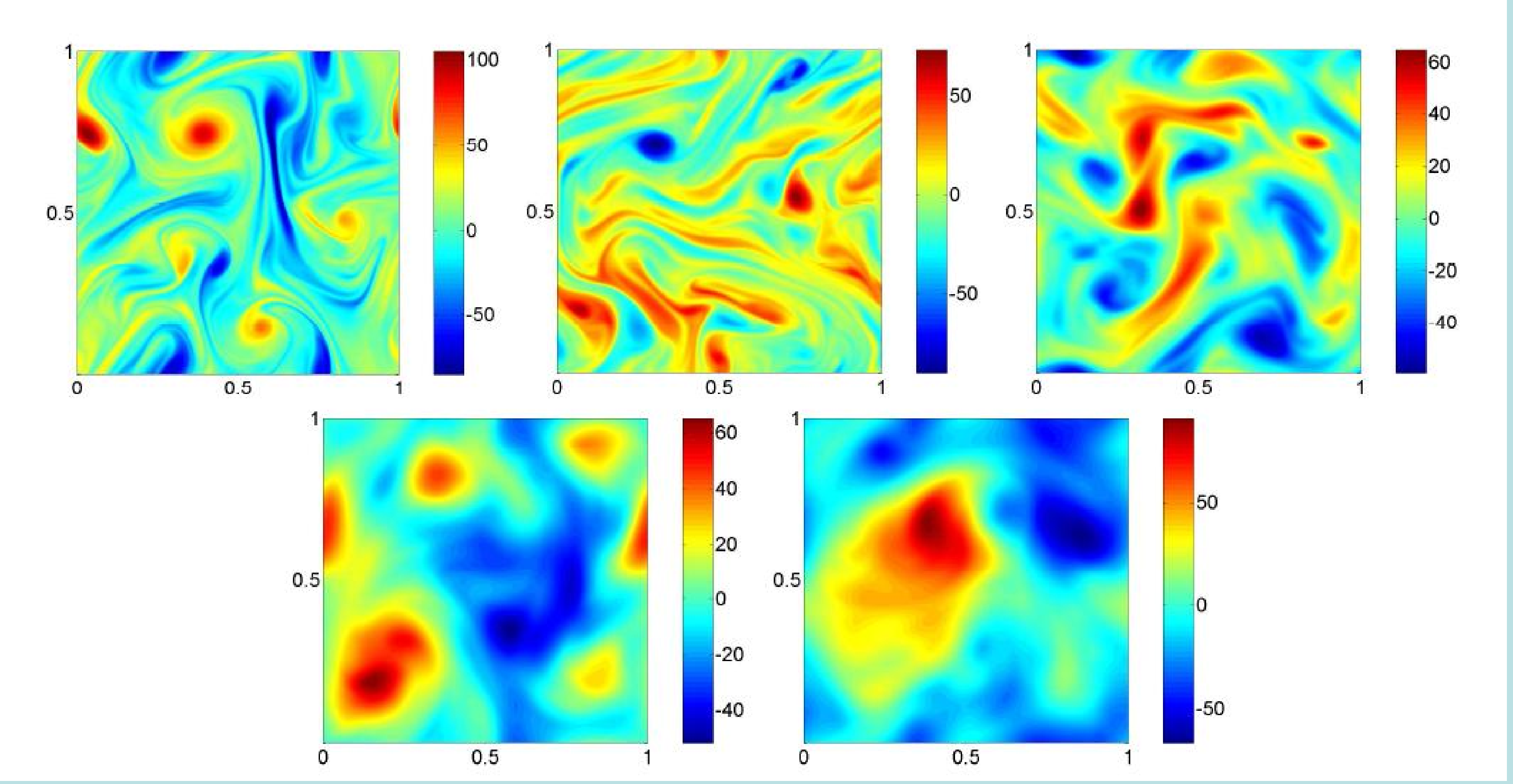


Figure 7. Isosurfaces of vorticity $\nabla \times u$ for the 1024^2 simulation. α = 0, 3,25, 15, 100, ∞, reading each row of figures from left to right. The structures become smoother with increasing α.



Thank You!